REMARKS ON FRAMES FROM PROJECTIVE REPRESENTATIONS OF LOCALLY COMPACT GROUPS

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Abstract. A projective representation of a locally compact group does phase retrieval if it admits a maximal spanning frame vector. In this paper, we provide a characterization of maximal spanning vectors for type I and square integrable irreducible projective representations of separable locally compact abelian groups. This generalizes the well-known criterion for the time-frequency case and unifies previous criteria for finite groups case and locally compact Gabor case. As an application, we show that irreducible projective representations of compact abelian groups do phase retrieval.

1. Introduction

The phase retrieval problem considers recovering a signal of interest from magnitudes of its linear or nonlinear measurements. Balan, Casazza and Edidin [2] initiated the investigation of the phase retrieval problem by using linear measurements against a frame. The study of the phase retrieval problem of frames has attracted attention of mathematicians and it has rich connections with abstract harmonic analysis, representation theory, number theory, algebraic geometry etc. (cf. [12, 17]).

As mentioned in [15], sources containing explicit constructions of frames with guaranteed phase retrieval properties are relatively scarce. A sufficient condition for a frame being phase retrievable is that it has maximal span (cf. [3]). Applying this idea, one could construct phase retrievable frames from orbits of vectors with respect to a group action. In the following, we consider Hilbert spaces over $\mathbb{C}$. By a locally compact group, we mean a topological group whose topology is locally compact and Hausdorff. The general setup goes as follows. Let G be a locally compact group with unity e. A *multiplier on* G is a measurable function from $G \times G$ to the unit circle $\alpha : G \times G \to \mathbb{T}$ which satisfies

(1) $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$;
(2) $\alpha(e, x) = \alpha(x, e) = 1$ for all $x \in G$.

A *projective representation* of G with respect to α (or an *α-representation*) is a map $\pi : G \to \mathbf{U}(V_\pi)$, where V_π is a complex Hilbert space and $\mathbf{U}(V_\pi)$ is the space of unitary operators on V_π, such that

(1) $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$ for all $x, y \in G$;
(2) for any $v \in V_\pi$, the map $x \mapsto \pi(x)v$ is a measurable function from G to V_π.

We study properties of the family of vectors $\{\pi(g)v \mid g \in G\}$ for a fixed $v \in V_\pi$. We refer to [20, 24] for properties of multipliers on locally compact groups, to [21, 22] for properties

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of projective representations (or multiplier representations) of locally compact groups. In particular, we know that the matrix coefficients are measurable by [21, Theorem 1].

Recall that $v \in V_\pi$ is a *frame vector* for (π, V_π) if the map $G \to V_\pi$ $(g \mapsto \pi(g)v)$ is a frame (cf. [25, Definition 2.1]); a frame vector $v \in V_\pi$ is *phase retrievable* if the associated frame is phase retrievable, i.e., the map

$$\begin{aligned} T_v : V_\pi/\mathbb{T} &\to L^2(G) \\ u &\mapsto (g \mapsto |\langle u, \pi(g)v\rangle|) \end{aligned}$$

is injective; if there exists a phase retrievable frame vector for (π, V_π), we say that the representation π *does phase retrieval.*

An element $v \in V_\pi$ is called *maximal spanning* if

$$\overline{\mathrm{Span}}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = \mathrm{HS}(V_\pi),$$

where $x \otimes y : V_\pi \to V_\pi$ $(u \mapsto \langle u, y\rangle x)$ is the one-dimensional projection and $\mathrm{HS}(V_\pi)$ is the space of Hilbert-Schmidt operators on V_π.

Maximal spanning frame vectors are special as they are phase retrievable. This property provides a method to explicitly construct phase retrievable group frames. We refer to [3, 23] and [6, Section 3.2.2] for more information on the relation between phase retrievable vectors and maximal spanning vectors. In [5, 23] Li, Han and etc. proved that irreducible projective representations of finite abelian groups do phase retrieval. In [6, 8] Cheng, Xu, and etc. proved that the Weyl-Heisenberg representation of $\widehat{H} \times H$ does phase retrieval for a large class of locally compact abelian groups H, where $\widehat{H}$ is the dual of H with the Plancherel measure. In [15], Führ and Oussa proved that irreducible representations of nilpotent Lie groups and certain nilpotent p-groups do phase retrieval. Continuing in this direction, in this paper, we provide the following characterization of maximal spanning vectors for type I and square integrable irreducible projective representations of separable locally compact abelian groups. This result generalizes the well-known criterion for the time-frequency case. (Cf. [4], [23, Theorem 1.7], [5, Proposition 3.11] for finite abelian groups case, [7, Section 4] for compact abelian groups case, [6, Proposition 3.3] for the case of locally compact abelian groups of type $\widehat{H} \times H$.)

Theorem 1.1. *Let G be a separable locally compact abelian group and α be a type I multiplier of G. Let $\pi : G \to \mathbf{U}(V_\pi)$ be a square integrable irreducible α-representation of G. Then $v \in V_\pi$ is a maximal spanning frame vector if and only if the matrix coefficient $(g \mapsto \langle \pi(g)v, v\rangle)$ is almost nowhere vanishing.*

An ingredient in the proof of Theorem 1.1 is the projective Plancherel theorem [22, Theorem 7.1], where we require that the locally compact groups are separable. An immediate consequence is the following result.

Corollary 1.2. *Let G and π be as in Theorem 1.1. If $v \in V_\pi$ is a vector such that the matrix coefficient $(g \mapsto \langle \pi(g)v, v\rangle)$ is almost nowhere vanishing, then $(g \mapsto \pi(g)v)$ is a phase retrievable frame and π does phase retrieval.*

By showing the existence of maximal spanning vectors, we have the following result.

Theorem 1.3. *Let G be a compact abelian group and $\pi : G \to \mathbf{U}(V_\pi)$ be an irreducible projective representation of G. Then the set of maximal spanning frame vectors for (π, V_π) is open dense in V_π. In particular, π does phase retrieval.*

The content of the paper is organised as follows. In Section 2, we explain that the wavelet transform for projective representations works as well as for linear representations. In particular, we show that square integrable irreducible α-representations are the same as irreducible sub α-representations of the regular α-representation and every nontrivial square integrable vector of such an α-representation is a frame vector. Moreover, we have the Duflo-Moore operator for discrete series α-representations and it gives the frame bounds for the tight frames generated by square integrable vectors. In Section 3, by reducing to the totally skew multipliers, we prove Theorem 1.1. One of the key ingredients is the structure of projective representations of locally compact abelian groups obtained by Baggett and Kleppner [1], which holds true without separability condition on the group. Applying the projective Plancherel theorem [22, Theorem 7.1] to the special case of locally compact abelian groups, we translate the maximal spanning property in $\mathrm{HS}(V_\pi)$ to a maximal spanning property in $L^2(G)$ and use the Fourier transformation to obtain the expected result.

2. The wavelet transform

In this section we study square integrable projective representations of locally compact groups via the wavelet transform. The strategy is the same as that in linear representations case (cf. [10, Chap. 12] and [14]). There is little doubt that the following results were known to the experts, but we could not find the statements in the literature. We sketch the idea of proofs and state them explicitly.

Let G be a locally compact group with left Haar measure $\mu = \int_G \cdot \,\mathrm{d}\, g$. Let $\alpha \in Z^2(G, \mathbb{T})$ be a multiplier and $\pi : G \to \mathbf{U}(V_\pi)$ be an α-representation of G on a Hilbert space V_π. Given any fixed vector $\xi \in V_\pi$, we obtain the *wavelet transform* corresponding to π and ξ

$$\begin{aligned} W_\xi : V_\pi &\to \mathrm{Map}(G, \mathbb{C}) \\ \eta &\mapsto (x \mapsto \langle \eta, \pi(x)\xi \rangle). \end{aligned}$$

Note that this transformation is injective if and only if ξ is a cyclic vector for π, i.e. the closed linear span of $\{\pi(x)\xi \mid x \in G\}$ is V_π. If π is irreducible, this is true for any nontrivial ξ and the injectivity of W_ξ is much easier than the injectivity of T_ξ.

Let D_ξ be the subspace of V_π given by

$$D_\xi = \{\eta \in V_\pi \mid W_\xi(\eta) \in L^2(G)\}.$$

The restriction of W_ξ induces a linear closed operator $W_\xi : D_\xi \to L^2(G)$ (cf. [10, Lemma 12.1.2]).

Definition 2.1.
(1) A vector $\xi \in V_\pi$ is called *square integrable* if $D_\xi = V_\pi$.
(2) Denote by D_π the subspace of V_π consisting of square integrable vectors. The α-representation (π, V_π) is called a *square integrable α-representation* if D_π is dense in V_π.
(3) A square integrable vector $\xi \in V_\pi$ is called an *admissible vector* if $W_\xi : D_\xi \to L^2(G)$ is isometric.

The results in [10, Section 12.1] generalize to α-representations easily. In particular, we have the following result (cf. [10, Example 12.1.7]), which shows the existence of nontrivial square integrable α-representations.

Lemma 2.2. *Let $L : G \to \mathbf{U}(L^2(G))$ be the left regular α-representation of G, i.e. $(L(y)f)(x) = \frac{\alpha(y,y^{-1})}{\alpha(y^{-1},x)} f(y^{-1}x)$ for $f \in L^2(G)$. Then every $\xi \in C_c(G) \subset L^2(G)$ is square integrable. In particular, as $C_c(G)$ is dense in $L^2(G)$, the left regular α-representation is square integrable.*

Proof. Let $R' : G \to \mathbf{U}(L^2(G))$ be the right regular α^{-1}-representation of G with respect to the left Haar measure, i.e. $R'(y)f(x) = \sqrt{\Delta(y)}\alpha(x,y)^{-1}f(xy)$ for $f \in L^2(G)$, where Δ is the modular function of G. Put $\tilde{\xi}(y) = \Delta(y)^{-1/2}\overline{\xi(y)}$. Then $\tilde{\xi} \in C_c(G) \subset L^1(G)$. For any $\eta \in L^2(G)$, we have

$$\begin{aligned}
\langle \eta, L(x)\xi\rangle &= \int_G \eta(y)\frac{\alpha(x^{-1},y)}{\alpha(x,x^{-1})}\overline{\xi(x^{-1}y)}\,\mathrm{d}\,y \\
&= \int_G \tilde{\xi}(y)\frac{\alpha(x^{-1},xy)}{\alpha(x,x^{-1})}\sqrt{\Delta(y)}\eta(xy)\,\mathrm{d}\,y \\
&= \int_G \tilde{\xi}(y)R'(y)\eta(x)\,\mathrm{d}\,y = (R')^*(\tilde{\xi})\eta(x).
\end{aligned}$$

Here the third identity follows from $\alpha(x^{-1},xy)\alpha(x,y) = \alpha(x^{-1},x)\alpha(e,y) = \alpha(x^{-1},x) = \alpha(x,x^{-1})$, $(R')^*$ is the representation of the Banach $*$-algebra $L^1(G,\alpha)$ associated with R'. See Remark 2.3 for details. Hence the map $(x \mapsto \langle \eta, L(x)\xi\rangle)$ is just $(R')^*(\tilde{\xi})\eta$, which is square integrable. The lemma follows. □

Remark 2.3. In the above computation, we used a relation between projective representations of G and modules of the *twisted group algebra* which we recall here. Let $L^1(G,\alpha)$ be the set of complex-valued integrable functions on G with multiplication (convolution) defined by

$$(f_1 * f_2)(x) = \int_G f_1(g)f_2(g^{-1}x)\alpha(g,g^{-1}x)\,\mathrm{d}\,g$$

and a $*$-operator (involution) defined by

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})\alpha(x,x^{-1})^{-1},$$

where $\bar{\ }$ is the complex conjugation and Δ is the modular function. Then $L^1(G,\alpha)$ is a Banach $*$-algebra. Let $\pi : G \to \mathbf{U}(V_\pi)$ be an α-representation of G. The map $\pi \mapsto \pi^*$, where

$$\begin{aligned}
\pi^* : L^1(G,\alpha) &\to \mathbf{B}(V_\pi) \\
f &\mapsto \int_G f(g)\pi(g)\,\mathrm{d}\,g,
\end{aligned}$$

induces a bijection between the set of equivalent classes of α-representations of G and the set of equivalent classes of representations of the Banach $*$-algebra $L^1(G,\alpha)$. See [11, Section 13.3.5].

Suppose that (π, V_π) is an α-representation of G and let $\xi \in V_\pi$ be any vector. For $y \in G$, we have

$$\begin{aligned}(W_\xi(\pi(y)\eta))(x) &= \langle \pi(y)\eta, \pi(x)\xi \rangle \\ &= \langle \eta, \pi(y)^{-1}\pi(x)\xi \rangle \\ &= \langle \eta, \alpha(y, y^{-1})^{-1}\alpha(y^{-1}, x)\pi(y^{-1}x)\xi \rangle \\ &= (L(y)W_\xi(\eta))(x).\end{aligned}$$

Therefore D_ξ is a $\pi(G)$-invariant subspace of V_π and W_ξ intertwines the representation $\pi|_{D_\xi}$ with the left regular α-representation. In particular if ξ is admissible, W_ξ establishes an equivalence between (π, V_π) and a subrepresentation of $(L, L^2(G))$.

An irreducible α-representation is called a *discrete series α-representation* of G if it is a subrepresentation of $(L, L^2(G))$. The results in [10, Section 12.2] hold for α-representations by the same argument. We have the following result that characterizes irreducible square integrable α-representations (cf. [10, Corollary 12.2.4]).

Proposition 2.4. *Let (π, V_π) be an irreducible α-representation of the locally compact group G. Then the following are equivalent:*

(1) *(π, V_π) is square integrable.*
(2) *There exists an admissible vector $\xi \in V_\pi$.*
(3) *(π, V_π) is a discrete series α-representation, i.e. it is equivalent to a subrepresentation of the left regular α-representation.*

The following result on the Duflo-Moore operator is important for our study on frames (cf. [10, Theorem 12.2.5]).

Theorem 2.5. *Let (π, V_π) be a discrete series α-representation of the locally compact group G. Let D_π be the set of square integrable vectors in V_π.*

(1) *There exists a closed densely defined operator $C_\pi : D_\pi \to V_\pi$ satisfying the orthogonality relation*

$$\langle C_\pi \xi', C_\pi \xi \rangle \langle \eta, \eta' \rangle = \langle W_\xi(\eta), W_{\xi'}(\eta') \rangle$$

for all $\xi, \xi' \in D_\pi$ and $\eta, \eta' \in V_\pi$.
(2) *The operator $C_\pi : D_\pi \to V_\pi$ is injective and $\xi \in V_\pi$ is admissible if and only if $\xi \in D_\pi$ with $||C_\pi \xi|| = 1$.*
(3) *If G is unimodular, then all $\xi \in V_\pi$ are square integrable and $D_\pi = V_\pi$. In this case, there exists a unique constant $c_\pi \in \mathbb{R}_{>0}$ such that C_π can be chosen equal to $c_\pi \,\mathrm{id}_{V_\pi}$.*

Remark 2.6. (1) Let $G(\alpha)$ be the extension of G by $\mathbb{T}$ given by α. Let $\pi_\alpha : G(\alpha) \to \mathbf{U}(V_\pi)$ be the linear unitary representation of $G(\alpha)$ associated with (π, V_π) given by $\pi_\alpha(t, x) = t\pi(x)$ for all $(t, x) \in G(\alpha)$ (cf. [21, Page 220]). Then the sets of square integrable vectors with respect to π and π_α are the same and the corresponding Duflo-Moore operators coincide.
(2) If G is compact, then irreducible α-representations of G are finite dimensional. The twisted Peter-Weyl Theorem (cf. [7, Section 2]) shows that $c_\pi = d_\pi^{-1/2}$, where d_π is the dimension of the representation space V_π.

(3) Let H be a locally compact abelian group with the Pancherel dual $\widehat{H}$. Let $G = \widehat{H} \times H$ and $\pi : \widehat{H} \times H \to \mathbf{U}(L^2(H))$ be the Weyl-Heisenberg representation, i.e. $(\pi(h^*, h)f)(h') = h^*(h')f(h'h)$ for $f \in L^2(H)$. Then from the computation in [6, Theorem 2.1], we have $c_\pi = 1$.

An immediate consequence of Theorem 2.5 is the following result.

Corollary 2.7. *Let (π, V_π) be a discrete series α-representation of G and $v \in V_\pi$ be a nontrivial vector. Then $\Phi_v : G \to V_\pi$ ($g \mapsto \pi(g)v$) is a continuous frame if and only if $v \in D_\pi$. Moreover, if $v \in D_\pi$, then Φ_v is a tight frame with frame bounds $A = B = ||C_\pi v||^2$.*

If G is unimodular, then $D_\pi = V_\pi$ and every nontrivial vector of V_π is a frame vector. Therefore to find maximal spanning frame vectors, we only need to focus on the maximal spanning property.

Combining Corollary 2.7 and the argument of [5, Section 2.2] (see also [7, Section 3.2], [9], [14, Theorem 2.31]), we have the following result.

Proposition 2.8. *Let (π, V_π) be an α-representation of G. Assume that (π, V_π) is a finite direct sum of discrete series α-representations. Write the irreducible decomposition as $(\pi, V_\pi) = \oplus_{i\in I}(\pi_i, V_i)$. Let $v \in V_\pi$ be a nontrivial vector and let v_i be the projection of v in V_i. Then $\Phi_v : G \to V$ ($g \mapsto \pi(g)v$) is a continuous tight frame if and only if $v_i \in D_{\pi_i}$ for $i \in I$ and the following two conditions are satisfied*

(1) $||C_{\pi_i} v_i|| = ||C_{\pi_j} v_j||$ *for all* $i, j \in I$;
(2) $\langle C_{\pi_j} \sigma v_i, C_{\pi_j} v_j\rangle = 0$ *for* $i, j \in I$ *and* $\sigma \in \mathrm{Hom}_{\mathrm{Rep}_\alpha}(V_i, V_j)$.

It is possible to generalize the above proposition from direct sum case to direct integral case (cf. [14, Section 4.3]). Since here we are interested in the phase retrieval property of irreducible α-representations, we leave the generalization to the readers.

3. Locally compact abelian groups

In this section G is a separable locally compact abelian group. Let $\alpha \in Z^2(G, \mathbb{T})$ be a type I multiplier, i.e. all the α-representations of G are of type I (cf. [13, Page 229], [18, 19]). Let $\pi : G \to \mathbf{U}(V_\pi)$ be a discrete series α-representation of G. For u, $v \in V_\pi$, denote by $c^\pi_{u,v}$ the matrix coefficient ($g \mapsto \langle\pi(g)u, v\rangle$). We have the following result.

Theorem 3.1. *With the above notation, $v \in V_\pi$ is a maximal spanning frame vector if and only if $c^\pi_{v,v}(g) \neq 0$ for almost all $g \in G$.*

Proof. Because G is unimodular, every nontrivial element $v \in V_\pi$ is a frame vector by Theorem 2.5(3) and we focus on the maximal spanning property. The following two observations enable us to simplify the situation.

(1) Suppose that $\pi : G \to \mathbf{U}(V_\pi)$ and $\pi' : G \to \mathbf{U}(V_{\pi'})$ are equivalent, i.e. there exist a measurable function $\mu : G \to \mathbb{T}$ and a unitary isomorphism $M : V_\pi \to V_{\pi'}$ with $M\pi(g)M^{-1} = \mu(g)\pi'(g)$. Let $v \in V_\pi$ and $v' = Mv \in V_{\pi'}$, then

$$\pi'(g)v' \otimes \pi'(g)v' = M(\pi(g)v \otimes \pi(g)v)M^{-1}, \ \langle\pi'(g)v', v'\rangle = \mu(g)^{-1}\langle\pi(g)v, v\rangle.$$

Therefore

$$M(\overline{\mathrm{Span}}\{\pi(g)v \otimes \pi(g)v \mid g \in G\})M^{-1} = \overline{\mathrm{Span}}\{\pi'(g)v' \otimes \pi'(g)v' \mid g \in G\},$$

and

$$c^{\pi}_{v,v}(g) = 0 \text{ if and only if } c^{\pi'}_{v',v'}(g) = 0.$$

Hence proving the theorem for π is equivalent to proving the theorem for π'.

(2) If $H \lhd G$ is a closed subgroup, $\pi_1 : G/H \to \mathbf{U}(V)$ is a projective representation and $\pi : G \to \mathbf{U}(V)$ is the composition of π_1 with the natural projection $G \to G/H$, then $\pi(g) = \pi_1(gH)$. Therefore

$$\overline{\operatorname{Span}}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = \overline{\operatorname{Span}}\{\pi_1(\bar{g})v \otimes \pi_1(\bar{g})v \mid \bar{g} \in G/H\}$$

and

$$c^{\pi}_{v,v}(g) = 0 \text{ if and only if } c^{\pi_1}_{v,v}(gH) = 0.$$

By the quotient integral formula (cf. [10, Theorem 1.5.3]), to prove the theorem for (π, G, V), it suffices to prove the theorem for $(\pi_1, G/H, V)$.

Since replacing α with a similar multiplier gives us equivalent projective representations, by the first observation, we may assume that α is normalized (i.e. $\alpha(x, x^{-1}) = 1$ for all $x \in G$) as in [1]. Let $\lambda : G \times G \to \mathbb{T}$ be the map $\lambda(x, y) = \frac{\alpha(y,x)}{\alpha(x,y)}$. Then λ is a bicharacter and it induces a homomorphism $\lambda_\alpha : G \to \widehat{G}$ with $\lambda_\alpha(x)(y) = \lambda(x, y)$. Let S_α be the kernel of λ_α. We call α *totally skew* if S_α is trivial. By [1, Theorem 3.1], α is similar to a multiplier which is lifted from a totally skew multiplier α' on G/S_α, i.e. α is similar to the composition of $G \times G \to G/S_\alpha \times G/S_\alpha$ and $\alpha' : G/S_\alpha \times G/S_\alpha \to \mathbb{T}$. Moreover, π is equivalent to a projective representation of the form $\gamma \otimes \pi_1'$, where $\gamma \in \widehat{G}$ is a linear character of G, π_1 is an α'-representation of G/S_α, π_1' is the projective representation of G induced from π_1 via the natural quotient map $G \to G/S_\alpha$. By the second observation, we may assume that α is totally skew at the beginning. In this case $\lambda_\alpha : G \to \widehat{G}$ is injective and has dense image. Moreover, because of the type I assumption, λ_α is bicontinuous by [1, Theorem 3.2] and the image of λ_α is open dense in $\widehat{G}$. Therefore, λ_α is an isomorphism.

By [1, Theorem 3.3], up to isomorphism, (π, V_π) is the unique α-representation of G. The projective Plancherel theorem [22, Theorem 7.1] tells us that in this special case we have an isomorphism

$$\begin{aligned} V_\pi \otimes V_\pi &\to L^2(G) \\ u \otimes v &\mapsto c^{\pi}_{u,v}. \end{aligned} \tag{3.1}$$

Here we identify V with V^* in the usual way if V is a Hilbert space. Hence for $v \in V_\pi$,

$$\overline{\operatorname{Span}}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = V_\pi \otimes V_\pi \iff \overline{\operatorname{Span}}\{c^{\pi}_{\pi(g)v,\pi(g)v} \mid g \in G\} = L^2(G). \tag{3.2}$$

Since α is normalized and G is abelian, the cocycle condition tells us

$$\alpha(g^{-1}, hg)\alpha(g, h) = \alpha(g^{-1}, gh)\alpha(g, h) = \alpha(g^{-1}, g)\alpha(e, h) = 1.$$

We then have

$$\begin{aligned} \pi(g)^*\pi(h)\pi(g) = \pi(g^{-1})\pi(h)\pi(g) &= \alpha(g^{-1}, hg)\alpha(h, g)\pi(g^{-1}hg) \\ &= \frac{\alpha(h, g)}{\alpha(g, h)}\pi(h) = \lambda_\alpha(g)(h)\pi(h). \end{aligned}$$

Hence

$$\begin{aligned} c^{\pi}_{\pi(g)v,\pi(g)v}(h) = \langle \pi(h)\pi(g)v, \pi(g)v\rangle &= \langle \pi(g)^*\pi(h)\pi(g)v, v\rangle \\ &= \langle \lambda_\alpha(g)(h)\pi(h)v, v\rangle = \lambda_\alpha(g)(h)c^{\pi}_{v,v}(h), \end{aligned}$$

and we have $c^{\pi}_{\pi(g)v,\pi(g)v} = \lambda_\alpha(g)c^{\pi}_{v,v}$. Therefore the second identity in (3.2) is equivalent to

$$\overline{\mathrm{Span}}\{\lambda_\alpha(g)c^{\pi}_{v,v} \mid g \in G\} = L^2(G). \tag{3.3}$$

Via Fourier transform, identity (3.3) is equivalent to

$$\overline{\mathrm{Span}}\{(\lambda_\alpha(g)c^{\pi}_{v,v})\widehat{\ } \mid g \in G\} = L^2(\widehat{G}) \Longleftrightarrow \overline{\mathrm{Span}}\{\widehat{c^{\pi}_{v,v}}(\lambda_\alpha(g)\bullet) \mid g \in G\} = L^2(\widehat{G}). \tag{3.4}$$

Since $\lambda_\alpha : G \to \widehat{G}$ is surjective, by Lemma 3.2, the second identity in (3.4) is equivalent to

$$c^{\pi}_{v,v}(g) \neq 0 \text{ for almost all } g \in G. \tag{3.5}$$

The theorem follows. $\square$

Lemma 3.2. *Let $D \subset G$ be a subset with $\mu(G - D) = 0$ and $f \in L^2(G)$. Let $S_D \subset L^2(G)$ be the closed linear span of the translates of f by elements of D. Then $S_D = L^2(G)$ if and only if $\widehat{f}(\chi) \neq 0$ for almost all $\chi \in \widehat{G}$.*

Proof. This is essentially [13, Proposition 4.72]. Let $g \in L^2(G)$. Let L_x be the left translation operator on $L^2(G)$, i.e. $(L_x f)(y) = f(x^{-1}y)$. Then $g \perp S_D$ if and only if $\int (L_x f)\overline{g} = 0$ for all $x \in D$. This is equivalent to

$$\begin{aligned} 0 &= \int_{\widehat{G}} (L_x f)\widehat{\ }(\chi)\overline{\widehat{g}(\chi)}\,\mathrm{d}\,\chi \\ &= \int_{\widehat{G}} \overline{\chi(x)}\widehat{f}(\chi)\overline{\widehat{g}(\chi)}\,\mathrm{d}\,\chi \\ &= (\widehat{f}\overline{\widehat{g}})\widehat{\ }(x) \end{aligned}$$

for all $x \in D$. Then it is equivalent to $\widehat{f}\overline{\widehat{g}} = 0$ almost everywhere. The lemma follows. $\square$

To apply Theorem 3.1, we start with the following lemma.

Lemma 3.3. *Let V be a Hilbert space and $T \in \mathbf{U}(V)$ be a unitary operator. Let $N(T) = \{u \in V \mid \langle Tu, u\rangle = 0\}$. Then $V - N(T)$ is open dense in V.*

Proof. If V is finite dimensional, the lemma is easy as by fixing a basis, the condition $\langle Tu, u\rangle = 0$ is given by polynomials on the real and imaginary parts of the coordinates of u (cf. [23, Lemma 2.2]). We assume that V is infinite dimensional.

Suppose that $\langle Tu, u\rangle = 0$ for all $u \in V$. Then by the polarization identity (cf. [13, A.1 Theorem]), $\langle Tu, v\rangle = 0$ for all $u, v \in V$, which is impossible as T is unitary. Therefore $N(T)$ is a proper subset of V. The same argument shows that for any proper subspace V' of V, $N(T) \cap V'$ is a proper subset of V'.

If $\langle Tu, u\rangle \neq 0$, then for any $w \in V$ with $||w||$ sufficiently small, we have

$$\begin{aligned} |\langle T(u+w), u+w\rangle| &\geq |\langle Tu, u\rangle| - |\langle Tu, w\rangle| - |\langle Tw, u\rangle| - |\langle Tw, w\rangle| \\ &\geq |\langle Tu, u\rangle| - 2||w|| \cdot ||u|| - ||w||^2 > 0. \end{aligned}$$

Therefore $N(T)$ is closed.

If $\langle Tu, u\rangle = 0$, from the above discussion, there exists a $w \in V$ such that $\langle Tw, w\rangle \neq 0$ and $\langle T^*u, w\rangle = \langle Tu, w\rangle = 0$, where T^* is the adjoint of T. For any positive integer n, we have

$$\langle T(u + \frac{1}{n}w), u + \frac{1}{n}w\rangle = \langle T(\frac{1}{n}w), \frac{1}{n}w\rangle \neq 0.$$

Therefore $V - N(T)$ is dense. The lemma follows. $\square$

Proof of Theorem 1.3. Assume first that G is a finite abelian group. Let $g \in G$ and define

$$P(g) = \{u \in V_\pi \mid \langle \pi(g)u, u\rangle \neq 0\}.$$

Then $P(g) \subset V_\pi$ is open dense by Lemma 3.3. Since G is finite, by the Baire category theorem, $\cap_{g\in G} P(g) \subset V_\pi$ is open dense in V_π. By Theorem 3.1, $\cap_{g\in G} P(g)$ is the set of maximal spanning frame vectors and the theorem follows in this case.

Assume that G is compact. Then $\widehat{G}$ is discrete. In the proof of Theorem 3.1, the kernel S_α of $\lambda_\alpha : G \to \widehat{G}$ is open and hence G/S_α is a finite group. By the second observation in the proof of Theorem 3.1, to prove the property of maximal spanning vectors for projective representations of compact abelian groups, it suffices to prove the property of maximal spanning vectors for projective representations of finite abelian groups. The theorem then follows from the above discussion or [23, Theorem 1.7]. $\square$

Note that if $v \in V_\pi$ is a maximal spanning vector for (π, V_π), then $v \otimes v \in V_\pi \otimes V_\pi$ is a cyclic vector for the linear representation $\pi \otimes \pi^* : G \to V_\pi \otimes V_\pi$, where π^* is the contragredient representation of π. We hence obtain the following result.

Corollary 3.4. *Let $\pi : G \to \mathbf{U}(V_\pi)$ be an irreducible projective representation of a compact abelian group G. The representation $\pi \otimes \pi^*$ is cyclic and it admits cyclic vectors of the form $v \otimes v$.*

Remark 3.5 (A question on zero sets of matrix coefficients)*.* Let G be a separable locally compact group and $\pi : G \to \mathbf{U}(V_\pi)$ be a discrete series projective representation. Let $c_{v,v}$ be the matrix coefficient associated with $v \in D_\pi$. Suppose that $c_{v,v}$ is nonzero in a dense subset of G, is it true that $c_{v,v}^{-1}(0)$ has measure zero? The motivation of this question is the close relation with phase retrieval property of certain group frames. More precisely, assume that G is abelian as in Theorem 1.1 and suppose that the question has a positive answer in this case. Let $I \subset G$ be a countable dense subset and consider $\cap_{g\in I} P(g)$. Then the same proof of Theorem 1.3 shows that the set of maximal spanning frame vectors for (π, V_π) is dense in V_π and in particular π does phase retrieval.

Example 3.6 (The Heisenberg group)**.** Let R be a commutative topological ring. Let $H(R)$ be the Heisenberg group over R, i.e. $H(R) = R^3$ with group law

$$(j, k, l)(j', k', l') = (j + j', k + k', l + l' + jk').$$

The center and the commutator subgroup of $H(R)$ are both equal to

$$Z = \{(0, 0, l) \mid l \in R\} \cong R.$$

Let $\chi : R \to \mathbb{T}$ be a character of R and we also regard it as a character of Z. Let $\mathcal{H}(R) := R \times R \times \mathbb{T}$ with group law

$$(j, k, t)(j', k', t') = (j + j', k + k', tt'\chi(jk')).$$

Let $G = R \times R$ be the direct product of two copies of R. Then $G(\alpha) = \mathcal{H}(R)$, where $\alpha : G \times G \to \mathbb{T}$ is the multiplier defined by

$$\alpha((j, k), (j', k')) = \chi(jk').$$

Let $\rho : H(R) \to \mathbf{U}(V_\rho)$ be an irreducible linear representation of $H(R)$ with central character χ. Then it induces an irreducible linear representation $\rho' : R \times R \times \mathbb{T} \to \mathbf{U}(V_\rho)$

with $\rho'(j,k,t) = t\rho(j,k,0)$ and an irreducible α-representation $\pi : R \times R \to \mathbf{U}(V_\rho)$ with $\pi(j,k) = \rho(j,k,0)$.

To check the phase retrieval property of ρ and ρ', it suffices to check the phase retrieval property of π. We discuss some special cases in the following.

(1) $R = \mathbb{F}_q$ is a finite field with q elements. If the central character χ is trivial, then ρ is one-dimensional and it is from a character of $\mathbb{F}_q \times \mathbb{F}_q \times \{0\}$. If the central character χ is not trivial, then ρ is induced from a character of $\{0\} \times \mathbb{F}_q \times \mathbb{F}_q$ and is q-dimensional. This is the Gabor case. Fix an isomorphism $V_\rho \cong \mathbb{C}^q$ such that each $\pi(g) \in \mathbb{C}^{q\times q}$ ($g \in \mathbb{F}_q \times \mathbb{F}_q$) has algebraic entries. Then the Lindemann-Weierstrass theorem provides an easy way to write down a maximal spanning vector. E.g. let p_i ($1 \leq i \leq q$) be different prime numbers, then $v = (e^{\sqrt{p_1}}, \ \ldots, \ e^{\sqrt{p_q}})' \in \mathbb{C}^q$ satisfies $\langle \pi(j,k)v, v\rangle \neq 0$ for all $(j,k) \in \mathbb{F}_q \times \mathbb{F}_q$, hence it is a maximal spanning vector.

(2) $R = \mathbb{Z}$ is the additive group of integers with discrete topology. In this case $\chi(l) = w^l$ for some $w \in \mathbb{T}$. If w has infinite order, then α is not of type I (cf. [13, Section 6.8, Chap. 7 (Example 4)]).

If w has finite order, i.e. it is a root of unity, say $w^q = 1$. In this case ρ is a finite dimensional representation from a representation of the quotient $\mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/q\mathbb{Z})$ (cf. [13, Theorem 6.58]). It is easy to see that the associated α-representation π of $\mathbb{Z} \times \mathbb{Z}$ is not square integrable.

In other words, the results in this paper does not apply to the discrete Heisenberg group $H(\mathbb{Z})$.

(3) R is a local field. If $\chi : Z \to \mathbb{T}$ is trivial, then $\rho : H(R) \to \mathbf{U}(V_\rho)$ factors through $R \times R \times \{0\}$, hence it is one-dimensional. In this case ρ is not square integrable.

If $\chi : Z \to \mathbb{T}$ is not trivial, then $R \to \widehat{R}$ ($r \mapsto \chi(r\cdot)$) is an isomorphism of topological groups. In this case, by the Markey machine, ρ is induced from a character of $\{0\} \times R \times R$ and $\pi : R \times R \to \mathbf{U}(V_\rho)$ is an irreducible square integrable α-representation (cf. [13, Section 6.6] and [10, Proposition 12.3.2]). We may take $V_\rho = L^2(R)$ and take the group action to be

$$\pi(j,k)f(l) = \chi(kl)f(j+l),$$

for $f \in L^2(R)$. Then $f \in L^2(R)$ is a maximal spanning vector if and only if $\int \chi(bx)f(a+x)\overline{f(x)}\,\mathrm{d}\,x \neq 0$ for almost all $(a,b) \in R \times R$. Such f has been constructed explicitly in [6, 8]. Therefore if ρ is an infinite dimensional irreducible representation of $H(R)$, then ρ does phase retrieval. If $R = \mathbb{R}$ or $R = \mathbb{C}$, this is a special case of [15]. Note that the case $R = \mathbb{R}$ is well-known (cf. [6, 14, 16]), the Gauss and Hermite windows satisfy the condition from Theorem 3.1.

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