SOME REMARKS ON THE STONE-VON NEUMANN THEOREM FOR
LOCALLY COMPACT ABELIAN GROUPS

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Abstract. In this paper we construct and study continuous frames from projective representations of locally compact abelian groups. In particular, using Wiener’s theorem, we obtain a characterization of maximal spanning vectors. As an application, we prove the existence of maximal spanning vectors, hence the phase retrievability, for the associated \((G \times \hat{G})\)-frames when \(G\) is a compactly generated locally Euclidean locally compact abelian group.

1. Introduction

Being phase retrievable is an important property of frames in the study of signal analysis. In case of finite frames with symmetries, e.g., group frames and twisted group frames, there are many results connecting maximal spanning vectors, vectors in general position, and finite frames which are phase retrievable (cf. [1, 2, 3, 4, 5, 6, 9, 10, 20, 21] etc.). In this paper, we study similar properties for continuous frames. Let \(V\) be a complex Hilbert space and \((\Omega, \mu)\) be a measure space with positive measure \(\mu\). Let \(F : \Omega \to V\) be a continuous frame with respect to \((\Omega, \mu)\), i.e.,

1. \(F\) is weakly-measurable, i.e., for all \(v \in V\), the map \(\omega \mapsto \langle v, F(\omega) \rangle\) is a measurable function on \(\Omega\);
2. there exist constants \(A, B > 0\) such that

\[
A\|v\|^2 \leq \int_{\Omega} |\langle v, F(\omega) \rangle|^2 d\mu(\omega) \leq B\|v\|^2, \quad \text{for all } v \in V.
\]

We then have a map

\[
t : V \to L^2(\Omega),
\]

\[
v \mapsto (\omega \mapsto |\langle v, F(\omega) \rangle|).
\]

This gives us a map \(T : V/\mathbb{T} \to L^2(\Omega)\). We say that the frame \((V, \Omega, F)\) is phase retrievable if the map \(T\) is injective. In case \(V\) is a finite dimensional space and \((\Omega, \mu)\) is a finite set with counting measure, a number of developments enable us to construct phase retrievable frames with symmetries. In particular, Balan, Casazza, and Edidin obtained the following important characterization of phase retrievable finite frames [1, 2, 3]. Note that their results contain more information and here we only state the part in our setting.

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Theorem 1.1. Let $V$ be a complex Hilbert space with dimension $n$. Let $\{f_i\}_{i=1}^N$ be a frame for $V$. Then $\{f_i\}_{i=1}^N$ is phase retrievable if $\{f_i\}_{i=1}^N$ is generic and $N \geq 4n - 2$.

Now let $G$ be a group and $\pi : G \to U(V)$ be a (projective) representation of $G$ in the Hilbert space $V$. Take $v \in V$ a non-zero vector and obtain $\Phi_v = \{\pi(g)v \mid g \in G\} \subset V$. In case $\pi$ is irreducible, $\Phi_v$ is a frame for $V$. Using Theorem 1.1, Lawrence, Pfander, and Walnut [20] proved the following result.

Theorem 1.2. Let $p$ be a prime number. Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\pi : G \to U(V)$ be the projective representation of $G$ that gives us the Gabor frame. Hence $\dim_C V = p$. Then the set of $v \in V$ for which $\Phi_v$ is phase retrievable is dense in $V$.

On the other hand, using maximal spanning vectors and tools from representation theory, Li, Han, and etc. [21, 9] proved the following result, which generalizes Theorem 1.2.

Theorem 1.3. Let $G$ be any finite abelian group and $\pi : G \to U(V)$ be an irreducible projective representation of $G$. Then the set of $v \in V$ for which $\Phi_v$ is phase retrievable is dense in $V$.

The above results concern phase retrievability of finite frames. In [10], Cheng and Li proved a version of Theorem 1.3 for compact groups and explained the similarity between compact continuous frame case and finite Gabor frame case. In this paper, we study continuous frames with locally compact symmetries. The situation is different as the Hilbert space are now usually of infinite dimension and we need to check the convergence all the time. On one hand, as a complementary part of paper [10], we construct explicit examples of continuous frames with symmetries via projective representations of locally compact abelian groups. The tool we use here is the Stone-von Neumann theorem and its variation in terms of projective representations. On the other hand, for the vector space we constructed, we define the notion of a maximal spanning vector. Motivated by the results for finite abelian groups, we propose a conjecture on the existence of maximal spanning vectors and prove the conjecture for compactly generated locally Euclidean locally compact abelian groups, i.e. groups of the form $\mathbb{R}^a \times \mathbb{Z}^b \times \mathbb{T}^c \times F$, where $a, b, c$ are non-negative integers and $F$ is a finite abelian group. The tool we use here is Wiener’s theorem, which provides us a characterization of maximal spanning vectors.

More precisely, let $\hat{G}$ be a locally compact abelian group with a fixed Haar measure. Let $\hat{G}$ be the dual group of $G$, which is also considered as a locally compact group with Plancherel measure. For $(v^*, v) \in \hat{G} \times G$, define an operator $\pi(v^*, v)$ on $L^2(G)$ via

$$(\pi(v^*, v)f)(u) = v^*(u)f(uv),$$

where $f \in L^2(G)$. Then this $\pi$ defines an $\alpha$-representation $\pi : \hat{G} \times G \to U(L^2(G))$, where $\alpha$ is the 2-cocycle in $Z^2(\hat{G} \times G, \mathbb{T})$ given by

$$\alpha((v^*_1, v_1), (v^*_2, v_2)) = v^*_2(v_1).$$

The following result is a variation of the Stone-von Neumann theorem.

Theorem 1.4. The $\alpha$-representation $\pi$ is irreducible. Moreover, if $G$ is separable, then up to isomorphism, $\pi$ is the unique irreducible $\alpha$-representation of $\hat{G} \times G$ over a separable Hilbert space.
Take any non-zero vector $f \in L^2(G)$, because of the irreducibility of $\pi$, we have
\[
\overline{\text{Span}}\{\pi(v^*, v)f \mid (v^*, v) \in \hat{G} \times G\} = L^2(G).
\]
Moreover, we show that the vectors in $\{\pi(v^*, v)f \mid (v^*, v) \in \hat{G} \times G\}$ form a tight continuous frame with bound $||f||_2$ (Theorem 3.1). Therefore, this provides us explicit examples of locally compact continuous frames with symmetries.

For $f \in L^2(G)$, we construct a function $c_{f,f} : \hat{G} \times G \to \mathbb{C}$ via
\[
c_{f,f}(v^*, v) = \langle \pi(v^*, v)f, f \rangle.
\]
One shows that $c_{f,f} \in L^2(\hat{G} \times G)$ (Lemma 3.2). Moreover, we call $f$ a maximal spanning vector if
\[
\overline{\text{Span}}\{c_{\pi(v^*, v)f, \pi(v^*, v)f} \mid (v^*, v) \in \hat{G} \times G\} = L^2(\hat{G} \times G).
\]

The relation between maximal spanning vectors and phase retrieval problems is explained in Section 3.3.2. Motivated by the results for finite abelian groups (cf. [21, Conjecture], [9, Section 3], [10, Section 3.3]), we propose the following conjecture.

**Conjecture 1.5.** With the above notation, there exist maximal spanning vectors in $L^2(G)$.

Using Wiener’s theorem, we show that $f \in L^2(G)$ is a maximal spanning vector if and only if $\langle \pi(v^*, v)f, f \rangle \neq 0$ for almost all $(v^*, v) \in \hat{G} \times G$ (Proposition 3.5). Note that this is consistent with [9, Proposition 3.11]. As a consequence, by explicit construction, we prove the conjecture for some special groups.

**Theorem 1.6.** Conjecture 1.5 holds for $G = \mathbb{R}^a \times \mathbb{Z}^b \times \mathbb{T}^c \times F$, where $\mathbb{R}$ is the additive group of real numbers, $\mathbb{Z}$ is the additive group of integers, $\mathbb{T}$ is the multiplicative group of norm one numbers, $F$ is a finite abelian group, $a$, $b$, $c$ are non-negative integers.

**Notation:** In this paper, $G$ is always a locally compact group. Denote by $\hat{G}$ the dual group of $G$ if $G$ is an abelian group. Moreover, we fix a Haar measure on $G$ and equip $\hat{G}$ with the Plancherel measure. Denote by $\hat{f}$ the Fourier transform of $f$ (in some appropriate setting) if $f$ is a function.

In this paper, $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{T}$ is the set of complex numbers of modulus 1.

Let $X$ be a measure space. Denote by $L^2(X)$ the space of measurable functions on $X$ for which $\int_X |f(x)|^2 \, dx < \infty$. If $f \in L^2(X)$, define $||f||_2 = (\int_X |f(x)|^2 \, dx)^{1/2}$. Denote by $L^1(X)$ the space of measurable functions on $X$ for which $\int_X |f(x)| \, dx < \infty$. If $f \in L^1(X)$, define $||f||_1 = \int_X |f(x)| \, dx$. Given $f, f' \in L^2(X)$, define an inner product by
\[
\langle f, f' \rangle = \int_X f(x)\overline{f'(x)} \, dx.
\]
Denote by $C_c(X)$ the space of continuous functions on $X$ with compact support.

2. The Stone-von Neumann theorem

In this section, we prove some representation theoretic results concerning groups of the form $\hat{G} \times G$, where $G$ is a locally compact abelian group. Using the relation between projective representations of a group and linear representations of the associated representation group (cf. the paragraphs after Definition 2.1), it is easy to deduce Theorems 2.2.
and [2,4] from the classical results (cf. [7] Section 4.8 and [17] etc.). For completeness we give detail proofs in terms of projective representations in the following. The readers could skip the details and read only the remarks to understand the simplification in Section 3.

2.1. Projective representations of locally compact groups. In this section, we review some basic properties of projective representations of locally compact groups (cf. [16, 18, 19]). Let $G$ be a locally compact group. A multiplier (or 2-cocyle) on $G$ is a continuous function $\alpha : G \times G \to \mathbb{T}$, such that

1. $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for any $x, y, z \in G$;
2. $\alpha(x, 1) = \alpha(1, x) = 1$ for any $x \in G$.

Two multipliers $\alpha$ and $\beta$ are equivalent if there exists a continuous function $\gamma : G \to \mathbb{T}$ with

$$\alpha(x, y)\beta(x, y)^{-1} = \gamma(x)\gamma(y)\gamma(xy)^{-1}.$$ 

Denote by $Z^2(G, \mathbb{T})$ the set of multipliers on $G$.

**Definition 2.1.** Let $G$ be a locally compact group and $\alpha \in Z^2(G, \mathbb{T})$. A projective representation of $G$ with multiplier $\alpha$ (or an $\alpha$-representation of $G$) is a map $\pi : G \to \mathbb{U}(V)$, where $V$ is a Hilbert space, $\mathbb{U}(V)$ is the space of unitary operators on $V$, such that

1. the map $G \to V$ ($g \mapsto \pi(g)v$) is continuous for all $v \in V$;
2. $\pi(g)\pi(h) = \alpha(g, h)\pi(gh)$ for any $g, h \in G$.

There are several approaches to study projective representations of $G$. For later use in this paper, we review two approaches.

The first approach is to use the representation group and transfer a projective representation of $G$ into a linear representation of another group. Each multiplier $\alpha$ on $G$ defines an extension $G(\alpha)$ of $G$ by $\mathbb{T}$. As a set, $G(\alpha) = \mathbb{T} \times G$. The multiplication on $G(\alpha)$ is given by

$$(s, g)(t, h) = (sta(g, h), gh)$$
for any $s, t \in \mathbb{T}$ and $g, h \in G$.

We equip $G(\alpha)$ a topology in which a basis of neighbourhoods of the identity is composed of the sets $XX^{-1}$, where $X$ is a set of finite positive measure for the product of right Haar measure on $\mathbb{T}$ and $G$. The restriction of this measure on $\mathbb{T} \times \{1\}$ is the original measure on $\mathbb{T}$ via the obvious identification $\mathbb{T} = \mathbb{T} \times \{1\} \subset G(\alpha)$. This makes $G(\alpha)$ a locally compact group.

It is easy to check that $G(\alpha) \cong G(\beta)$ as locally compact groups if $\alpha$ and $\beta$ are equivalent multipliers. The map $\pi \mapsto \pi_\alpha$, where

$$\pi_\alpha : G(\alpha) = \mathbb{T} \times G \to \mathbb{U}(V)$$

$$\pi_\alpha(s, g) = s\pi(g),$$

is a bijection between the set of equivalent classes of $\alpha$-representations of $G$ and the set of equivalent classes of representations of $G(\alpha)$ with $\mathbb{T}$ acting as scalars. See [19, Corollary to Theorem 1].

The second approach is to transfer a projective representation of $G$ into a module over an algebra with “good” structure. More precisely, denote by $L^1(G, \alpha)$ the set of complex-valued integrable functions with multiplication (convolution) defined by

$$f_1 \ast f_2(x) = \int_G f_1(g)f_2(g^{-1}x)\alpha(g, g^{-1}x)\,dg$$

(2.1)
and a \(*\)-operator (involution) defined by
\[
\begin{equation}
 f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})\alpha(x,x^{-1})^{-1},
\end{equation}
\]
where \(\overline{\cdot}\) is the complex conjugation and \(\Delta\) is the modular function. Then \(L^1(G,\alpha)\) is a Banach \(*\)-algebra. The map \(\pi \mapsto \pi^*\), where
\[
\pi^* : L^1(G,\alpha) \to B(V)
\]
\[
f \mapsto \int_G \pi(g)f(g)\,dg,
\]
is a bijection between the set of equivalent classes of \(\alpha\)-representations of \(G\) and the set of equivalent classes of representations of the Banach \(*\)-algebra \(L^1(G,\alpha)\). See [13, Section 13.3.5].

2.2. The result of Segal, Shale and Weil.

2.2.1. The statement. Let \(G\) and \(G^*\) be locally compact abelian groups. Suppose we have a bi-homomorphism
\[
\beta : G^* \times G \to \mathbb{T}
\]
\[
(v^*,v) \mapsto v^*(v).
\]
Let \(H = G^* \times G\) and define
\[
\alpha : H \times H \to \mathbb{T}
\]
\[
((v^*_1,v_1),(v^*_2,v_2)) \mapsto v^*_2(v_1).
\]
Then \(\alpha\) is a 2-cocycle. Define \(\pi_\beta : H \to GL(L^2(G))\) by
\[
(\pi_\beta(v^*,v)f)(u) = v^*(u)f(uv).
\]
It is easy to check that \(\pi_\beta\) is unitary and \(\pi_\beta : H \to U(L^2(G))\) is an \(\alpha\)-representation of \(H\). Moreover, we have the following result.

**Theorem 2.2** (Segal, Shale, and Weil). If \(G^* = \hat{G}\) and \(\beta\) is the natural pairing, then the \(\alpha\)-representation \(\pi := \pi_\beta\) is irreducible, i.e., there is no non-trivial invariant closed subspace of \(L^2(G)\).

If we consider \(\pi\) as a representation of \(\mathbb{T} \times (\hat{G} \times G)\) as explained in Section 2.1, the theorem is part of a result of Segal, Shale, and Weil (cf. [7, Theorem 4.8.2]). For completeness and for later use, we provide a proof in terms of projective representations by adapting Weil’s argument.

2.2.2. \(L^2(\hat{G} \times G), L^2(G \times G), \) and \(L^2(G)\). In this section, we study properties of the spaces \(L^2(\hat{G} \times G), L^2(G \times G), \) and \(L^2(G)\).

Consider \(C_c(\hat{G} \times G)\), the space of compactly supported continuous functions on \(\hat{G} \times G\). Then \(C_c(\hat{G} \times G)\) is a sub-algebra of \(L^1(\hat{G} \times G,\alpha)\) (cf. Section 2.1). We may view \(L^2(G)\) as a \(C_c(\hat{G} \times G)\)-module via
\[
(2.3) \quad (\pi(\phi)\Phi)(u) = \int_{\hat{G} \times G} \phi(w)(\pi(w)\Phi)(u)\,dw,
\]
where \( u \in G, \phi \in C_c(\hat{G} \times G) \) and \( \Phi \in L^2(G) \). As [7] Page 526, define \( K_\phi : G \times G \to \mathbb{C} \) by
(2.4) \[
K_\phi(u, v) = \int_{\hat{G}} \phi(v^*, vu^{-1}) v^*(u) \, dv^*,
\]
then we have
(2.5) \[
(\pi(\phi)\Phi)(u) = \int_{G} K_\phi(u, v) \Phi(v) \, dv.
\]
The map \( \phi \mapsto K_\phi \) extends to an \( L^2 \)-isometry \( L^2(\hat{G} \times G) \to L^2(G \times G) \). Denote by \( \lambda \) the inverse of this isometry (cf. Section 3.3.2).

Moreover, if we define the multiplication on \( L^2(G \times G) \) by
(2.6) \[
(K_1 \times K_2)(u, v) = \int_{G} K_1(u, x)K_2(x, v) \, dx,
\]
for \( K_1, K_2 \in L^2(G \times G) \), then we have
(2.7) \[
K_{\phi_1 \ast \phi_2} = K_{\phi_1} \times K_{\phi_2}.
\]
Here \( \ast \) is the convolution defined by (2.1). Note that the convolution \( \ast \) is defined for the \( L^1 \)-space at first. But the above multiplication makes \( L^2(G \times G) \) into a Banach algebra without unit (cf. [7] Page 527) and we may extend \( \ast \) to \( L^2 \)-space using \( \lambda \). Moreover, \( L^2(G) \) is both a left and right module for \( L^2(G \times G) \) with module structure given by
(2.8) \[
(K \times P)(u) = \int_{G} K(u, v)P(v) \, dv, \quad (Q \times K)(v) = \int_{G} Q(u)K(u, v) \, du,
\]
for all \( K \in L^2(G \times G), P, Q \in L^2(G) \). See [7] Page 527 for more information on the space \( L^2(G \times G) \). In particular, the following identities are important for us:
(2.9) \[
(P \otimes Q) \times K = P \otimes (Q \times K), \quad K \times (P \otimes Q) = (K \times P) \otimes Q,
\]
and
(2.10) \[
(P \otimes Q) \times (S \otimes T) = (Q, S)P \otimes T.
\]
Here \( P, Q, S, T \in L^2(G), K \in L^2(G \times G), P \otimes Q \in L^2(G \times G) \) is defined by \( (P \otimes Q)(u, v) = P(u)Q(v) \).

2.2.3. Proof of Theorem 2.2. In order to show that \( \pi \) is irreducible, it suffices to show that any endomorphism \( T \in \text{End}(L^2(G)) \) commuting with \( \pi \) is a scalar. Indeed, \( T \) commutes with \( \pi(\phi) \) for any \( \phi \in C_c(\hat{G} \times G) \). In particular, take \( \phi = \lambda(P \otimes Q) \), where \( P, Q \in C_c(G) \). Then for \( \Phi \in L^2(G) \),
(2.11) \[
(\pi(\phi)\Phi)(u) = (\int \Phi(v)Q(v) \, dv)P(u),
\]
i.e., \( \pi(\phi)\Phi = \langle \Phi, Q \rangle P \). Therefore,
\[
\langle T\Phi, Q \rangle P = \pi(\phi)\langle T\Phi \rangle = T(\pi(\phi)\Phi) = \langle \Phi, Q \rangle (TP).
\]
Choose \( \Phi \) and \( Q \) so that \( \langle \Phi, Q \rangle \neq 0 \), then \( TP = \langle \Phi, Q \rangle^{-1}\langle T\Phi, Q \rangle P \). Hence \( T \) is a scalar and the theorem follows.
2.2.4. Some variations. The pairing \( \beta : G^* \times G \to \mathbb{T} \) induces a homomorphism \( \iota : G^* \to \hat{G} \) and the representation \( \pi_\beta : G^* \times G \to \mathbf{U}(L^2(G)) \) factors through

\[
G^* \times G \xrightarrow{\iota \times \text{id}} \hat{G} \times G \xrightarrow{\pi} \mathbf{U}(L^2(G)).
\]

In particular, if \( \iota : G^* \to G \) is surjective, then \( \pi_\beta \) is an irreducible \( \alpha \)-representation. On the other hand, if \( \iota \) is injective with nontrivial cokernel, then \( \pi_\beta \) is reducible in general as we shall see next.

2.2.5. The case \( \hat{G} \times G \) is essential. We continue the discussion in Section 2.2.4. Assume now that the pairing \( \beta \) induces an injection \( \iota : G^* \to \hat{G} \). Let \( K = \{ x \in G \mid v^*(k) = 1 \) for all \( v^* \in \text{Im}(\iota) \} \). Assume that \( \text{Im}(\iota) \) is an open subgroup of \( \hat{G} \) and \( K \) is compact. Then we have \( \text{Im}(\iota) = \hat{G}/K \). The natural map \( G \to G/K \) induces an injection \( \kappa : L^2(G/K) \to L^2(G) \). Denote by \( \tilde{l}^2(G/K) \) the image of \( \kappa \). Then

\[
\tilde{l}^2(G/K) = \{ f \in L^2(G) \mid f(gk) = f(g) \text{ for all } g \in G, k \in K \}.
\]

For any \((v^*, v) \in \text{Im}(\iota) \times G\), we have an operator \( \varpi(v^*, v) \) on \( L^2(G) \) via

\[
(\varpi(v^*, v)f)(g) = v^*(g) f(gv).
\]

If \( f \in \tilde{l}^2(G/K) \), it is clear that \( \varpi(v^*, v)f \in \tilde{l}^2(G/K) \). Moreover, for \( v \in K \), we have \( \varpi(1, v)f = f \). Therefore, \( \varpi \) induces a projective representation \( \varpi : \text{Im}(\iota) \times G/K \to \mathbf{U}(\tilde{l}^2(G/K)) \). This is nothing but the projective representation in Theorem 2.2 for the group \( G/K \). It is closely related to \((\pi, \tilde{l}^2(G)\)) as follows.

The projective representation \( \varpi : \text{Im}(\iota) \times G/K \to \mathbf{U}(\tilde{l}^2(G/K)) \) induces a projective representation \( \varpi' : \text{Im}(\iota) \times G \to \text{Im}(\iota) \times G/K \to \mathbf{U}(\tilde{l}^2(G/K)) \). Note that \( \text{Im}(\iota) \times G \) is a closed subgroup of \( \hat{G} \times G \) and \( \varpi' \) is an \( \alpha \)-representation. Take the induced representation with respect to \( \alpha \), we obtain \( \alpha \text{Ind}_{\text{Im}(\iota) \times G}(\varpi', \tilde{l}^2(G/K)) \), an \( \alpha \)-representation of \( \hat{G} \times G \).

(See [H] Section 3.4 for the construction of \( \alpha \text{Ind} \).) Then we have the following result.

**Proposition 2.3.** With the above notation, we have

\[
\alpha \text{Ind}_{\text{Im}(\iota) \times G}^{\hat{G} \times G}(\varpi', \tilde{l}^2(G/K)) \cong (\pi, \tilde{l}^2(G)).
\]

In particular, \( \pi_\beta : G^* \times G \to \mathbf{U}(L^2(G)) \) is reducible as \( \text{Re} \text{Ind}_{\text{Im}(\iota) \times G}^{\hat{G} \times G}(\varpi', \tilde{l}^2(G/K)) \) is reducible.

2.2.6. Relation with Fourier transform. By symmetry, we have an action of \( \hat{G} \times G \) on the space \( L^2(\hat{G}) \) via

\[
(\pi'(v^*, v)\Psi)(u^*) = u^*(v)\Psi(u^*v^*),
\]

where \((v^*, v) \in \hat{G} \times G\) and \( \Psi \in L^2(\hat{G}) \). The associated multiplier is \( \alpha' : (\hat{G} \times G) \times (\hat{G} \times G) \to \mathbb{T} \) given by

\[
((v_1^*, v_1), (v_2^*, v_2)) \mapsto v_1^*(v_2).
\]

As explained in Section 2.2.4, we obtain a projective representation of \( \hat{G} \times G \) by composition with \( \hat{G} \to \hat{G} \) \((v^* \mapsto (v^*)^{-1})\). Changing the resulting projective representation by the coboundary \( \beta \), we obtain an irreducible projective representation \( \rho : \hat{G} \times G \to \mathbf{U}(L^2(\hat{G})) \), where

\[
(\rho(v^*, v)\Psi)(u^*) = v^*(v)^{-1}u^*(v)^{-1}\Psi(u^*v^*).
\]
An easy computation shows that $\rho$ is an $\alpha$-representation. Note that we have an isometry between $L^2(G)$ and $L^2(\hat{G})$ given by the Fourier transform
\[
F : L^2(G) \to L^2(\hat{G})
\]
\[
\Phi \mapsto (\hat{\Phi} : u^* \mapsto \int_G \Phi(u) u^*(u) \, d\mu(u)).
\]

Then $\rho(v^*, v)\hat{\Phi} = (\pi(v^*, v)\Phi)\hat{\pi}$. In other words, $F \in \text{Hom}_{\text{Rep}_{\alpha}^G}(\pi, \rho)$ is an isomorphism of $\alpha$-representations.

### The uniqueness

From the results on finite group case (cf. [8] Section 2.3, [9] Section 3.2) and the $G = \mathbb{R}$ case (cf. [12] Theorem 10.2.1), we prove the following result.

**Theorem 2.4.** Assume that $G$ is separable, i.e., $L^2(G)$ admits an orthonormal basis with countable many elements. Then up to isomorphism, $\pi$ is the unique irreducible $\alpha$-representation of $\hat{G} \times G$ over a separable Hilbert space.

We start with some lemmas. Let $\rho : \hat{G} \times G \to U(W)$ be an irreducible $\alpha$-representation of $\hat{G} \times G$. As explained in Section 2.2.2, we have an action of $C_c(\hat{G} \times G)$ on $W$ given by
\[
(\rho(\phi))w = \int_{\hat{G} \times G} \phi(v^*, v) \rho(v^*, v) w \, dv(v^*, v).
\]

Hence we obtain a morphism $\rho : C_c(\hat{G} \times G) \to B(W)$.

**Lemma 2.5.** The morphism $\rho : C_c(\hat{G} \times G) \to B(W)$ is injective.

**Proof.** Assume that $\rho(\phi) = 0$ for $\phi \in C_c(\hat{G} \times G)$. Then for any $x, y \in W$, $(u^*, u) \in \hat{G} \times G$,
\[
0 = \langle \rho(u^*, u)\rho(\phi)\rho(u^*, u)^{-1}x, y \rangle
= \int_{\hat{G} \times G} \langle \rho(u^*, u)\phi(v^*, v)\rho(v^*, v)\rho(u^*, u)^{-1}x, y \rangle \, dv(v^*, v, y)
= \int_{\hat{G} \times G} \langle \rho(v^*, v)x, y \rangle \rho(v^*, v) \frac{v^*(u)}{u^*(v)} \, dv(v^*, v)
= \langle (\hat{\phi})\hat{\pi}(u, u^*) \rangle.
\]

Here $\phi(v^*, v) = \langle \rho(v^*, v)x, y \rangle \phi(v^*, v)$, the last identity is the Fourier transform for $\phi \in L^2(\hat{G} \times G)$. Therefore, we must have $\langle \rho(v^*, v)x, y \rangle \phi(v^*, v) = 0$ for any $x, y \in W$. This is true only if $\phi(v^*, v) = 0$ and the lemma follows. \hfill $\square$

From the construction in Section 2.2.2 we have
\[
\pi(\phi)\Phi = K_\phi \times \Phi,
\]
for $\phi \in C_c(\hat{G} \times G)$ and $\Phi \in L^2(G)$. Denote by $\phi_{P,Q} \in L^2(\hat{G} \times G)$ the element $\lambda(P \otimes Q)$. Then
\[
(\pi(\phi_{P,Q})f)(u) = \int_G K_{\phi_{P,Q}}(u, v) f(v) \, dv
= \int_G (P \otimes Q)(u, v) f(v) \, dv
= \langle f, Q \rangle P(u).
\]
Let $L : \hat{G} \times G \to \mathbf{U}(L^2(\hat{G} \times G))$ be the left regular $\alpha$-representation of $\hat{G} \times G$, i.e.,

\[(L(h)\phi)(h') = \frac{\alpha(h,h^{-1})}{\alpha(h^{-1},h')} \phi(h^{-1}h'), \tag{2.15}\]

for $\phi \in L^2(\hat{G} \times G)$ and $h, h' \in \hat{G} \times G$. Then we have the following lemma.

**Lemma 2.6.** With the above notation, we have

$\rho(L(h)\phi) = \rho(h)\rho(\phi)$

and

$L(h)\phi_{P,Q} = \phi_{\pi(h)P,Q}$.

**Proof.** This follows from direct computation. Let $w \in W$ be any vector, we have

\[
\begin{align*}
\rho(L(h)\phi)w &= \int_{H} (L(h)\phi)(h')\rho(h')w \, dh' \\
&= \int_{H} \frac{\alpha(h,h^{-1})}{\alpha(h^{-1},h')} \phi(h^{-1}h')\rho(h')w \, dh' \\
&= \int_{H} \rho(h)\phi(h^{-1}h')\rho(h^{-1}h')w \, dh' \\
&= \rho(h)\rho(\phi)w.
\end{align*}
\]

Here the third equality follows from

$\pi(h^{-1})\pi(h') = \alpha(h^{-1},h')\pi(h^{-1}h')$ and $\pi(h)\pi(h^{-1}) = \alpha(h,h^{-1}) \text{id}$.

The first identity follows. For the second identity, it suffices to show that

$\pi(L(h)\phi_{P,Q}) = \pi(\phi_{\pi(h)P,Q})$.

But this follows from the first identity (by taking $\rho = \pi$) and equation (2.14). \(\square\)

By our assumption on $G$, there exist functions $P_i \in C_c(G)$ ($i \in \mathbb{N}$) that form an orthonormal basis of $L^2(G)$. For simplicity, write $\phi_{i,j}$ for the function $\phi_{P_i}\phi_{P_j} = \lambda(P_i \otimes P_j)$. Then from equation (2.9) and identity $K_{\phi^*} = K_{\phi}$, we have

\[
\phi_{i,j} * \phi_{k,l} = \delta_{j,k}\phi_{i,l}, \tag{2.16}\]

and

\[
\phi_{i,j} = \phi_{j,i}^*. \tag{2.17}\]

Combining the above equations and Lemma 2.6, we have the following result.

**Lemma 2.7.** Let $\rho : \hat{G} \times G \to \mathbf{U}(W)$ be any irreducible $\alpha$-representation of $\hat{G} \times G$. Then $\rho(\phi_{i,i}) : W \to W$ are non-zero projections and they are pairwise orthonormal. Moreover, $\rho(\phi_{i,j})$ is an isometry from $V_j$ to $V_i$ and it annihilates every $V_k$ for $k \neq j$, where $V_i$ is the image of $\rho(\phi_{i,i})$.

**Proof of Theorem 2.4.** Take any $w_1 \in W$ with norm one. Define $w_j = \rho(\phi_{j,1})w_1$. Then $\{w_j \mid j \in \mathbb{N}\}$ is an orthonormal system in $W$. Define

\[
T : L^2(G) \to W \\
P_i \mapsto w_i. \tag{2.18}
\]
We show that $T$ is an isomorphism of $\alpha$-representations. Since $\{P_j \mid j \in \mathbb{N}\}$ is an orthonormal basis, we have

$$\pi(h)P_j = \sum_k \langle \pi(h)P_j, P_k \rangle P_k.$$ 

Therefore,

$$\rho(h)T(P_j) = \rho(h)w_j = \rho(h)\rho(\phi_{j,j})w_j = \rho(L(h)\phi_{j,j})w_j = \rho(\phi_{\pi(h)P_j,P_j})w_j \quad \text{(by Lemma 2.6)}$$

$$= \rho(\phi_{\sum_k \langle \pi(h)P_j, P_k \rangle P_k, P_j})w_j$$

$$= \sum_k \langle \pi(h)P_j, P_k \rangle \rho(\phi_{P_k, P_j})w_j$$

$$= \sum_k \langle \pi(h)P_j, P_k \rangle w_k$$

$$= T(\pi(h)P_j).$$

The theorem then follows as $\pi$ and $\rho$ are both irreducible $\alpha$-representations. □

Remark 2.8. Combining the discussion in Sections 2.2.5, 2.2.6, and Theorem 2.4, we basically obtained all the $\alpha$-representations for pairs of type $(G^* \times G, \alpha)$ with $\alpha$ coming from a bi-homomorphism $G^* \times G \rightarrow \mathbb{T}$. In particular, the standard representation $\pi : \hat{G} \times G \rightarrow \mathbb{U}(L^2(G))$ in Theorem 2.2 is the essential case and we will only consider this case in the rest of this paper.

3. Continuous $(\hat{G} \times G)$-frames and maximal spanning vectors in $L^2(G)$

In this section, $\pi : \hat{G} \times G \rightarrow \mathbb{U}(L^2(G))$ is the $\alpha$-representation as in Theorem 2.2. We study continuous $\hat{G} \times G$-frames of the Hilbert space $L^2(G)$ constructed from $\pi$ and prove the existence of maximal spanning vectors for compactly generated locally Euclidean locally compact abelian groups $G = \mathbb{R}^a \times \mathbb{T}^b \times \mathbb{Z}^c \times F$. We refer to the paper [15, 22] for basic properties of continuous frames.

3.1. The frame condition. Let $f \in L^2(G)$ be a non-zero element. Since $\pi$ is irreducible, we have

$$\overline{\text{Span}} \{ \pi(v^*, v)f \mid (v^*, v) \in \hat{G} \times G \} = L^2(G).$$

Even better, we have the following stronger result, which is a generalization of [11, Corollary 11.1.4].

**Theorem 3.1.** Let $f \in L^2(G)$ be non-zero. Then the set $\{ \pi(v^*, v)f \mid (v^*, v) \in \hat{G} \times G \}$ is a tight $(\hat{G} \times G)$-frame of $L^2(G)$ with bound $\|f\|_2^2$. 
The argument is adapted from the idea in [11, Section 11.1]. Let \( f_1, f_2, g_1, g_2 \) be elements in \( L^2(G) \). Then
\[
\left\langle \int_{\hat{G} \times G} (f_1, \pi(v^*, v)g_1)(f_2, \pi(v^*, v)g_2) \, d(v^*, v) \right\rangle = \int_{\hat{G} \times G} \int_G f_1(x)(\pi(v^*, v)g_1(x)) \, d x \int_G f_2(y)(\pi(v^*, v)g_2(y)) \, d y \, d(v^*, v).
\]

Let \( F_i(x) = f_i(x)\overline{g_i(xv)} \) for \( i = 1, 2 \). Then
\[
\left\langle \int_{\hat{G} \times G} (f_1, \pi(v^*, v)g_1)(f_2, \pi(v^*, v)g_2) \, d(v^*, v) \right\rangle = \int_{\hat{G} \times G} \hat{F}_1(v^*)\overline{\hat{F}_2(v^*)} \, d v^* \, d v
\]
\[
= \int_G \int_G F_1(x)\overline{F_2(x)} \, d x \, d v
\]
\[
= \int_G \int_G f_1(x)\overline{g_1(xv)}f_2(x)\overline{g_2(xv)} \, d x \, d v
\]
\[
= \langle f_1, f_2 \rangle \langle g_2, g_1 \rangle.
\]

Therefore, for any \( F \in L^2(G) \), we have
\[
\left\langle \int_{\hat{G} \times G} |\langle F, \pi(v^*, v)f \rangle|^2 \, d(v^*, v) \right\rangle = ||f||_2^2||F||_2^2.
\]

In particular, we see that \( \{\pi(v^*, v)f \mid (v^*, v) \in \hat{G} \times G\} \) is a tight \((\hat{G} \times G)\)-frame of \( L^2(G) \) with frame bounds \( A = B = ||f||_2^2 \).

### 3.2. Maximal spanning vectors

Let \( f, g \in L^2(G) \). Then we have a map \( c_{f,g} : \hat{G} \times G \to \mathbb{C} \) given by
\[
c_{f,g}(v^*, v) = \langle \pi(v^*, v)f, g \rangle.
\]
We call such a map a matrix coefficient of \( \pi \). In equation (3.2), take \( f_1 = f_2 = f \) and \( g_1 = g_2 = g \), we obtain the following result.

**Lemma 3.2.** Every matrix coefficient \( c_{f,g} \) is an element in \( L^2(\hat{G} \times G) \).

**Definition 3.3.** Let \( f, g \in L^2(G) \). We say that \( (f, g) \) is a maximal spanning pair if
\[
\text{Span}\{c_{\pi(v^*, v)f, \pi(v^*, v)g} \mid (v^*, v) \in \hat{G} \times G\} = L^2(\hat{G} \times G).
\]
We call \( f \in L^2(G) \) a maximal spanning vector if \( (f, f) \) is a maximal spanning pair.

Motivated by the results for finite groups (cf. [21, Conjecture], [9, Section 3], [10, Section 3.3]), we propose the following conjecture.

**Conjecture 3.4.** \( \pi : \hat{G} \times G \to U(L^2(G)) \) admits maximal spanning vectors.
In the following, we provide a characterization for maximal spanning vectors and prove the conjecture for groups $G = \mathbb{R}^a \times T^b \times \mathbb{Z}^c \times F$, where $F$ is a finite abelian group.

We have

$$c_{\pi(v^*,v)f,\pi(v^*,v)g}(u^*,u) = \langle \pi(u^*,u)\pi(v^*,v)f, \pi(v^*,v)g \rangle$$

$$= \int_G (\pi(u^*,u)\pi(v^*,v)f)(x)(\pi(v^*,v)g)(x) \, dx$$

$$= \int_G v^*(u)u^*v^*(x)f(xuv)v^*(x)g(xv) \, dx$$

$$= \frac{v^*(u)}{u^*(v)}c_{f,g}(u^*,u).$$

Note that the pairing $(\hat{G} \times G) \times (\hat{G} \times G) \to \mathbb{T}$ given by

$$((v^*,v),(u^*,u)) \mapsto \frac{v^*(u)}{u^*(v)}$$

induces an isomorphism $\hat{G} \times G \cong (\hat{G} \times G)^\sim$. Hence

$$C_{f,g} = \overline{\text{Span}}\{c_{\pi(v^*,v)f,\pi(v^*,v)g} \mid (v^*,v) \in \hat{G} \times G\}$$

$$= \overline{\text{Span}}\{\chi_{c_{f,g}} \mid \chi \in (\hat{G} \times G)^\sim \}.$$

Then we have the following proposition, which generalizes the results for finite abelian groups (cf. [6, 21]).

Proposition 3.5. With the notation as above, $(f,g)$ is a maximal spanning pair for $(\pi, L^2(G))$ if and only if $c_{f,g}(u^*,u) \neq 0$ for almost all $(u^*,u) \in \hat{G} \times G$.

In particular, $f$ is a maximal spanning vector for $(\pi, L^2(G))$ if and only if $c_{f,f}(u^*,u) \neq 0$ for almost all $(u^*,u) \in \hat{G} \times G$.

Proof. The Fourier transform for $\hat{G} \times G$ is an isometry between $L^2(\hat{G} \times G) \to L^2((\hat{G} \times G)^\sim)$. Therefore

$$C_{f,g} = L^2(\hat{G} \times G)$$

$$\iff \overline{\text{Span}}\{(\chi_{c_{f,g}})^\sim \mid \chi \in (\hat{G} \times G)^\sim \} = L^2((\hat{G} \times G)^\sim)$$

$$\iff c_{f,g}(u^*,u) \neq 0 \text{ for almost all } (u^*,u) \in \hat{G} \times G.$$

Note that $(\chi_{c_{f,g}})^\sim$ is the translation of $(c_{f,g})$ by $\chi$, the last equivalence follows from Wiener’s theorem (see for example [14, Proposition 4.72]). The proposition follows.

Corollary 3.6. If there exists one maximal spanning vector $f$ for $(\pi, L^2(G))$, then up to scalars, there are infinitely many maximal spanning vectors in $L^2(G)$.

Proof. If $G$ is finite, then the set of maximal spanning vectors is dense in $L^2(G)$ by [9, Theorem 1.7, Lemma 2.2]. If $G$ is infinite, then each $\pi(u^*,u)f$ is a maximal spanning vector and $\overline{\text{Span}}\{\pi(u^*,u)f \mid (u^*,u) \in \hat{G} \times G\} = L^2(G)$. The claim then follows.

As an application of the characterization, we prove the following theorem.
Theorem 3.7. Conjecture 3.4 holds for groups of type \( G = \mathbb{R}^a \times \mathbb{T}^b \times \mathbb{Z}^c \times F \), where \( F \) is a finite abelian group.

Proof. Note that if the conjecture holds for \( G_1 \) and \( G_2 \), then it holds for \( G_1 \times G_2 \) as well. Indeed, it is easy to check that if \( f_i \in L^2(G_i) \) \((i = 1, 2)\), then \( f_1 \otimes f_2 \in L^2(G_1 \times G_2) \) is a maximal spanning vector for \( L^2(G_1 \times G_2) \), where \((f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)\). Moreover, if the conjecture holds for \( \mathbb{Z} \), then it holds for \( \mathbb{T} \) as well by Section 2.2.6. Since the conjecture holds for finite abelian groups by [9 Proposition 3.11], we only need to prove the theorem for \( G = \mathbb{R}, \mathbb{Z} \). This follows from the following two lemmas.

Lemma 3.8. Let \( G = \mathbb{Z} \). Let \( f \in L^2(\mathbb{Z}) \) given by \( f(n) = e^{-|n|} \). Then \( f \) is a maximal spanning vector for \( (\pi, L^2(\mathbb{Z})) \).

Proof. We need to show that \( c_{f,f}(\theta, N) \neq 0 \) for almost all \((\theta, N) \in \mathbb{T} \times \mathbb{Z}\). By definition, we have

\[
c_{f,f}(\theta, N) = (\pi(\theta, N)f, f) = \sum_{n \in \mathbb{Z}} (\pi(\theta, N)f)(n)\bar{f}(n) = \sum_{n \in \mathbb{Z}} \theta^n f(n + N)f(n) = \sum_{n \in \mathbb{Z}} \theta^n e^{-|n+N|}e^{-|n|}.
\]

(3.5)

We divide the discussion into the following cases.

1. If \( \theta = 1 \), then \( c_{f,f}(1, N) = \sum_{n \in \mathbb{Z}} e^{-|n+N|}e^{-|n|} \neq 0 \).

2. If \( \theta \neq 1 \) and \( N = 0 \), then

\[
c_{f,f}(\theta, 0) = \sum_{n \in \mathbb{Z}} \theta^n e^{-2|n|} = \sum_{n \geq 0} \theta^n e^{-2n} + \sum_{n \geq 0} \theta^{-n} e^{-2n} - 1 = \frac{1}{1 - \theta e^{-2}} + \frac{1}{1 - \theta^{-1}e^{-2}} - 1 = \frac{1 - e^{-4}}{(1 - \theta e^{-2})(1 - \theta^{-1}e^{-2})} \neq 0.
\]

3. If \( \theta \neq 1 \) and \( N \geq 1 \), then

\[
c_{f,f}(\theta, N) = \sum_{n \in \mathbb{Z}} \theta^n e^{-|n+N|}e^{-|n|}
\]

\[
= \sum_{n \geq 0} \theta^n e^{-(n+N)}e^{-n} + \sum_{n \leq -N} \theta^n e^{n+N}e^{n} + \sum_{n = -N+1}^{\infty} \theta^{n-n}e^{-(n+N)}e^{n}
\]

\[
= e^{-N} \sum_{n \geq 0} \theta^n e^{-2n} + e^N \sum_{n \geq N} \theta^{-n} e^{-2n} + e^{-N} \sum_{n = 1}^{N-1} \theta^{-n}
\]

\[
= e^{-N} \left( \frac{1}{1 - \theta e^{-2}} + \frac{\theta^{-N}}{1 - \theta^{-1}e^{-2}} + \frac{\theta^{-1} - \theta^{-N}}{1 - \theta^{-1}} \right)
\]

Then \( c_{f,f}(\theta, N) = 0 \) if and only if \( \frac{1}{1 - \theta e^{-2}} + \frac{\theta^{-N}}{1 - \theta^{-1}e^{-2}} + \frac{\theta^{-1} - \theta^{-N}}{1 - \theta^{-1}} = 0 \). Multiplying both sides with \((1 - \theta e^{-2})(1 - \theta^{-1}e^{-2})(1 - \theta^{-1})\), direct computation shows that
\[ c_{f,f}(\theta, N) = 0 \] if and only if
\[ \frac{1 - \theta e^{-2}}{1 - \theta^{-1}e^{-2}} = \theta^{N+1}. \]  

For each \( N \geq 1 \), there are at most \( N \) solutions to the equation (3.6).

(4) If \( \theta \neq 1 \) and \( N \leq -1 \), then from the identity
\[ c_{f,f}(\theta, -N) = \theta^N c_{f,f}(\theta, N), \]
we also have for each \( N \leq -1 \), there are at most \(|N|\) solutions with \( c_{f,f}(\theta, N) = 0 \).

From the above discussion, the set \( \{(\theta, N) \in \mathbb{T} \times \mathbb{Z} \mid c_{f,f}(\theta, N) = 0\} \) is countable. The lemma then follows.

\[ \square \]

Lemma 3.9. Let \( G = \mathbb{R} \). Let \( f \in L^2(\mathbb{R}) \) given by \( f(x) = e^{-x^2} \). Then \( f \) is a maximal spanning vector for \((\pi, L^2(\mathbb{R}))\).

Proof. We show that \( c_{f,f}(a, b) \neq 0 \) for all \((a, b) \in \mathbb{R} \times \mathbb{R}\). We have the well-known identity (see for example [23, Page 42])
\[ \int_{\mathbb{R}} e^{-2\pi i \xi x} e^{-\pi x^2} \, dx = e^{-\pi \xi^2}. \]  

Therefore,
\[ c_{f,f}(a, b) = \langle \pi(a, b) f, f \rangle \]
\[ = \int_{\mathbb{R}} e^{2\pi i a x} e^{-(x+b)^2} - x^2 \, dx \]
\[ = e^{-\frac{1}{2}b^2} \int_{\mathbb{R}} e^{2\pi i a x} e^{-\pi(\sqrt{x^2} + \sqrt{1/2b})^2} \, dx \]
\[ = e^{-\frac{1}{2}b^2} \frac{\pi}{2} e^{-\pi i a b} \int_{\mathbb{R}} e^{-2\pi i y(-a \sqrt{x^2} + \sqrt{1/2b})} e^{-\pi y^2} \, dy \quad (\text{here } y = \sqrt{\frac{2}{\pi} x + \sqrt{\frac{1}{2\pi} b}}) \]
\[ = e^{-\frac{1}{2}b^2} \frac{\pi}{2} e^{-\pi i a b} e^{-\frac{1}{2}a^2 \pi^2} \neq 0. \]

The lemma follows. \( \square \)

3.3. Some remarks on maximal spanning vectors.

3.3.1. The Bessel property. Suppose that \((f, g) \in L^2(G)\) is a maximal spanning pair. We then have
\[ \overline{\text{Span}}\{c_{\pi(v^*, v)f, \pi(v^*, v)g} \mid (v^*, v) \in \hat{G} \times G\} = L^2(\hat{G} \times G). \]

We may ask the following question: do the vectors in the set \( \{c_{\pi(v^*, v)f, \pi(v^*, v)g} \mid (v^*, v) \in \hat{G} \times G\} \) form a continuous \((\hat{G} \times G)\)-frame for \( L^2(\hat{G} \times G)\)? For \( G \) finite, the answer is certainly yes. In general this is not true. For simplicity, we write \( c \) for the function \( c_{f,g} \) and \( c_{v^*, v} \) for the function \( c_{\pi(v^*, v)f, \pi(v^*, v)g} \). For any \( \psi \in L^2(\hat{G} \times G) \), we have
\[ \int_{\hat{G} \times G} |\langle \psi, c_{v^*, v} \rangle|^2 \, dv^* \, v \]
\[ = \int_{\hat{G} \times G} \langle \psi, c_{v^*, v} \rangle \langle \psi, c_{v^*, v} \rangle \, dv^* \, v \]
\[ = \int_{\hat{G} \times G} \langle \psi, c_{v^*, v} \rangle \langle \psi, c_{v^*, v} \rangle \, dv^* \, v \]
Phase retrieval.

3.3.2. This gives another proof for the \( \psi \leq \) U \( \subset \) 

Remark

Then \( T(3.8) \) 

The vector \( f \in L^2(G) \) is injective. For any \( x \in L^2(G) \), define a rank one projection \( x \otimes x : L^2(G) \rightarrow L^2(G) \) 

\( F \mapsto \langle F, x \rangle x \)

It is clear that \( L^2(G) \rightarrow \) End\( (L^2(G)) \) \( (x \mapsto x \otimes x) \) is injective. We view \( x \otimes x \) as an element in \( L^2(\hat{G} \times G) \) via the following isometries

\( \text{End}(L^2(G)) \rightarrow L^2(G) \otimes L^2(G) \overset{\sigma}{\rightarrow} L^2(G \times G) \overset{\lambda}{\rightarrow} L^2(\hat{G} \times G) \).
Here $\sigma$ is defined by $\sigma(P \otimes Q)(u,v) = \overline{P(u)Q(v)}$, $\lambda$ is the morphism in Section 2.2.2 and is defined by

\[(3.10) \quad \phi \in L^2(G \times G) \mapsto ((v^*, v) \mapsto \int_u \phi(u, uv)v^*(u) \, du).\]

Direct computation shows that the image of $x \otimes x$ in $L^2(\hat{G} \times G)$ is nothing but the function $c_{x,x} : (v^*, v) \mapsto \langle \pi(v^*, v)x, x \rangle$. Moreover, from equation (3.2), we have

\[\langle c_{x,x}, c_{\pi(v^*, v)}f, \pi(v^*, v)f \rangle = |\langle x, \pi(v^*, v)f \rangle|^2.\]

Therefore we obtain the following result.

**Proposition 3.11.** The frame $\{\pi(u^*, u)f \mid (u^*, u) \in \hat{G} \times G\} \subset L^2(G)$ is phase retrievable if $f \in L^2(G)$ is a maximal spanning vector.

**REFERENCES**


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