Banach J. Math. Anal. (2021) 15:32
https://doi.org/10.1007/s43037-020-00118-2

Tusi Mathematical Research Group

ORIGINAL PAPER

Phase retrieval for continuous Gabor frames on locally compact abelian groups

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Received: 22 July 2020 / Accepted: 15 December 2020

Abstract

In this paper, we study continuous frames from projective representations of locally compact abelian groups of type $\widehat{G} \times G$. In particular, using the Fourier transform on locally compact abelian groups, we obtain a characterization of maximal spanning vectors. As an application, for G, a compactly generated locally Euclidean locally compact abelian group or a local field with odd residue characteristic, we prove the existence of maximal spanning vectors, hence the phase retrievability, for the associated $(\widehat{G} \times G)$-frames.

Keywords The Stone–von Neumann theorem · Continuous frame · Maximal spanning vector · Phase retrieval · Fourier analysis on local fields

Mathematics Subject Classification 20C25 · 20C35 · 11S80 · 42C15 · 43A70

1 Introduction

Being phase retrievable is an important property of frames in the study of signal analysis. In the case of finite frames with symmetries, e.g., group frames and twisted group frames, there are many results connecting maximal spanning vectors, vectors with the Haar property, and phase retrievable finite frames (cf. [1–7, 10, 11, 23, 24] etc.). In this paper, we study similar properties for continuous frames generated by one vector from projective representations of locally

Communicated by Michael Frank.

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Published online: 20 January 2021

compact abelian groups of type $\widehat{G} \times G$. In this paper, all locally compact groups are assumed Hausdorff and second countable, and all Hilbert spaces are assumed separable.

Let V be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. Let $F : \Omega \to V$ be a *continuous frame* with respect to (Ω, μ), i.e.,

1. F is weakly measurable, i.e., for all $v \in V$, the map $\omega \mapsto \langle v, F(\omega)\rangle$ is a measurable function on Ω;
2. there exist constants $A, B > 0$ such that

$$A||v||^2 \leq \int_\Omega |\langle v, F(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \leq B||v||^2, \quad \text{for all } v \in V. \tag{1.1}$$

For simplicity, we say that the set of vectors $\{F(\omega) \mid \omega \in \Omega\}$ is a continuous frame. We refer to [15, 26] for basic properties of continuous frames. The triple (V, Ω, F) gives us a map

$$\begin{aligned} t : V &\to L^2(\Omega) \\ v &\mapsto (\omega \mapsto |\langle v, F(\omega)\rangle|). \end{aligned}$$

Then, we obtain a map $T : V/\mathbb{T} \to L^2(\Omega)$. We say that the frame F with respect to (Ω, μ) is *phase retrievable* if the map T is injective. Let $\mathcal{S}_F$ be the set of operators $F(\omega) \otimes F(\omega)$ $(\omega \in \Omega)$, where

$$\begin{aligned} F(\omega) \otimes F(\omega) : V &\to V \\ v &\mapsto \langle v, F(\omega)\rangle F(\omega). \end{aligned}$$

We say that $\mathcal{S}_F$ has *maximal span* if $\overline{\mathrm{Span}}(\mathcal{S}_F) = \mathrm{HS}(V)$. Here $\mathrm{HS}(V)$ denotes the space of Hilbert-Schmidt operators on V. In this case, the operator $x \otimes x$ is determined by $\langle x \otimes x, F(\omega) \otimes F(\omega)\rangle_{\mathrm{HS}} = |\langle x, F(\omega)\rangle|^2$. Hence the frame F is phase retrievable (cf. [6, Section 3]). In the case where V is finite dimensional and (Ω, μ) is a finite set with counting measure, there have been many results about the phase retrievability of a finite frame in the literature, see for example [1–3, 6] and the references there.

Now, let G be a group and $\pi : G \to \mathbf{U}(V)$ be a (projective) representation of G on the Hilbert space V. Let $v \in V$ be a nonzero vector and $\Phi_v = \{\pi(g)v \mid g \in G\}$. In the simple case where G is finite and π is irreducible, Φ_v is a frame for V. By showing that the associated frame satisfies the Haar property ([23, Definition 2]), Lawrence, Pfander, and Walnut [23, Theorems 1, 2] proved the following result.

Theorem 1.1 (Lawrence, Pfander, Walnut) *Let p be a prime number. Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\pi : G \to \mathbf{U}(V)$ be the Weyl-Heisenberg representation* ([23, *Definition* 1]). *Then the set of $v \in V$ for which Φ_v is phase retrievable is dense in V.*

One important ingredient of the proof of the above theorem is a nonvanishing property of the generalized Vandermonder determinants ([23, Lemma 4]), which holds only for prime numbers p. On the other hand, using maximal spanning vectors and tools from representation theory, Li, Han, etc. [10, 24] proved the following result, which generalizes Theorem 1.1.

Theorem 1.2 (Li, Han, etc.) *Let G be any finite abelian group and $\pi : G \to \mathbf{U}(V)$ be an irreducible projective representation of G. Then the set of $v \in V$ for which Φ_v is phase retrievable is dense in V.*

The above results focused on phase retrievability of finite frames. In [11], Cheng and Li proved a version of Theorem 1.2 for compact groups and explained the similarity between compact continuous frame case and finite Gabor frame case. In this paper, we study continuous frames with locally compact symmetries. The situation is different as the representation spaces are now usually of infinite dimension and divergence causes trouble (e.g. Sect. 3.2.1). On one hand, as a complementary part of paper [11], we construct explicit examples of continuous frames with symmetries via projective representations of locally compact abelian groups. The tool we use here is the Stone–von Neumann theorem and its variation in terms of projective representations. On the other hand, for the vector space we constructed, we define the notion of a maximal spanning vector. Motivated by results for finite abelian groups, we propose a conjecture (Conjecture 1.4) on the existence of maximal spanning vectors and prove the conjecture for: (a) compactly generated locally Euclidean locally compact abelian groups, i.e. groups of the form $\mathbb{R}^a \times \mathbb{Z}^b \times \mathbb{T}^c \times E$, where a, b, c are non-negative integers and E is a finite abelian group; and (b) local fields with residue characteristic p ($p > 2$). The tool we use here is the Fourier transform, which provides us a characterization of maximal spanning vectors.

We outline the contents of the paper in what follows. Let G be a locally compact abelian group with a fixed Haar measure. Let $\widehat{G}$ be the dual group of G, which is also considered as a locally compact group with the Plancherel measure. For $(v^*, v) \in \widehat{G} \times G$, define an operator $\pi(v^*, v)$ on $L^2(G)$ via

$$(\pi(v^*, v)f)(u) = v^*(u)f(uv),$$

where $f \in L^2(G)$. Then, this π defines an α-representation $\pi : \widehat{G} \times G \to \mathbf{U}(L^2(G))$, where α is the 2-cocycle in $Z^2(\widehat{G} \times G, \mathbb{T})$ given by

$$\alpha((v_1^*, v_1), (v_2^*, v_2)) = v_2^*(v_1).$$

We refer to [11, 19, 21, 22] for basic properties of projective representations. Using the correspondence between projective representations of $\widehat{G} \times G$ and linear representations of the group $G(\alpha) \in \mathrm{Ext}^1(\widehat{G} \times G, \mathbb{T})$ (cf. [22]), we obtain the following result, which is a variation of the Stone-von Neumann theorem (cf. [8, Theorem 4.8.2], [9, Section 2.3], [10, Section 3.2], [13, Theorem 10.2.1], [20]).

Theorem 1.3 (Segal, Shale, Weil) *The α-representation π is irreducible. Moreover, up to isomorphism, π is the unique irreducible α-representation of $\widehat{G} \times G$ on a Hilbert space.*

Take any nonzero vector $f \in L^2(G)$, because of the irreducibility of π, we have

$$\overline{\text{Span}}\{\pi(v^*, v)f \mid (v^*, v) \in \widehat{G} \times G\} = L^2(G).$$

Moreover, the set $\Phi_f := \{\pi(v^*, v)f \mid (v^*, v) \in \widehat{G} \times G\}$ forms a tight continuous frame with bound $||f||_2$ (Theorem 2.1). In time-frequency analysis, Φ_f is called a *Gabor system* (cf [17]). Note that the construction does not work for projective representations of general locally compact groups as equation (1.1) may fail.

For $f \in L^2(G)$, we construct a function $c_{f,f} : \widehat{G} \times G \to \mathbb{C}$ via

$$c_{f,f}(v^*, v) = \langle \pi(v^*, v)f, f \rangle. \tag{1.2}$$

It turns out that $c_{f,f} \in L^2(\widehat{G} \times G)$ (cf. Section 3.1). We call f a *maximal spanning vector* if

$$\overline{\text{Span}}\{c_{\pi(v^*,v)f,\pi(v^*,v)f} \mid (v^*, v) \in \widehat{G} \times G\} = L^2(\widehat{G} \times G).$$

The relation between maximal spanning vectors and phase retrieval problems is explained in Sect. 3.2.2, where we show that f is a maximal spanning vector if and only if the set of operators $\mathcal{S}_{\Phi_f}$ has maximal span. Motivated by the results for finite abelian groups (cf. [24, Conjecture], [10, Section 3], [11, Section 3.3]), we propose the following conjecture.

Conjecture 1.4 *With the above notation, there exist maximal spanning vectors in $L^2(G)$.*

Applying the result on the linear span of translates of an element in $L^2(G)$ (e.g. [14, Proposition 4.72]), we show that $f \in L^2(G)$ is a maximal spanning vector if and only if $\langle \pi(v^*, v)f, f \rangle \neq 0$ for almost all $(v^*, v) \in \widehat{G} \times G$ (Proposition 3.3). Note that this is consistent with [10, Proposition 3.11]. As a consequence, by explicit construction, we prove the conjecture for some special groups.

Theorem 1.5 *Conjecture 1.4 holds in the following cases. Here K is a local field with residue characteristic p ($p > 2$).*

1. *G is a finite abelian group with discrete topology.*
2. *$G = \mathbb{Z}$ is the additive group of integers with discrete topology.*
3. *$G = \mathbb{T}$ is the multiplicative group of norm one numbers with the Euclidean topology.*
4. *$G = \mathbb{R}$ is the additive group of real numbers with the Euclidean topology.*
5. *$G = \mathbb{R}^\times$ is the multiplicative group of nonzero real numbers with the Euclidean topology.*
6. *$G = (K, +)$ is the additive group of K with its non-archimedean topology.*
7. *$G = (\mathcal{O}_K, +)$ is the ring of integers of K.*
8. *$G = K/\mathcal{D}_K$ with discrete topology, where $\mathcal{D}_K$ is the difference of K.*
9. *$G = (K^\times, \times)$ is the multiplicative group of nonzero elements of K.*
10. *$G = (\mathcal{O}_K^\times, \times)$ is the group of units of K.*

Moreover, Conjecture 1.4 holds for G which is a finite product of groups as listed above.

Remark 1.6 (Idea of proof) Applying Proposition 3.3, it is easy to check that if $f_i \in L^2(G_i)$ is a maximal spanning vector for $(\pi_i, L^2(G_i))$ $(i = 1, 2)$, then $f_1 \otimes f_2 \in L^2(G_1 \times G_2)$ is a maximal spanning vector for $L^2(G_1 \times G_2)$, where $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)$. Therefore if the conjecture holds for G_1 and G_2, then it holds for $G_1 \times G_2$ as well. Moreover, if the conjecture holds for $\mathbb{Z}$ (resp. $\mathcal{O}_K$), then it holds for $\mathbb{T}$ (resp. $K/\mathcal{D}_K$) as well by the discussion in Sect. 2.1.2. Because the conjecture holds for finite abelian groups by [10, Proposition 3.11], using the structure decomposition of $K^\times$, $\mathcal{O}_K^\times$ (cf. [25, Chap 2, Section 5, (5.7)]) and $\mathbb{R}^\times$, we only need to verify the theorem for $G = \mathbb{R}, \mathbb{Z}, K, \mathcal{O}_K$. The detailed proof is the content of Sect. 4.

Remark 1.7 As a consequence, combining Theorem 1.5, Corollary 3.4, and Proposition 3.6, for G as in Theorem 1.5, we have

$$\overline{\mathrm{Span}}\{f \in L^2(G) \mid \Phi_f \text{ is phase retrievable}\} = L^2(G).$$

Notation: In this paper, G is always a locally compact group. Denote by $\widehat{G}$ the dual group of G if G is an abelian group. Moreover, we fix a Haar measure on G and equip $\widehat{G}$ with the Plancherel measure. Denote by $\widehat{f}$ the Fourier transform of f if $f \in L^2(G)$ ([14, Section 4.2]).

In this paper, $\mathbb{R}$ is the additive group of real numbers, $\mathbb{Z}$ is the additive group of integers, $\mathbb{T}$ is the multiplicative group of norm one numbers.

Let X be a measure space. Denote by $L^2(X)$ the space of measurable functions on X for which $\int_X |f(x)|^2 \mathrm{d}x < \infty$. Given $f, f' \in L^2(X)$, the inner product is defined by

$$\langle f, f' \rangle = \int_X f(x)\overline{f'(x)}\mathrm{d}x.$$

For $f \in L^2(X)$, define $||f||_2 = (\langle f, f \rangle)^{1/2}$.

2 Continuous ($\widehat{G} \times G$)-frames in $L^2(G)$

2.1 Remarks on Theorem 1.3

In the following, we make some remarks on the Stone-von Neumann theorem. These observations enable us to reduce the proof of Theorem 1.5 to some easy cases and to generalize Theorem 1.5 to other groups (not of the form $\widehat{G} \times G$).

2.1.1 Some variations

Let G and G^* be locally compact abelian groups. Suppose that we have a bi-homomorphism

$$\begin{aligned}\beta : G^* \times G &\to \mathbb{T}\\ (v^*, v) &\mapsto v^*(v).\end{aligned}$$

Let $H = G^* \times G$ and define

$$\begin{aligned}\alpha : H \times H &\to \mathbb{T}\\ ((v_1^*, v_1), (v_2^*, v_2)) &\mapsto v_2^*(v_1).\end{aligned}$$

Then, α is a 2-cocycle. Define $\pi_\beta : H \to \mathrm{GL}(L^2(G))$ by

$$(\pi_\beta(v^*, v)f)(u) = v^*(u)f(uv).$$

It is easy to check that π_β is unitary and $\pi_\beta : H \to \mathbf{U}(L^2(G))$ is an α-representation of H.

The pairing $\beta : G^* \times G \to \mathbb{T}$ induces a homomorphism $\iota : G^* \to \widehat{G}$ and the α -representation $\pi_\beta : G^* \times G \to \mathbf{U}(L^2(G))$ factors through

$$G^* \times G \xrightarrow{\iota \times \mathrm{id}} \widehat{G} \times G \xrightarrow{\pi} \mathbf{U}(L^2(G)). \tag{2.1}$$

In particular, if $\iota : G^* \to \widehat{G}$ is surjective, then π_β is an irreducible α-representation. This factorization is part of the idea in [10, 24]. More precisely, assume that the kernel of $\iota : G^* \to \widehat{G}$ has finite volume C. If $\{\pi(g)f \mid g \in \widehat{G} \times G\}$ is a continuous frame with frame bounds A and B, then $\{\pi_\beta(g)f \mid g \in G^* \times G\}$ is a continuous frame with frame bounds CA and CB. Moreover, if $f \in L^2(G)$ is a maximal spanning vector for $\pi : \widehat{G} \times G \to \mathbf{U}(L^2(G))$, then

$$\overline{\mathrm{Span}}\{c_{\pi_\beta(v^*,v)f, \pi_\beta(v^*,v)f} \mid (v^*, v) \in \widehat{G} \times G\} = L^2(\widehat{G} \times G),$$

and the frame $\{\pi_\beta(v^*, v)f \mid (v^*, v) \in G^* \times G\}$ is phase retrievable by Proposition 3.6.

Assume now that the pairing β induces an injection $\iota : G^* \to \widehat{G}$ with nontrivial cokernel. Let $K = \{x \in G \mid v^*(k) = 1 \text{ for all } v^* \in \mathrm{Im}(\iota)\}$. Assume that $\mathrm{Im}(\iota)$ is an open subgroup of $\widehat{G}$ and K is compact. Then we have $\mathrm{Im}(\iota) = \widehat{G/K}$. The natural map $G \to G/K$ induces an injection $\kappa : L^2(G/K) \to L^2(G)$. Denote by $l^2(G/K)$ the image of κ. Then

$$l^2(G/K) = \{f \in L^2(G) \mid f(gk) = f(g) \text{ for all } g \in G,\ k \in K\}.$$

For any $(v^*, v) \in \mathrm{Im}(\iota) \times G$, we have an operator $\varpi(v^*, v)$ on $L^2(G)$ via

$$(\varpi(v^*, v)f)(g) = v^*(g)f(gv).$$

If $f \in l^2(G/K)$, it is clear that $\varpi(v^*, v)f \in l^2(G/K)$. This tells us that π_β is reducible and the reducibility causes trouble to find maximal spanning vectors (cf. [10, Proposition 3.4(3)]).

2.1.2 Relation with the Fourier transform

By symmetry, we have an action of $\widehat{G} \times G$ on the space $L^2(\widehat{G})$ via

$$(\pi'(v^*,v)\Psi)(u^*) = u^*(v)\Psi(u^*v^*),$$

where $(v^*,v) \in \widehat{G}\times G$ and $\Psi \in L^2(\widehat{G})$. The associated multiplier is $\alpha' : (\widehat{G}\times G)\times(\widehat{G}\times G) \to \mathbb{T}$ given by

$$((v_1^*,v_1),(v_2^*,v_2)) \mapsto v_1^*(v_2).$$

As explained in Sect. 2.1.1, we obtain a projective representation of $\widehat{G}\times G$ by composition with $\widehat{G} \to \widehat{G}$ ($v^* \mapsto (v^*)^{-1}$). Changing the resulting projective representation by the coboundary β, which is the natural pairing between $\widehat{G}$ and G, we obtain an irreducible projective representation $\rho : \widehat{G}\times G \to \mathbf{U}(L^2(\widehat{G}))$, where

$$(\rho(v^*,v)\Psi)(u^*) = v^*(v)^{-1}u^*(v)^{-1}\Psi(u^*v^*).$$

An easy computation shows that ρ is an α-representation. Define

$$\begin{aligned} F' &: L^2(G) \to L^2(\widehat{G}) \\ \Phi &\mapsto \left(\tilde{\Phi} : u^* \mapsto \int_G \Phi(u)u^*(u)\mathrm{d}u\right) \text{ for } \Phi \in L^1(G)\cap L^2(G). \end{aligned}$$

Then, $\tilde{\Phi}(u^*) = \widehat{\Phi}((u^*)^{-1})$ and F' is a unitary isomorphism. Moreover, it is easy to check that $\rho(v^*,v)\tilde{\Phi} = (\pi(v^*,v)\Phi)\tilde{}$. In other words, F' is an isomorphism of α-representations. By the discussion in Sect. 2.1.1 (in the case where ι is an isomorphism), if Conjecture 1.4 holds for G, then it holds for $\widehat{G}$.

2.2 The frame condition

Let $f \in L^2(G)$ be a nonzero element. Since π is irreducible, we have

$$\overline{\mathrm{Span}}\{\pi(v^*,v)f \mid (v^*,v) \in \widehat{G}\times G\} = L^2(G).$$

Even better, we have the following stronger result, known as the Plancherel formula for the short-time Fourier transform (cf. [16, Theorem 6.2.1] and [12, Corollary 11.1.4]). We give a proof for completeness.

Theorem 2.1 *Let $f \in L^2(G)$ be a nonzero element. Then the set $\Phi_f = \{\pi(v^*,v)f \mid (v^*,v) \in \widehat{G}\times G\}$ is a tight $(\widehat{G}\times G)$-frame of $L^2(G)$ with bound $||f||_2^2$. In particular, if $||f||_2 = 1$, then Φ_f is a Parseval frame.*

Proof The following argument is adapted from the idea in [12, Section 11.1]. Let f_1, f_2, g_1, g_2 be elements in $L^2(G)$. Then

$$\begin{aligned}
&\int_{\widehat{G}\times G}\langle f_1,\pi(v^*,v)g_1\rangle\overline{\langle f_2,\pi(v^*,v)g_2\rangle}\mathrm{d}(v^*,v)\\
&=\int_{\widehat{G}\times G}\int_G f_1(x)\overline{(\pi(v^*,v)g_1)(x)}\mathrm{d}x\int_G\overline{f_2(y)}(\pi(v^*,v)g_2)(y)\mathrm{d}y\mathrm{d}(v^*,v)\\
&=\int_{\widehat{G}\times G}\int_G f_1(x)\overline{g_1(xv)v^*(x)}\mathrm{d}x\int_G\overline{f_2(y)}g_2(yv)v^*(y)\mathrm{d}y\mathrm{d}(v^*,v).
\end{aligned}\tag{2.2}$$

Let $F_i(x)=f_i(x)\overline{g_i(xv)}$ for $i=1,2$. Then

$$\begin{aligned}
&\int_{\widehat{G}\times G}\langle f_1,\pi(v^*,v)g_1\rangle\overline{\langle f_2,\pi(v^*,v)g_2\rangle}\mathrm{d}(v^*,v)\\
&=\int_{\widehat{G}\times G}\widehat{F}_1(v^*)\overline{\widehat{F}_2(v^*)}\mathrm{d}v^*\mathrm{d}v\\
&=\int_G\int_G F_1(x)\overline{F_2(x)}\mathrm{d}x\mathrm{d}v\\
&=\int_G\int_G f_1(x)\overline{g_1(xv)f_2(x)}g_2(xv)\mathrm{d}x\mathrm{d}v\\
&=\langle f_1,f_2\rangle\langle g_2,g_1\rangle.
\end{aligned}\tag{2.3}$$

Therefore, for any $F\in L^2(G)$, we have

$$\int_{\widehat{G}\times G}|\langle F,\pi(v^*,v)f\rangle|^2\mathrm{d}(v^*,v)=||f||_2^2||F||_2^2.$$

In particular, we see that $\{\pi(v^*,v)f\mid(v^*,v)\in\widehat{G}\times G\}$ is a tight $(\widehat{G}\times G)$-frame of $L^2(G)$ with frame bounds $A=B=||f||_2^2$. □

3 Maximal spanning vectors in $L^2(G)$

3.1 Definition and characterization

Let $f,g\in L^2(G)$. Then we have a map $c_{f,g}:\widehat{G}\times G\to\mathbb{C}$ given by

$$c_{f,g}(v^*,v)=\langle\pi(v^*,v)f,g\rangle.$$

We call such a map a *matrix coefficient of* π. In time-frequency analysis, $c_{f,g}$ is called the *short-time Fourier transform* of f with respect to g (cf. [17]). In equation (2.3), taking $f_1=f_2=f$ and $g_1=g_2=g$, we obtain that the matrix coefficient $c_{f,g}$ is an element in $L^2(\widehat{G}\times G)$. Hence, the α-representation π is square-integrable.

Definition 3.1 Let $f,g\in L^2(G)$. We say that (f,g) is a *maximal spanning pair* if

$$\overline{\mathrm{Span}}\{c_{\pi(v^*,v)f,\pi(v^*,v)g}\mid(v^*,v)\in\widehat{G}\times G\}=L^2(\widehat{G}\times G).$$

We call $f \in L^2(G)$ a *maximal spanning vector* if (f, f) is a maximal spanning pair.

Motivated by the results for finite groups (cf. [24, Conjecture], [10, Section 3], [11, Section 3.3]), we propose the following conjecture.

Conjecture 3.2 $\pi : \widehat{G} \times G \to \mathbf{U}(L^2(G))$ *admits maximal spanning vectors.*

We provide a characterization for maximal spanning vectors in the following and then apply this characterization to prove Theorem 1.5 in next section. We have

$$\begin{aligned} c_{\pi(v^*,v)f,\pi(v^*,v)g}(u^*, u) &= \langle \pi(u^*, u)\pi(v^*, v)f, \pi(v^*, v)g\rangle \\ &= \int_G (\pi(u^*, u)\pi(v^*, v)f)(x)\overline{(\pi(v^*, v)g)(x)}\mathrm{d}x \\ &= \int_G v^*(u)u^*v^*(x)f(xuv)\overline{v^*(x)g(xv)}\mathrm{d}x \\ &= \int_G v^*(u)u^*(v)^{-1}u^*(xv)f(xvu)\overline{g(xv)}\mathrm{d}x \\ &= \frac{v^*(u)}{u^*(v)}c_{f,g}(u^*, u). \end{aligned}$$

Note that the pairing $(\widehat{G} \times G) \times (\widehat{G} \times G) \to \mathbb{T}$ given by

$$((v^*, v), (u^*, u)) \mapsto \frac{v^*(u)}{u^*(v)}$$

induces an isomorphism $\widehat{G} \times G \cong (\widehat{G} \times G)\widehat{\ }$. Hence,

$$\begin{aligned} C_{f,g} := &\ \overline{\mathrm{Span}}\{c_{\pi(v^*,v)f,\pi(v^*,v)g} \mid (v^*, v) \in \widehat{G} \times G\} \\ = &\ \overline{\mathrm{Span}}\{\chi c_{f,g} \mid \chi \in (\widehat{G} \times G)\widehat{\ }\}. \end{aligned} \tag{3.1}$$

We then have the following proposition, which generalizes the result for finite abelian groups (cf. [7, 24]).

Proposition 3.3 *With the notation as above, (f, g) is a maximal spanning pair for* $(\pi, L^2(G))$ *if and only if* $c_{f,g}(u^*, u) \neq 0$ *for almost all* $(u^*, u) \in \widehat{G} \times G$.

In particular, f is a maximal spanning vector for $(\pi, L^2(G))$ *if and only if* $c_{f,f}(u^*, u) \neq 0$ *for almost all* $(u^*, u) \in \widehat{G} \times G$.

Proof The Fourier transform for $\widehat{G} \times G$ is an isometry between $L^2(\widehat{G} \times G) \to L^2((\widehat{G} \times G)\widehat{\ })$. Therefore,

$$
\begin{aligned}
& C_{f,g} = L^2(\widehat{G} \times G) \\
\Longleftrightarrow\ & \overline{\mathrm{Span}}\{(\chi c_{f,g})\widehat{}\ \mid \chi \in (\widehat{G} \times G)\widehat{}\,\} = L^2((\widehat{G} \times G)\widehat{}\,) \\
\Longleftrightarrow\ & c_{f,g}(u^*, u) \neq 0 \text{ for almost all } (u^*, u) \in \widehat{G} \times G.
\end{aligned}
\tag{3.2}
$$

Note that $(\chi c_{f,g})\widehat{}$ is the translation of $(c_{f,g})\widehat{}$ by χ, the last equivalence follows from [14, Proposition 4.72]. The proposition follows. ☐

Corollary 3.4 *If there exists one maximal spanning vector f for $(\pi, L^2(G))$, then there are infinitely many maximal spanning vectors of length one for $(\pi, L^2(G))$.*

Proof If G is finite, then the set of maximal spanning vectors is dense in $L^2(G)$ by [24, Theorem 1.7, Lemma 2.2]. If G is infinite, then each $\pi(u^*, u)f$ is a maximal spanning vector and $\overline{\mathrm{Span}}\{\pi(u^*, u)f \mid (u^*, u) \in \widehat{G} \times G\} = L^2(G)$. Note that if f is a maximal spanning vector, so is $f/||f||_2$. The claim then follows. ☐

3.2 Some remarks on maximal spanning vectors

3.2.1 The Bessel property

Suppose that $(f, g) \in L^2(G)$ is a maximal spanning pair. We then have

$$
\overline{\mathrm{Span}}\{c_{\pi(v^*,v)f,\pi(v^*,v)g} \mid (v^*, v) \in \widehat{G} \times G\} = L^2(\widehat{G} \times G).
$$

One may ask the following question: Do the vectors $c_{\pi(v^*,v)f,\pi(v^*,v)g}$ $((v^*, v) \in \widehat{G} \times G)$ form a continuous $(\widehat{G} \times G)$-frame for $L^2(\widehat{G} \times G)$? For G finite, the answer is certainly yes. In general this is not true. For simplicity, we write c for the function $c_{f,g}$ and $c_{v^*,v}$ for the function $c_{\pi(v^*,v)f,\pi(v^*,v)g}$. For any $\psi \in L^2(\widehat{G} \times G)$, we have

$$
\begin{aligned}
&\int_{\widehat{G}\times G} |\langle \psi, c_{v^*,v}\rangle|^2 \mathrm{d}(v^*,v)\\
=&\int_{\widehat{G}\times G} \langle \psi, c_{v^*,v}\rangle\overline{\langle \psi, c_{v^*,v}\rangle}\mathrm{d}(v^*,v)\\
=&\int_{\widehat{G}\times G}\int_{\widehat{G}\times G} \psi(u^*,u)\overline{c_{v^*,v}(u^*,u)}\mathrm{d}(u^*,u)\\
&\qquad\int_{\widehat{G}\times G} \overline{\psi(u^*,u)}c_{v^*,v}(u^*,u)\mathrm{d}(u^*,u)\mathrm{d}(v^*,v)\\
=&\int_{\widehat{G}\times G}\int_{\widehat{G}\times G} \psi(u^*,u)\overline{c(u^*,u)}\bar{(\frac{v^*(u)}{u^*(v)})}\mathrm{d}(u^*,u)\\
&\qquad\int_{\widehat{G}\times G} \overline{\psi(u^*,u)}c(u^*,u)\frac{v^*(u)}{u^*(v)}\mathrm{d}(u^*,u)\mathrm{d}(v^*,v)\\
=&\int_{\widehat{G}\times G} \widehat{\Psi}(v,v^*)\overline{\widehat{\Psi}(v,v^*)}\mathrm{d}(v,v^*) \quad (\text{here } \Psi(u^*,u)=\psi(u^*,u)\overline{c(u^*,u)})\\
=&\int_{\widehat{G}\times G} \Psi(u^*,u)\overline{\Psi(u^*,u)}\mathrm{d}(u^*,u)\\
=&||\psi\cdot\bar{c}||_2^2.
\end{aligned}
$$

Hence, the set $\{c_{\pi(v^*,v)f,\pi(v^*,v)g} \mid (v^*,v)\in\widehat{G}\times G\}$ is a continuous frame for $L^2(\widehat{G}\times G)$ if and only if (f, g) is a maximal spanning pair for $L^2(G)$ and there exist positive numbers A and B with

$$
A||\psi||_2 \le ||\psi\cdot c_{f,g}||_2 \le B||\psi||_2
$$

for all $\psi\in L^2(\widehat{G}\times G)$.

If $c_{f,g}$ is bounded, e.g. it is continuous, then the existence of B is clear and the set is a so called *Bessel set*. On the other hand, the existence of A is troublesome. To explain the idea, let us consider the case $G=\mathbb{R}$ and $c_{f,f}$ from Lemma 4.3. On one hand, we may take $B=\sqrt{\frac{\pi}{2}}$. On the other hand, for $n\in\mathbb{Z}_{\ge 1}$, define $\psi_n:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ by

$$
\psi_n(x,y)=\begin{cases}\frac{1}{n} & \text{if } n\le|x|\le 2n,\ n\le|y|\le 2n,\\ 0 & \text{otherwise.}\end{cases}
$$

Then, $||\psi_n||_2^2=4$. But $\lim_{n\to\infty}||\psi_n\cdot c_{f,f}||_2^2=0$ and A does not exist.

Remark 3.5 From the computation, we see that if $c_{f,g}$ is zero for $(v^*,v)\in U$, where $U\subset\widehat{G}\times G$ has positive measure, then (f, g) cannot be a maximal spanning pair. Indeed, let ψ be the characteristic function for U, then ψ is orthogonal to every $c_{\pi(v^*,v)f,\pi(v^*,v)g}$. This proves the *only if* part of Proposition 3.3.

3.2.2 Phase retrieval

Let $f\in L^2(G)$ be a nonzero vector. Then, we have the following map

$$\begin{aligned} t_f : L^2(G) &\to L^2(\widehat{G} \times G) \\ x &\mapsto ((u^*, u) \mapsto |\langle x, \pi(u^*, u)f \rangle|). \end{aligned}$$

Certainly, t_f factors through $L^2(G)/\mathbb{T}$ and we obtain a map

$$T_f : L^2(G)/\mathbb{T} \to L^2(\widehat{G} \times G). \tag{3.3}$$

The frame $\Phi_f = \{\pi(u^*, u)f \mid (u^*, u) \in \widehat{G} \times G\}$ is *phase retrievable* if T_f in (3.3) is injective. Let $\mathcal{S}_{\Phi_f}$ be the set of operators $x \otimes x : L^2(G) \to L^2(G)$ $(x \in \Phi_f)$ as defined in Section 1. We have the following result.

Proposition 3.6 *The vector f is a maximal spanning vector if and only if $\mathcal{S}_{\Phi_f}$ has maximal span. In particular, if f is a maximal spanning vector, then Φ_f is phase retrievable.*

Proof Let $\phi \in C_c(\widehat{G} \times G)$ be a compactly supported continuous function on $\widehat{G} \times G$. Define $K_\phi : G \times G \to \mathbb{C}$ by

$$K_\phi(u, v) = \int_{\widehat{G}} \phi(v^*, vu^{-1})\overline{v^*(u)}\mathrm{d}v^*.$$

Then,

$$K_\phi(u, uv) = \int_{\widehat{G}} \phi(v^*, v)\overline{v^*(u)}\mathrm{d}v^*. \tag{3.4}$$

Hence, $K_\phi(u, uv)$ is the Fourier transform of $\phi(v^*, v)$ in the first variable. Thus,

$$\begin{aligned} \int_{G\times G} |K_\phi(u, v)|^2 \mathrm{d}u\mathrm{d}v &= \int_{G\times G} |K_\phi(u, uv)|^2 \mathrm{d}u\mathrm{d}v \\ &= \int_{\widehat{G}\times G} |\phi(v^*, v)|^2 \mathrm{d}v^*\mathrm{d}v. \end{aligned}$$

The map $\phi \mapsto K_\phi$ extends to an L^2-isometry from $L^2(\widehat{G} \times G)$ to $L^2(G \times G)$. It is invertible as

$$\phi(v^*, v) = \int_G K_\phi(u, uv)v^*(u)\mathrm{d}u$$

by the Fourier inversion formula and equation (3.4). Let $\lambda : L^2(G \times G) \to L^2(\widehat{G} \times G)$ be the inversion of this isometry (cf. [8, Page 526]).

For any Hilbert space V with dual space V^*, since $\mathrm{HS}(V) \cong V^* \otimes V$ (cf. [13, Section 5.3]) and the Riesz representation theorem provides a bijective conjugate-linear isometry between V and V^*, we may identify $\mathrm{HS}(V)$ with $V \otimes V$. Consider the following unitary isomorphisms

$$\mathrm{HS}(L^2(G)) = L^2(G) \otimes L^2(G) \xrightarrow{\sigma} L^2(G \times G) \xrightarrow{\lambda} L^2(\widehat{G} \times G),$$

where σ is defined by $\sigma(P \otimes Q)(u, v) = \overline{P(u)}Q(v)$. Let $x \in C_c(G)$. Then

$$\begin{aligned} \lambda\circ\sigma(x \otimes x)(v^*, v) &= \int_G \sigma(x \otimes x)(u, uv)v^*(u)\mathrm{d}u \\ &= \int_G \overline{x(u)}x(uv)v^*(u)\mathrm{d}u \\ &= \langle \pi(v^*, v)x, x\rangle = c_{x,x}(v^*, v). \end{aligned}$$

Hence, for all $x \in L^2(G)$, the image of $x \otimes x$ in $L^2(\widehat{G} \times G)$ is nothing but the function $c_{x,x}$. The proposition then follows. □

4 Proof of Theorem 1.5

In this section, we prove Theorem 1.5 by explicit construction of maximal spanning vectors. As explained in Remark 1.6, we only need to prove the result for four cases. We divide the proof into two parts.

4.1 Euclidean case

We prove the following result in this section.

Theorem 4.1 *Conjecture 1.4 holds for groups of type $G = \mathbb{R}^a \times \mathbb{T}^b \times \mathbb{Z}^c \times E$, where E is a finite abelian group.*

Groups as in the theorem are called *compactly generated locally Euclidean locally compact abelian group*. The theorem follows from the following two lemmas and Remark 1.6.

Lemma 4.2 *Let $G = \mathbb{Z}$. Let $f \in L^2(\mathbb{Z})$ given by $f(n) = e^{-|n|}$. Then f is a maximal spanning vector for $(\pi, L^2(\mathbb{Z}))$.*

Proof We need to show that $c_{f,f}(\theta, N) \neq 0$ for almost all $(\theta, N) \in \mathbb{T} \times \mathbb{Z}$. By definition, we have

$$\begin{aligned} c_{f,f}(\theta, N) &= \langle \pi(\theta, N)f, f\rangle = \sum_{n\in\mathbb{Z}} (\pi(\theta, N)f)(n)\bar{f}(n) \\ &= \sum_{n\in\mathbb{Z}} \theta^n f(n + N)f(n) = \sum_{n\in\mathbb{Z}} \theta^n e^{-|n+N|}e^{-|n|}. \end{aligned} \tag{4.1}$$

We divide the discussion into the following cases.

1. If $\theta = 1$, then $c_{f,f}(1, N) = \sum_{n\in\mathbb{Z}} e^{-|n+N|} e^{-|n|} \neq 0$.
2. If $\theta \neq 1$ and $N = 0$, then

$$c_{f,f}(\theta, 0) = \sum_{n\in\mathbb{Z}} \theta^n e^{-2|n|} = \sum_{n\geq 0} \theta^n e^{-2n} + \sum_{n\geq 0} \theta^{-n} e^{-2n} - 1$$
$$= \frac{1}{1-\theta e^{-2}} + \frac{1}{1-\theta^{-1}e^{-2}} - 1$$
$$= \frac{1-e^{-4}}{(1-\theta e^{-2})(1-\theta^{-1}e^{-2})} \neq 0.$$

3. If $\theta \neq 1$ and $N \geq 1$, then

$$c_{f,f}(\theta, N) = \sum_{n\in\mathbb{Z}} \theta^n e^{-|n+N|} e^{-|n|}$$
$$= \sum_{n\geq 0} \theta^n e^{-(n+N)} e^{-n} + \sum_{n\leq -N} \theta^n e^{n+N} e^n + \sum_{n=-N+1}^{-1} \theta^n e^{-(n+N)} e^n$$
$$= e^{-N} \sum_{n\geq 0} \theta^n e^{-2n} + e^N \sum_{n\geq N} \theta^{-n} e^{-2n} + e^{-N} \sum_{n=1}^{N-1} \theta^{-n}$$
$$= e^{-N}\left(\frac{1}{1-\theta e^{-2}} + \frac{\theta^{-N}}{1-\theta^{-1}e^{-2}} + \frac{\theta^{-1}-\theta^{-N}}{1-\theta^{-1}}\right).$$

Hence, $c_{f,f}(\theta, N) = 0$ if and only if $\frac{1}{1-\theta e^{-2}} + \frac{\theta^{-N}}{1-\theta^{-1}e^{-2}} + \frac{\theta^{-1}-\theta^{-N}}{1-\theta^{-1}} = 0$. Multiplying both sides with $(1-\theta e^{-2})(1-\theta^{-1}e^{-2})(1-\theta^{-1})$, direct computation shows that $c_{f,f}(\theta, N) = 0$ if and only if

$$\frac{1-\theta e^{-2}}{1-\theta^{-1}e^{-2}} = \theta^{N+1}. \tag{4.2}$$

Since $\theta \neq 1$, for each $N \geq 1$, there are at most N solutions to the equation (4.2).
4. If $\theta \neq 1$ and $N \leq -1$, then from the identity

$$c_{f,f}(\theta, -N) = \theta^N c_{f,f}(\theta, N),$$

we have that, for each $N \leq -1$, there are at most $|N|$ solutions with $c_{f,f}(\theta, N) = 0$
.

From the above discussion, the set $\{(\theta, N) \in \mathbb{T} \times \mathbb{Z} \mid c_{f,f}(\theta, N) = 0\}$ is countable. The lemma then follows. □

The following lemma is well known (e.g. [18]). We provide a proof for completeness.

Lemma 4.3 *Let $G = \mathbb{R}$. Let $f \in L^2(\mathbb{R})$ given by $f(x) = e^{-x^2}$. Then f is a maximal spanning vector for $(\pi, L^2(\mathbb{R}))$.*

Proof We show that $c_{f,f}(a, b) \neq 0$ for all $(a, b) \in \mathbb{R} \times \mathbb{R}$. We have the well-known identity (see for example [27, Page 42])

$$\int_{\mathbb{R}} e^{-2\pi i\xi x} e^{-\pi x^2} \mathrm{d}x = e^{-\pi\xi^2}. \tag{4.3}$$

Therefore,

$$\begin{aligned} c_{f,f}(a,b) &= \langle \pi(a,b)f, f\rangle \\ &= \int_{\mathbb{R}} e^{2\pi iax} e^{-(x+b)^2 - x^2} \mathrm{d}x \\ &= e^{-\frac{1}{2}b^2} \int_{\mathbb{R}} e^{2\pi iax} e^{-\pi(\sqrt{\frac{2}{\pi}}x + \sqrt{\frac{1}{2\pi}}b)^2} \mathrm{d}x \\ &= e^{-\frac{1}{2}b^2} \sqrt{\frac{\pi}{2}} e^{-\pi iab} \int_{\mathbb{R}} e^{-2\pi iy(-a\sqrt{\frac{\pi}{2}})} e^{-\pi y^2} \mathrm{d}y \\ &= e^{-\frac{1}{2}b^2} \sqrt{\frac{\pi}{2}} e^{-\pi iab} e^{-\frac{1}{2}a^2\pi^2} \neq 0. \end{aligned}$$

The lemma follows. □

4.2 Non-archimedean case

In the following, K is a non-archimedean local field with residue characteristic p, $\mathcal{O}$ is the ring of integers of K. Fix π a uniformizer of K. Denote by ord the valuation on K with $\mathrm{ord}(\pi) = 1$. Let q be the cardinality of the residue field $\mathcal{O}/\pi\mathcal{O}$. For $n \in \mathbb{Z}$, let $\mathcal{O}_n$ be the fractional ideal $\pi^n\mathcal{O}$ and $A_n = \mathcal{O}_n - \mathcal{O}_{n+1}$. Fix a Haar measure μ on K with $\mu(\mathcal{O}) = 1$.

Let $\psi : K \to \mathbb{T}$ be the non-trivial character of $(K, +)$ as in Tate's thesis [28]. More precisely, it is given as follows.

1. If $K = \mathbb{Q}_p$, then ψ is the composition

$$\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \sim \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z} \cong \mathbb{T},$$

   which is characterized by $\psi|_{\mathbb{Z}_p} = 1$ and $\psi(p^{-n}) = e^{2\pi i/p^n}$ for all $n \geq 1$.
2. If $K = \mathbb{F}_p(T)$, then $\psi(\sum a_n T^n) := e^{2\pi i a_{-1}/p}$. Here, we lift $a_{-1} \in \mathbb{F}_p$ to $\mathbb{Z}$ to make sense of the definition.
3. If K is a finite extension of K_0, where $K_0 = \mathbb{Q}_p$ or $\mathbb{F}_p(T)$, then ψ for K is defined to be the composition $K \xrightarrow{\mathrm{Tr}_{K/K_0}} K_0 \xrightarrow{\psi_0} \mathbb{T}$, where ψ_0 is the additive character for K_0 as constructed above.

We could and do identity K with $\widehat{K}$ via $a \mapsto (\psi_a : x \mapsto \psi(ax))$. The *conductor* of a character $\phi : K \to \mathbb{T}$ is the integer l such that $\phi|_{\mathcal{O}_l}$ is not trivial and $\phi|_{\mathcal{O}_{l+1}}$ is trivial. Define $\mathrm{cond}(a)$ to be the conductor of ψ_a for $a \in K$. The difference between $\mathrm{cond}(a)$ and $\mathrm{ord}(a)$ is given by the difference of K over K_0. The following lemma will be used repeatedly.

Lemma 4.4 *Let $\phi : K \to \mathbb{T}$ be a character of conductor l. We have*

$$\int_{A_n} \phi(x)\mathrm{d}x = \begin{cases} 0 & \text{if } n+1 \leq l, \\ -\mu(\mathcal{O}_{l+1}) & \text{if } n+1 = l+1, \\ \mu(A_n) & \text{if } n+1 \geq l+2. \end{cases}$$

Proof Note that $\int_{A_n} \phi(x)\mathrm{d}x = \int_{\mathcal{O}_n} \phi(x)\mathrm{d}x - \int_{\mathcal{O}_{n+1}} \phi(x)\mathrm{d}x$. In the first case, both terms are 0. In the second case, the first term is 0 and the second term is $-\mu(\mathcal{O}_{l+1})$. In the third case, ϕ is trivial on A_n. The lemma follows. □

For $n \in \mathbb{Z}$, define $\delta(n) = \begin{cases} q^{2n} & \text{if } n < 0, \\ q^{-n} & \text{if } n \geq 0. \end{cases}$ Then,

$$\begin{aligned} \delta(n)\mu(A_n) &= \begin{cases} q^{n}(1-q^{-1}) & \text{if } n < 0, \\ q^{-2n}(1-q^{-1}) & \text{if } n \geq 0. \end{cases} \\ \delta(n)^2\mu(A_n) &= \begin{cases} q^{3n}(1-q^{-1}) & \text{if } n < 0, \\ q^{-3n}(1-q^{-1}) & \text{if } n \geq 0. \end{cases} \end{aligned} \tag{4.4}$$

Lemma 4.5 *For any $m \in \mathbb{Z}$, the number*

$$D_m := -\frac{1}{q-1}\delta(m)\mu(A_m) + \sum_{n=m+1}^{+\infty} \delta(n)\mu(A_n)$$

is a nonzero rational number.

Proof The rationality follows from equation (4.4). It is also easy to check that $D_m > 0$ if $m < 0$ and $D_m < 0$ if $m \geq 0$. The lemma follows. □

4.2.1 Case $G = (K, +)$

Let $f : K \to \mathbb{C}$ be given by

$$f(x) = \sum_{n \in \mathbb{Z}} \delta(n)\mathbf{1}_{A_n}(x),$$

where $\mathbf{1}_{A_n}$ is the characteristic function for the set A_n. It is easy to see that $f \in L^2(K)$. Denote by c the function c_{ff} given by equation (1.2). We have the following result.

Proposition 4.6 *With the notation as above,*

1. *if $p \neq 2$, then $c(a, b) \neq 0$ for all $(a, b) \in K \times K$;*
2. *if $p = 2$, then $c(a, b) = 0$ if and only if $\mathrm{cond}(a) \geq \mathrm{ord}(b)$ and $\psi(-ab) = -1$.*

Proof Let l and m be the conductor of a and the order of b, respectively. Then,

$$\begin{aligned} c(a,b) = \langle \pi(a,b)f, f\rangle &= \int_K \psi(ax)f(x+b)\overline{f(x)}\mathrm{d}x \\ &= \sum_{n\in\mathbb{Z}} \int_{A_n} \psi(ax)f(x+b)\overline{f(x)}\mathrm{d}x \\ &= \sum_{n\in\mathbb{Z}} \delta(n) \int_{A_n} \psi(ax)f(x+b)\mathrm{d}x. \end{aligned} \tag{4.5}$$

If $n > m$, then $\mathrm{ord}(x+b) = m$ for $x \in A_n$ and

$$\delta(n)\int_{A_n} \psi(ax)f(x+b)\mathrm{d}x = \delta(n)\delta(m)\int_{A_n} \psi(ax)\mathrm{d}x.$$

If $n < m$, then $\mathrm{ord}(x+b) = n$ for $x \in A_n$ and

$$\delta(n)\int_{A_n} \psi(ax)f(x+b)\mathrm{d}x = \delta(n)^2\int_{A_n} \psi(ax)\mathrm{d}x.$$

Therefore,

$$\begin{aligned} c(a,b) = \sum_{n=-\infty}^{m-1} \delta(n)^2 \int_{A_n} \psi(ax)\mathrm{d}x + \delta(m)\int_{A_m} \psi(ax)f(x+b)\mathrm{d}x \\ + \delta(m)\sum_{n=m+1}^{+\infty} \delta(n)\int_{A_n} \psi(ax)\mathrm{d}x. \end{aligned} \tag{4.6}$$

Let $\mathcal{O}_{m,-b}$ be the set $\{x \in \mathcal{O}_m \mid x \equiv -b \pmod{\pi^{m+1}}\}$ and let $\mathcal{O}^c_{m,-b} = \mathcal{O}_m - \mathcal{O}_{m,-b}$. Then $\mu(\mathcal{O}^c_{m,-b}) = \mu(A_m)$. Moreover, as $(b + \mathcal{O}_{m,-b}) = \mathcal{O}_{m+1} = \bigsqcup_{n\geq m+1} A_n$, we have

$$\mu(\{x \in \mathcal{O}_{m,-b} \mid \mathrm{ord}(x+b) = n\}) = \mu(A_n) \text{ for } n \geq m+1. \tag{4.7}$$

Case $l \leq m-1$. In this case, $\psi(ax) = 1$ for $x \in A_m$. By Lemma 4.4, we have

$$\begin{aligned} c(a,b) = &- \delta(l)^2\mu(\mathcal{O}_{l+1}) + \sum_{n=l+1}^{m-1} \delta(n)^2\mu(A_n) + \delta(m)\int_{A_m} f(x+b)\mathrm{d}x \\ &+ \delta(m)\sum_{n=m+1}^{+\infty} \delta(n)\mu(A_n). \end{aligned} \tag{4.8}$$

Here, the term $\sum_{n=l+1}^{m-1} \delta(n)^2\mu(A_n) = 0$ if $l = m-1$. Note that

$$
\begin{aligned}
\int_{A_m} f(x+b)\mathrm{d}x &= \int_{\mathcal{O}_m} f(x+b)\mathrm{d}x - \int_{\mathcal{O}_{m+1}} f(x+b)\mathrm{d}x \\
&= \int_{\mathcal{O}_{m,-b}} f(x+b)\mathrm{d}x + \int_{\mathcal{O}^c_{m,-b}} f(x+b)\mathrm{d}x - \delta(m)\mu(\mathcal{O}_{m+1}) \\
&= \int_{\mathcal{O}_{m,-b}} f(x+b)\mathrm{d}x + \delta(m)\mu(\mathcal{O}^c_{m,-b}) - \delta(m)\mu(\mathcal{O}_{m+1}) \\
&= \sum_{n=m+1}^{+\infty} \delta(n)\mu(A_n) + \delta(m)\mu(A_m) - \delta(m)\mu(\mathcal{O}_{m+1}).
\end{aligned}
$$

Here, the last identity follows from equation (4.7). Therefore, $\int_{A_m} f(x+b)\mathrm{d}x$ is a positive rational number as $\mu(A_m) \geq \mu(\mathcal{O}_{m+1})$.

- If $m-1<0$, then $l<0$. As $\mu(A_l) = (q-1)\mu(\mathcal{O}_{l+1})$, we have

$$
\begin{aligned}
c(a,b) &\geq \delta(m)\sum_{n=1}^{+\infty}\delta(n)\mu(A_n) - \frac{1}{q-1}\delta(l)^2\mu(A_l) \\
&= q^{2l-1}\left(\frac{q^{2(m-l)}}{q+1} - q^l\right) > 0.
\end{aligned}
$$

- If $m-1 \geq 0$ and $l<0$, then

$$
c(a,b) > \delta(1)^2\mu(A_1) - \frac{1}{q-1}\delta(l)^2\mu(A_l) \geq 0.
$$

- If $m-1\geq 0$ and $l \geq 0$, then $\delta(l) > \delta(l+1) > \cdots$ and

$$
\begin{aligned}
c(a,b) =& -\frac{1}{q-1}\delta(l)^2\mu(A_l) - \delta(m)^2\mu(\mathcal{O}_{m+1}) + \sum_{n=l+1}^{m}\delta(n)^2\mu(A_n) \\
&+ 2\delta(m)\sum_{n=m+1}^{+\infty}\delta(n)\mu(A_n) \\
<& -\frac{1}{q-1}\delta(l)^2\mu(A_l) + 2\delta(l+1)\sum_{n=l+1}^{+\infty}\delta(n)\mu(A_n) \\
=& q^{-3l-2}\left(-q + \frac{2}{q+1}\right) < 0.
\end{aligned}
$$

Thus, the claims hold if $l<m$.

Case $l=m$. In this case, by Lemma 4.4, we have

$$
\delta(m)^{-1}c(a,b) = \int_{A_m}\psi(ax)f(x+b)\mathrm{d}x + \sum_{n=m+1}^{+\infty}\delta(n)\mu(A_n). \tag{4.9}
$$

Let $\mathcal{O}^n_{m,-b}$ $(n \geq m+1)$ be the set $\{x \in \mathcal{O}_{m,-b} \mid \mathrm{ord}(x+b) = n\}$. First, we have

$$\begin{aligned}
&\int_{A_m} \psi(ax)f(x+b)\mathrm{d}x \\
&= \int_{\mathcal{O}_m} \psi(ax)f(x+b)\mathrm{d}x - \int_{\mathcal{O}_{m+1}} \psi(ax)f(x+b)\mathrm{d}x \\
&= \int_{\mathcal{O}_{m,-b}} \psi(ax)f(x+b)\mathrm{d}x + \int_{\mathcal{O}^c_{m,-b}} \psi(ax)f(x+b)\mathrm{d}x - \delta(m)\mu(\mathcal{O}_{m+1})
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\mathcal{O}_{m,-b}} \psi(ax)f(x+b)\mathrm{d}x &= \psi(-ab) \sum_{n=m+1}^{+\infty} \int_{\mathcal{O}^n_{m,-b}} f(x+b)\mathrm{d}x \\
&= \psi(-ab) \sum_{n=m+1}^{+\infty} \delta(n)\mu(A_n),
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathcal{O}^c_{m,-b}} \psi(ax)f(x+b)\mathrm{d}x &= \delta(m) \int_{\mathcal{O}^c_{m,-b}} \psi(ax)\mathrm{d}x \\
&= \delta(m)\left(\int_{\mathcal{O}_m} \psi(ax)\mathrm{d}x - \int_{\mathcal{O}_{m,-b}} \psi(ax)\mathrm{d}x \right) \\
&= -\delta(m)\psi(-ab)\mu(\mathcal{O}_{m+1}).
\end{aligned}$$

Combining the above equations, we have

$$\delta(m)^{-1}c(a,b) = (1+\psi(-ab))\left(-\frac{1}{q-1}\delta(m)\mu(A_m) + \sum_{n=m+1}^{+\infty} \delta(n)\mu(A_n) \right).$$

If p is an odd prime, $1+\psi(-ab) \neq 0$. Hence if $l=m$, the claims hold by Lemma 4.5.

Case $l > m$. By Lemma 4.4 again, we have

$$\delta(m)^{-1}c(a,b) = \int_{A_m} \psi(ax)f(x+b)\mathrm{d}x - \delta(l)\mu(\mathcal{O}_{l+1}) + \sum_{n=l+1}^{+\infty} \delta(n)\mu(A_n).$$

Moreover,

$$
\begin{aligned}
&\int_{A_m} \psi(ax)f(x+b)\mathrm{d}x\\
=&\int_{\mathcal{O}_m} \psi(ax)f(x+b)\mathrm{d}x - \int_{\mathcal{O}_{m+1}} \psi(ax)f(x+b)\mathrm{d}x\\
=&\int_{\mathcal{O}_{m,-b}} \psi(ax)f(x+b)\mathrm{d}x + \int_{\mathcal{O}^c_{m,-b}} \psi(ax)f(x+b)\mathrm{d}x\\
&- \delta(m)\int_{\mathcal{O}_{m+1}} \psi(ax)\mathrm{d}x\\
=&\int_{\mathcal{O}_{m,-b}} \psi(ax)f(x+b)\mathrm{d}x + \delta(m)\int_{\mathcal{O}^c_{m,-b}} \psi(ax)\mathrm{d}x\\
=&\psi(-ab)\sum_{n=m+1}^{+\infty}\delta(n)\int_{A_n}\psi(ax)\mathrm{d}x\\
&+\delta(m)\left(\int_{\mathcal{O}_m}\psi(ax)\mathrm{d}x - \int_{\mathcal{O}_{m,-b}}\psi(ax)\mathrm{d}x\right)\\
=&\psi(-ab)\left(\sum_{n=m+1}^{+\infty}\delta(n)\int_{A_n}\psi(ax)\mathrm{d}x - \delta(m)\sum_{n=m+1}^{+\infty}\int_{A_n}\psi(ax)\mathrm{d}x\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=m+1}^{+\infty}\delta(n)\int_{A_n}\psi(ax)\mathrm{d}x - \delta(m)\sum_{n=m+1}^{+\infty}\int_{A_n}\psi(ax)\mathrm{d}x\\
=&-\delta(l)\mu(\mathcal{O}_{l+1}) + \sum_{n=l+1}^{+\infty}\delta(n)\mu(A_n) + \delta(m)\mu(\mathcal{O}_{l+1}) - \delta(m)\sum_{n=l+1}^{+\infty}\mu(A_n)\\
=&-\delta(l)\mu(\mathcal{O}_{l+1}) + \sum_{n=l+1}^{+\infty}\delta(n)\mu(A_n).
\end{aligned}
$$

Therefore,

$$
c(a,b) = \delta(m)(1+\psi(-ab))(-\delta(l)\mu(\mathcal{O}_{l+1}) + \sum_{n=l+1}^{+\infty}\delta(n)\mu(A_n)).
$$

If $l > m$, the claims hold as in the case $l = m$. The proposition then follows. □

Remark 4.7 If $p = 2$, the function constructed above is not a maximal spanning vector, as the set $\{(a,b) \in K \times K \mid \mathrm{cond}(a) \geq \mathrm{ord}(b) \text{ and } \psi(-ab) = -1\}$ has positive measure. Note that the problem remains if we change the values of $\delta(n)$, the functions of type $\sum_{n\in\mathbb{Z}} \beta(n)\mathbf{1}_{A_n}$ are not maximal spanning vectors.

4.2.2 Case $G = (\mathcal{O}, +)$

If $p \neq 2$ and $G = \mathcal{O}$, let $g \in L^2(\mathcal{O})$ be the restriction $f|_{\mathcal{O}}$, i.e.

$$g(x) = \sum_{n=0}^{+\infty} \delta(n) \mathbf{1}_{A_n}(x).$$

Then, similar argument as above shows that $c_{g,g}(a, b) \neq 0$ for all $(a, b) \in \widehat{\mathcal{O}} \times \mathcal{O}$.

Acknowledgements The authors would like to thank the referee for comments and suggestions, which improved the paper substantially. C. Cheng was supported by NSFC 11701272.

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