PHASE RETRIEVABLE VECTORS AND MAXIMAL SPANNING VECTORS

Abstract. Let $\pi : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbf{U}(\mathbb{C}^n)$ be the Weyl-Heisenberg representation of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. In this short note, we show that, if $n \geq 5$, there exist infinitely many $v \in \mathbb{C}^n$ such that the frame $\{\pi(g)v \mid g \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\}$ is phase retrievable but does not have maximal span. This answers a question proposed in [6, Problem C].

1. Introduction

Being phase retrievable is an important property of frames. In general it is not easy to determine directly whether a frame is phase retrievable. On the other hand, a frame is phase retrievable if it has maximal span (cf. [2, Section 3]), and this stronger condition is relatively easy to check. In [6], Li-Han-etc. proved the existence of phase retrievable vectors for irreducible projective representations of finite abelian groups by showing the existence of maximal spanning vectors. Then they asked the following: for group frames generated by one vector in the representation space, is it possible that it is phase retrievable but it is not maximal spanning (cf. [6, Problem C]). In this short note, we give an affirmative answer to that question using Fourier transform for projective representations of finite groups. More precisely, let n be a positive integer, $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\pi : G \to \mathbf{U}(\mathbb{C}^n)$ be the Weyl-Heisenberg representation of G. We show that, if $n \geq 5$, there exist infinitely many $v \in \mathbb{C}^n$ such that the frame $\{\pi(g)v \mid g \in G\}$ for $\mathbb{C}^n$ is phase retrievable but does not have maximal span.

Let us first review the basics of frames and [6, Problem C]. Let $n,\ m$ be positive integers and V be an n-dimensional Hilbert space over $\mathbb{C}$. Let $\Phi = \{\phi_i \in V \mid 1 \leq i \leq m\}$ be a frame for V. Recall the following definitions.

(1) The frame Φ is *phase retrievable* if the map

$$\begin{aligned} t_\Phi : V/\mathbb{T} &\to \mathbb{R}_{\geq 0}^{\oplus m}, \\ f &\mapsto (|\langle f, \phi_i \rangle|)_{1 \leq i \leq m} \end{aligned}$$

is injective. Here $\mathbb{T}$ is the set of complex numbers with absolute value one.

(2) The frame Φ is *maximal spanning* (or has *maximal span*) if

$$\mathrm{Span}\{\phi_i \otimes \phi_i \mid 1 \leq i \leq m\} = \mathrm{HS}(V).$$

Here $\mathrm{HS}(V)$ is the space of Hilbert-Schmidt operators on V, and $x \otimes x$ ($x \in V$) is the projection given by

$$\begin{aligned} x \otimes x : V &\to \mathbb{C}x \subset V, \\ f &\mapsto \langle f, x \rangle x. \end{aligned}$$

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It is well known that if Φ is maximal spanning, then Φ is phase retrievable (cf. [2, Section 3]).

Let G be a finite group and $\alpha \in Z^2(G, \mathbb{T})$ be a multiplier of G. Let $\pi : G \to \mathbf{U}(V)$ be an α-representation of G (i.e. $\pi(g)\pi(h) = \alpha(g,h)\pi(gh)$ for all $g,\ h \in G$). An element $v \in V$ is called a *frame vector* if $\Phi_v := \{\pi(g)v \mid g \in G\}$ is a frame for V. In particular, when π is irreducible, all nonzero vectors $v \in V$ are frame vectors, which is the case for the Weyl-Heisenberg representations. We have the following definitions.

(1) An element $v \in V$ is *phase retrievable* for (π, G, V) if v is a frame vector and the frame Φ_v is phase retrievable.
(2) An element $v \in V$ is *maximal spanning* for (π, G, V) if v is a frame vector and the frame Φ_v is maximal spanning.

In [6, Problem C], the authors asked the following question: does there exist a phase retrievable vector $v \in V$ for (π, G, V) which is not maximal spanning? We show that this question has an affirmative answer and prove the following result.

Theorem 1.1. *Let $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\pi : G \to \mathbf{U}(\mathbb{C}^n)$ be the Weyl-Heisenberg representation of G, i.e. $\pi(a,b) = T^a S^b$, where*

$$T = \mathrm{diag}(1, \xi, \xi^2, \cdots, \xi^{n-1}), \quad S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and $\xi = e^{\frac{2\pi i}{n}}$. The corresponding multiplier $\alpha : G \times G \to \mathbb{T}$ is $\alpha((a,b),(c,d)) = \xi^{bc}$. Then the following statements hold.

(1) *If $n = 2, 3$, then $v \in \mathbb{C}^n$ is phase retrievable for $(\pi, G, \mathbb{C}^n)$ if and only if v is maximal spanning.*
(2) *If $n = 4$, then $v \in \mathbb{C}^n$ is phase retrievable for $(\pi, G, \mathbb{C}^n)$ if and only if v is maximal spanning or the zero set of the matrix coefficient $c_{v,v}$ has the form $\{h\}$ for some $h \in G$ of order 2, where $c_{v,v} : G \to \mathbb{C}$ is defined as $c_{v,v}(g) := \langle \pi(g)v, v\rangle$.*
(3) *If $n \geq 5$, there exist infinitely many $v \in \mathbb{C}^n$ such that v is phase retrievable for $(\pi, G, \mathbb{C}^n)$ but v is not maximal spanning.*

The main tool we use is the Fourier transform for projective representations of finite groups, which in the setting of Theorem 1.1 gives an isometric isomorphism between $L^2(G)$ and $M_n(\mathbb{C})$. We shall see in the following that the method works for projective representations of general finite abelian groups.

2. Projective representations of finite abelian groups

In this section, we review the Fourier transform for projective representations of finite groups. As an application, for G abelian and (π, G, V) an irreducible projective representation of G, we give a practical criterion to check whether $v \in V$ is phase retrievable (cf. Corollary 2.4). This criterion is the key ingredient in the proof of Theorem 1.1.

2.1. **The Fourier transform.** Let G be a finite group and $\alpha \in Z^2(G,\mathbb{T})$ be a multiplier. Denote by $\widehat{G}_\alpha$ the set of isomorphism classes of finite-dimensional irreducible α-representations of G. Let (π, V_π, d_π) be a representative of an element in $\widehat{G}_\alpha$ where $d_\pi = \dim V_\pi$, and denote by $[\pi]$ the corresponding isomorphism class.

Fixing $[\pi] \in \widehat{G}_\alpha$ and an orthonormal basis $\{e_i^\pi \mid 1 \le i \le d_\pi\}$ of V_π, then $\mathrm{End}(V_\pi)$ is isomorphic to the vector space $M_{d_\pi}(\mathbb{C})$ of $d_\pi \times d_\pi$ matrices with complex entries. The *Fourier transform with respect to* π is the linear map

$$F_\pi : L^2(G) \to \mathrm{End}(V_\pi) \cong M_{d_\pi}(\mathbb{C})$$
$$f \mapsto \widehat{f}_\pi := \frac{1}{|G|}\sum_{g\in G} f(g)\pi(g)^*,$$

where $\pi(g)^*$ means the conjugate transpose of the matrix $\pi(g)$.

Define $F : L^2(G) \to \oplus_{[\pi]\in\widehat{G}_\alpha} M_{d_\pi}(\mathbb{C})$ by $F := \oplus_{[\pi]\in\widehat{G}_\alpha} F_\pi$. Note that both sides of this map are Hilbert spaces. For the left hand side, the inner product on $L^2(G)$ is defined by

$$\langle f,h\rangle := \frac{1}{|G|}\sum_{g\in G} f(g)\overline{h(g)}, \quad f,g \in L^2(G).$$

While for the right hand side, the inner product on $\oplus_{[\pi]\in\widehat{G}_\alpha} M_{d_\pi}(\mathbb{C})$ is the sum of inner products on all direct summands, where on $M_{d_\pi}(\mathbb{C})$ the inner product is defined by

$$\langle A,B\rangle = d_\pi \,\mathrm{Tr}(AB^*), \quad A,B \in M_{d_\pi}(\mathbb{C}).$$

Then the map F is an isometric isomorphism (cf. [4, Theorem 2] for the Peter-Weyl theorem for compact groups and [5, Theorem 7.1] for the Plancherel formula for locally compact groups).

Matrix coefficients are the elements in $L^2(G)$ which we are particularly interested in. Let $[\pi] \in \widehat{G}_\alpha$. For $u, v \in V_\pi$, define the *matrix coefficient* $c^\pi_{u,v} : G \to \mathbb{C}$ by

$$c^\pi_{u,v}(h) := \langle \pi(h)u, v\rangle,$$

and define the subspace $C^\pi_{u,v}$ of $L^2(G)$ as

$$C^\pi_{u,v} := \mathrm{Span}\{c^\pi_{\pi(g)u,\pi(g)v} \mid g \in G\}.$$

It is easy to see that $F_\pi(c^\pi_{u,v}) = u \otimes v$. From this we obtain the following result.

Lemma 2.1. *For $[\pi] \in \widehat{G}_\alpha$ and $v \in V_\pi$, the following statements are equivalent.*

1. *v is a maximal spanning vector for (π, G, V_π).*
2. $\dim \mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = d_\pi^2$.
3. $\dim \mathrm{Span}\{c^\pi_{\pi(g)v,\pi(g)v} \mid g \in G\} = d_\pi^2$.

2.2. **Phase retrievable vectors and maximal spanning vectors: abelian case.** In this part we assume that G is a finite abelian group and $\pi : G \to \mathbf{U}(V)$ is an irreducible α-representation of G. Let $\lambda : G \times G \to \mathbb{T}$ be the map defined by

$$\lambda(x,y) := \frac{\alpha(y,x)}{\alpha(x,y)}.$$

Then λ is a bicharacter and it induces a homomorphism $\lambda_\alpha : G \to \widehat{G}$ by

$$\lambda_\alpha(x)(y) := \lambda(x,y).$$

Let S_α be the kernel of λ_α. We call α *totally skew* if S_α is trivial. By Theorem [1, Theorem 3.1] (or [4, Remark 8]), α is similar to a multiplier which is lifted from a totally skew multiplier α' on G/S_α, i.e. α is similar to the composition of the quotient map $G \times G \to G/S_\alpha \times G/S_\alpha$ and $\alpha' : G/S_\alpha \times G/S_\alpha \to \mathbb{T}$. Moreover, π is equivalent to a projective representation of the form $\gamma \otimes \pi_1$, where $\gamma \in \widehat{G}$ is a linear character of G, and π_1 is the projective representation of G induced from an α'-representation π' of G/S_α via the natural quotient map $G \to G/S_\alpha$.

In this note the properties we study for π and those for π' determine each other. Hence we may and do assume that α *is totally skew in the following*. In this case $\lambda_\alpha : G \to \widehat{G}$ is an isomorphism and there is a unique α-representation of G up to isomorphism (cf. [1, Theorem 3.3] for the locally compact abelian groups case and [3, Section 3.2] for the finite abelian groups case). In particular, this is the case for the Weyl-Heisenberg representations.

Let $\pi : G \to \mathbf{U}(V)$ be the unique α-representation of G. Since there will be no confusion on the representation here, we omit the symbol π in the super and subscript for simplicity. In this case the Fourier transform $F : L^2(G) \to \mathrm{End}(V)$ is an isometric isomorphism and $(\dim V)^2 = |G|$. Note that for any $u,\ v \in V,\ g,\ h \in G$,

$$\begin{aligned} c_{\pi(g)u,\pi(g)v}(h) &= \langle \pi(h)(\pi(g)u), \pi(g)v\rangle \\ &= \alpha(g^{-1}, g)^{-1}\alpha(h, g)\alpha(g^{-1}, hg)c_{u,v}(g^{-1}hg) \\ &= \lambda_\alpha(g)(h)c_{u,v}(h). \end{aligned}$$

Hence we have $C_{u,v} = \mathrm{Span}\{\lambda_\alpha(g)c_{u,v} \mid g \in G\} = \mathrm{Span}\{\chi c_{u,v} \mid \chi \in \widehat{G}\}$.

Lemma 2.2. *Let $v \in V$ be a nontrivial vector and denote by $N(v) \subset G$ the zero set of the matrix coefficient $c_{v,v}$, i.e.*

$$N(v) := \{g \in G \mid c_{v,v}(g) = 0\}.$$

Then

$$\dim C_{v,v} + |N(v)| = |G|.$$

In particular, v is maximal spanning if and only if $N(v)$ is the empty set.

Proof. Let δ_g be the characteristic function of the set $\{g\}$. Then the lemma follows from the following identity:

$$\begin{aligned} C_{v,v} &= \mathrm{Span}\{\chi c_{v,v} \mid \chi \in \widehat{G}\} \\ &= \mathrm{Span}\{\delta_g c_{v,v} \mid g \in G\} \\ &= \mathrm{Span}\{\delta_g \mid g \notin N(v)\}. \end{aligned}$$

□

This result has interesting consequences.

Corollary 2.3. *Let $v \in V$ be a nontrivial vector. Then the orthogonal complement of* $\mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\}$ *is spanned by $\pi(g)$ for $g \in N(v)$.*

Proof. From the proof of Lemma 2.2, one sees that the orthogonal complement of $C_{v,v}$ in $L^2(G)$ is spanned by δ_g for $g \in N(v)$. Note that g and g^{-1} lie in $N(v)$ simultaneously. Then the result follows from the fact that $F(\delta_g) = \frac{1}{|G|}\pi(g)^{-1}$. □

Corollary 2.4. *For a nontrivial vector $v \in V$, the following statements are equivalent.*

(1) *v is not phase retrievable.*

(2) *There exist $x, y \in V$ and for all $g \in N(v)$, a complex number $a_g \in \mathbb{C}$, such that*

$$x \otimes x - y \otimes y = \sum_{g \in N(v)} a_g \pi(g) \neq 0.$$

Proof. By definition, v is not phase retrievable if and only if there exist $x, y \in \mathbb{C}^n$, $x \not\equiv y \pmod{\mathbb{T}}$ such that

$$|\langle x, \pi(g)v\rangle| = |\langle y, \pi(g)v\rangle| \text{ for all } g \in G.$$

Hence for all $g \in G$ we have

$$\begin{aligned}\langle x \otimes x, \pi(g)v \otimes \pi(g)v\rangle &= |\langle x, \pi(g)v\rangle|^2 \\ &= |\langle y, \pi(g)v\rangle|^2 = \langle y \otimes y, \pi(g)v \otimes \pi(g)v\rangle.\end{aligned}$$

This means that $x \otimes x - y \otimes y$ is orthogonal to $\mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\}$. By Corollary 2.3, this is equivalent to the existence of $a_g \in \mathbb{C}$ for all $g \in N(v)$ such that

$$x \otimes x - y \otimes y = \sum_{g \in N(v)} a_g \pi(g).$$

Note that $x \not\equiv y \pmod{\mathbb{T}}$ is equivalent to $x \otimes x - y \otimes y \neq 0$, this completes the proof. $\square$

Remark 2.5. Note that $\mathrm{rank}(x \otimes x - y \otimes y) \leq 2$, if one could find v such that $N(v)$ is not empty and the linear combinations $\sum_{g \in N(v)} a_g \pi(g)$ are either zero or of rank greater or equal to 3, then v is phase retrievable but not maximal spanning. This is the key observation behind the proof of Theorem 1.1.

3. Proof of Theorem 1.1

From now on, we go back to the notation as in Theorem 1.1, i.e. $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, and $\pi : G \to \mathbf{U}(\mathbb{C}^n)$ is the Weyl-Heisenberg representation of G.

3.1. **Case $n = 2$.** Although one could deduce the $n = 2$ case by similar arguments to those in the following sections, we give an explicit argument by direct computation. It also shows that for $n \geq 3$, direct computation is not applicable since we are dealing with determinants of matrices of size $n^2 \times n^2$.

Let $v = (x\ y)' \in \mathbb{C}^2$ be a nonzero vector. Then $\{\pi(g)v \otimes \pi(g)v \mid g \in G\} \subseteq \mathrm{HS}(\mathbb{C}^2)$ is the set of the following four projections

$$\begin{pmatrix} x\bar{x} & x\bar{y} \\ \bar{x}y & y\bar{y} \end{pmatrix}, \begin{pmatrix} x\bar{x} & -x\bar{y} \\ -\bar{x}y & y\bar{y} \end{pmatrix}, \begin{pmatrix} y\bar{y} & \bar{x}y \\ x\bar{y} & x\bar{x} \end{pmatrix}, \begin{pmatrix} y\bar{y} & -\bar{x}y \\ -x\bar{y} & x\bar{x} \end{pmatrix}.$$

The span of these projections has dimension 4 if and only if the following determinant Δ

$$\Delta = \begin{vmatrix} x\bar{x} & x\bar{y} & \bar{x}y & y\bar{y} \\ x\bar{x} & -x\bar{y} & -\bar{x}y & y\bar{y} \\ y\bar{y} & \bar{x}y & x\bar{y} & x\bar{x} \\ y\bar{y} & -\bar{x}y & -x\bar{y} & x\bar{x} \end{vmatrix}$$

is nonzero. Direct computation shows that

$$\Delta = 4(|x|^4 - |y|^4)(x\bar{y} + \bar{x}y)(x\bar{y} - \bar{x}y).$$

Hence $(x\ y)' \in \mathbb{C}^2$ is a maximal spanning vector for $(\pi, \mathbb{C}^2)$ if and only if

$$(|x| - |y|)(x\bar{y} + \bar{x}y)(x\bar{y} - \bar{x}y) \neq 0.$$

On the other side, let $t : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}^{\oplus 4}$ be the map defined by

$$f \mapsto (|\langle f, \pi(g)v\rangle|)_{g\in G}.$$

It is easy to check that the following three claims hold.

(1) If $|x| = |y|$, then $t(a\ 0)' = t(0\ a)'$ for all nonzero complex numbers $a \in \mathbb{C}$;
(2) If $x\bar{y} - \bar{x}y = 0$, then $t(1\ \mathrm{i})' = t(1\ \ -\mathrm{i})'$;
(3) If $x\bar{y} + \bar{x}y = 0$, then $t(1\ 1)' = t(1\ \ -1)'$.

Hence $(x\ y)'$ is not phase retrievable when $(|x| - |y|)(x\bar{y} - \bar{x}y)(x\bar{y} + \bar{x}y) = 0$. Therefore we obtain the following result, which covers Theorem 1.1 for case $n = 2$.

Proposition 3.1. *With the notation as above, the following three conditions are equivalent.*

(1) $(x\ y)' \in \mathbb{C}^2$ *is maximal spanning;*
(2) $(x\ y)' \in \mathbb{C}^2$ *is phase retrievable;*
(3) $(|x| - |y|)(x\bar{y} - \bar{x}y)(x\bar{y} + \bar{x}y) \neq 0.$

3.2. **Case** $n = 3, 4$**.** On one hand, by Lemma 2.2, a nontrivial vector $v \in V$ is not maximal spanning if and only if $N(v)$ is non-empty. On the other hand, by Corollary 2.4, a nontrivial vector $v \in V$ is not phase retrievable if and only if there exist $x, y \in V$ and for all $g \in N(v)$, a complex number $a_g \in \mathbb{C}$, such that

$$x \otimes x - y \otimes y = \sum_{g\in N(v)} a_g \pi(g) \neq 0.$$

Using the idea in Remark 2.5, we prove the following result, which covers Theorem 1.1 for case $n = 3, 4$.

Proposition 3.2. *Let $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\pi : G \to \mathbf{U}(\mathbb{C}^n)$ be the Weyl-Heisenberg representation of G. Then the following statements hold.*

(1) *If $n = 3$, then $v \in \mathbb{C}^n$ is phase retrievable for $(\pi, G, \mathbb{C}^n)$ if and only if v is maximal spanning.*
(2) *If $n = 4$, then $v \in \mathbb{C}^n$ is phase retrievable for $(\pi, G, \mathbb{C}^n)$ if and only if v is maximal spanning or $N(v) = \{h\}$ for some $h \in G$ of order 2. In other words, $v \in \mathbb{C}^n$ is phase retrievable if and only if $\dim C_{v,v} \geq 15$.*

Proof. For $n = 3$, we only need to show that if $N(v)$ is not empty, then v is not phase retrievable. There are four cases to consider, listed as follows.

(1) Suppose $N(v) \supseteq \{(1, 0), (2, 0)\}$. Then there exist

$$x = \begin{pmatrix} 0 \\ 3^{\frac{1}{4}} \\ 0 \end{pmatrix},\ y = \begin{pmatrix} 0 \\ 0 \\ 3^{\frac{1}{4}} \end{pmatrix}$$

such that $x \otimes x - y \otimes y = -\mathrm{i}T + \mathrm{i}T^2 \neq 0$. Hence v is not phase retrievable.

(2) Suppose $N(v) \supseteq \{(0,1),(0,2)\}$. Then there exist

$$x = \sqrt{\frac{e^{\frac{\pi i}{6}} + e^{-\frac{\pi i}{6}}}{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ y = \sqrt{-\frac{e^{\frac{5\pi i}{6}} + e^{-\frac{5\pi i}{6}}}{3}} \begin{pmatrix} 1 \\ e^{\frac{4\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{pmatrix}$$

such that $x \otimes x - y \otimes y = e^{-\frac{\pi i}{6}} S + e^{\frac{\pi i}{6}} S^2 \neq 0$. Hence v is not phase retrievable.

(3) Suppose $N(v) \supseteq \{(1,1),(2,2)\}$. Then there exist

$$x = \sqrt{\frac{e^{\frac{\pi i}{6}} + e^{-\frac{\pi i}{6}}}{3}} \begin{pmatrix} e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \\ 1 \end{pmatrix}, \ y = \sqrt{-\frac{e^{\frac{5\pi i}{6}} + e^{-\frac{5\pi i}{6}}}{3}} \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{pmatrix}$$

such that $x \otimes x - y \otimes y = e^{\frac{\pi i}{6}} TS + e^{\frac{\pi i}{2}} T^2 S^2 \neq 0$. Hence v is not phase retrievable.

(4) Suppose $N(v) \supseteq \{(2,1),(1,2)\}$. Then there exist

$$x = \sqrt{\frac{e^{\frac{\pi i}{6}} + e^{-\frac{\pi i}{6}}}{3}} \begin{pmatrix} e^{\frac{4\pi i}{3}} \\ e^{\frac{4\pi i}{3}} \\ 1 \end{pmatrix}, \ y = \sqrt{-\frac{e^{\frac{5\pi i}{6}} + e^{-\frac{5\pi i}{6}}}{3}} \begin{pmatrix} e^{\frac{4\pi i}{3}} \\ 1 \\ e^{\frac{4\pi i}{3}} \end{pmatrix}$$

such that $x \otimes x - y \otimes y = e^{\frac{\pi i}{6}} T^2 S + e^{-\frac{5\pi i}{6}} TS^2 \neq 0$. Hence v is not phase retrievable.

The list completes the proof for $n = 3$.

For $n = 4$, we need to prove the following two statements.

- If $N(v)$ is not of the form $\{h\}$ where $h \in G$ has order 2 and v is not maximal spanning, then v is not phase retrievable.
- If $N(v) = \{h\}$ for some $h \in G$ of order 2, then v is phase retrievable.

For the first statement, we do the same as in case $n = 3$. Suppose that $v \in \mathbb{C}^n$ is not maximal spanning. Then $N(v)$ is non-empty, and there are nine cases to consider, listed as follows.

(1) Suppose $N(v) \supseteq \{(1,0),(3,0)\}$. Then there exist

$$x = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ y = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

such that $x \otimes x - y \otimes y = T + T^3 \neq 0$. Hence v is not phase retrievable.

(2) Suppose $N(v) \supseteq \{(0,1),(0,3)\}$. Then there exist

$$x = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ y = \frac{\sqrt{2}}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

such that $x \otimes x - y \otimes y = S + S^3 \neq 0$. Hence v is not phase retrievable.

(3) Suppose $N(v) \supseteq \{(1,1),(3,3)\}$. Then there exist

$$x = \begin{pmatrix} e^{-\frac{\pi i}{4}} \\ -i \\ -e^{-\frac{\pi i}{4}} \\ -i \end{pmatrix}, \quad y = \begin{pmatrix} e^{-\frac{\pi i}{4}} \\ i \\ -e^{-\frac{\pi i}{4}} \\ i \end{pmatrix}$$

such that $x \otimes x - y \otimes y = 2e^{\frac{\pi i}{4}}(TS + T^3S^3) \neq 0$. Hence v is not phase retrievable.

(4) Suppose $N(v) \supseteq \{(2,1),(2,3)\}$. Then there exist

$$x = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad y = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

such that $x \otimes x - y \otimes y = -T^2S + T^2S^3 \neq 0$. Hence v is not phase retrievable.

(5) Suppose $N(v) \supseteq \{(3,1),(1,3)\}$. Then there exist

$$x = \begin{pmatrix} e^{\frac{\pi i}{4}} \\ i \\ -e^{\frac{\pi i}{4}} \\ i \end{pmatrix}, \quad y = \begin{pmatrix} e^{\frac{\pi i}{4}} \\ -i \\ -e^{\frac{\pi i}{4}} \\ -i \end{pmatrix}$$

such that $x \otimes x - y \otimes y = 2e^{-\frac{\pi i}{4}}(T^3S + TS^3) \neq 0$. Hence v is not phase retrievable.

(6) Suppose $N(v) \supseteq \{(1,2),(3,2)\}$. Then there exist

$$x = \begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ i \\ 0 \\ -1 \end{pmatrix}$$

such that $x \otimes x - y \otimes y = TS^2 - T^3S^2 \neq 0$. Hence v is not phase retrievable.

(7) Suppose $N(v) \supseteq \{(2,0),(0,2)\}$. Then there exist

$$x = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

such that $x \otimes x - y \otimes y = \frac{1}{2}(T^2 - S^2) \neq 0$. Hence v is not phase retrievable.

(8) Suppose $N(v) \supseteq \{(2,0),(2,2)\}$. Then there exist

$$x = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

such that $x \otimes x - y \otimes y = \frac{1}{2}(T^2 + T^2S^2) \neq 0$. Hence v is not phase retrievable.

(9) Suppose $N(v) \supseteq \{(0,2),(2,2)\}$. Then there exist

$$x = \begin{pmatrix} 0 \\ \mathrm{i} \\ 0 \\ \mathrm{i} \end{pmatrix}, \; y = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

such that $x \otimes x - y \otimes y = S^2 - T^2 S^2 \neq 0$. Hence v is not phase retrievable.

Therefore, whatever the case, v is not phase retrievable, and this proves the first statement for $n = 4$.

For the second statement, we prove by negation. Suppose that v is not phase retrievable. Then by Corollary 2.4, there exist $x, y \in \mathbb{C}^n$ and $a \in \mathbb{C}$ such that

$$x \otimes x - y \otimes y = a\pi(h) \neq 0.$$

However, $x \otimes x - y \otimes y$ has rank at most 2, while $a\pi(h)$ must be invertible if nonzero, which yields a contradiction. □

Remark 3.3. For the case $n = 4$, let

$$v = \begin{pmatrix} 1+2\mathrm{i} \\ -\frac{1}{3} \\ 1 \\ 3 \end{pmatrix}.$$

Then one easily calculates that $N(v) = \{(0,2)\}$. Therefore, the situation that $N(v) = \{h\}$ for some $h \in G$ of order 2 may necessarily occur. This is an explicit example of a phase retrievable vector that is not maximal spanning.

3.3. **Case** $n \geq 5$**.** We begin with the following lemma, which constructs a vector v that is not maximal spanning, but $C_{v,v}$ still has relatively large dimension.

Lemma 3.4. *Let $n \geq 5$, $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\pi : G \to \mathbf{U}(\mathbb{C}^n)$ be the Weyl-Heisenberg representation of G. Then for any $a_0 \in \mathbb{Z}/n\mathbb{Z}$ with $a_0 \neq 0$, there exist infinitely many $v \in \mathbb{C}^n$ such that*

$$N(v) = \{h, h^{-1}\}$$

where $h = (a_0, 0) \in G$. It is allowed that $h = h^{-1}$ in the case that n is even.

Proof. This is done by direct construction. Indeed, we may furthermore require that $v = (v_0, v_1, \cdots, v_{n-1})' \in \mathbb{C}^n$ satisfies an additional condition: $v_j \in \mathbb{R}_{>0} \cap \overline{\mathbb{Q}}$ for $j = 1, 2, \cdots, n-1$. Thus, we write $v_0 = r_0 \mathrm{e}^{\mathrm{i}\theta}$, $v_j = r_j$, $j = 1, 2, \cdots, n-1$ where $r_j \in \mathbb{R}_{>0} \cap \overline{\mathbb{Q}}$ for $j = 0, 1, \cdots, n-1$ and $0 \leq \theta < 2\pi$. Let $k = \lfloor \frac{n}{2} \rfloor$ and we may assume $0 < a_0 \leq k$. It suffices to show that there exist infinitely many such $v \in \mathbb{C}^n$ that the following two statements hold simultaneously:

(1) $c_{v,v}(g) = 0$ for $g = (a_0, 0), (n - a_0, 0)$ and $c_{v,v}(g) \neq 0$ for $g = (a, 0)$, $a \neq a_0, n - a_0$;
(2) $c_{v,v}(g) \neq 0$ for $g = (a, b) \in G$ where $b \neq 0 \in \mathbb{Z}/n\mathbb{Z}$.

For the first statement, one only needs to check for $0 < a \leq k$ by symmetry. Note that for $g = (a, 0)$,

$$\begin{aligned} c_{v,v}(g) &= v_0\overline{v_0} + \xi^a v_1 \overline{v_1} + \cdots + \xi^{(n-1)a} v_{n-1}\overline{v_{n-1}} \\ &= r_0^2 + \xi^a r_1^2 + \cdots + \xi^{(n-1)a} r_{n-1}^2 \end{aligned} \tag{3.1}$$

has nothing to do with the variable θ. Set

$$r_{n-1} = r_1,\ r_{n-2} = r_2,\ \cdots,\ r_{k+1} = \begin{cases} r_k, & \text{if } n \text{ is odd}, \\ r_{k-1}, & \text{if } n \text{ is even}. \end{cases}$$

Then the imaginary part of (3.1) vanishes for all $0 < a \leq k$. Therefore, the first statement is reduced to the verification for the real part. More explicitly, we claim that there exist infinitely many $(r_0, r_1, \cdots, r_k) \in \mathbb{R}^{k+1}$ such that the formulas

$$r_0^2 + (2\cos\frac{2\pi}{n}a)r_1^2 + \cdots + (2\cos\frac{2(k-1)\pi}{n}a)r_{k-1}^2 + (\lambda\cos\frac{2k\pi}{n}a)r_k^2,\ 0 < a \leq k \tag{3.2}$$

where

$$\lambda = \begin{cases} 2, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}, \end{cases}$$

equal to zero exactly when $a = a_0$.

Indeed, for $0 < a \leq k$, let

$$f_a(r_1, \cdots, r_k) = -(2\cos\frac{2\pi}{n}a)r_1^2 - \cdots - (2\cos\frac{2(k-1)\pi}{n}a)r_{k-1}^2 - (\lambda\cos\frac{2k\pi}{n}a)r_k^2.$$

These are distinct polynomials and these quadratic forms are not semi-negative definite. Therefore, there exists a non-empty open subset $U \subseteq \mathbb{R}^k_{>0}$ such that

$$f_a(r_1, \cdots, r_k) \neq f_{a_0}(r_1, \cdots, r_k) > 0$$

for all $a \neq a_0$ and $(r_1, r_2, \cdots, r_k) \in U$. For any $(r_1, r_2, \cdots, r_k) \in U$, set

$$r_0 = \sqrt{f_{a_0}(r_1, \cdots, r_k)},$$

then one easily verifies that these values of $r_0, r_1, \cdots, r_k$ satisfy the required property.

For the second statement, fix any choice of $r_0, r_1, \cdots, r_{n-1}$ as above. We may furthermore require that these r_i's are algebraic. Let $K = \mathbb{Q}(r_0, r_1, \cdots, r_k, \xi)$. Then there exist infinitely many prime numbers p such that $K \cap \mathbb{Q}(\xi_p) = \mathbb{Q}$, and so $[K(\xi_p) : K] = [\mathbb{Q}(\xi_p) : \mathbb{Q}] = p - 1$. For any such p with $p \geq 5$, set $\theta = \frac{2\pi}{p}$, then for $g = (a, b)$, $g \neq 0$, one has

$$\begin{aligned} c_{v,v}(g) &= v_b\overline{v_0} + \xi^a v_{b+1}\overline{v_1} + \cdots + \xi^{(n-1-b)a} v_{n-1}\overline{v_{n-1-b}} \\ &\quad \xi^{(n-b)a} v_0\overline{v_{n-b}} + \xi^{(n-b+1)a} v_1\overline{v_{n-b+1}} + \cdots + \xi^{(n-1)a} v_{b-1}\overline{v_{n-1}} \\ &= r_b r_0 \overline{\xi_p} + \xi^{(n-b)a} r_0 r_{n-b}\xi_p + r \end{aligned} \tag{3.3}$$

where $r \in K$. Noting that $\overline{\xi_p} = \xi_p^{p-1}$ and $p \geq 5$, equation (3.3) cannot be zero, since the minimal polynomial for ξ_p is $X^{p-1} + X^{p-2} + \cdots + X + 1$. This finishes the proof. $\square$

By Lemma 2.2, the vectors $v \in \mathbb{C}^n$ given by Lemma 3.4 are not maximal spanning. We claim that they are phase retrievable (cf. Remark 2.5), hence obtain the following proposition and finish the proof of Theorem 1.1.

Proposition 3.5. *Let $n \geq 5$, $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\pi : G \to \mathbf{U}(\mathbb{C}^n)$ be the Weyl-Heisenberg representation of G. Then there exist infinitely many $v \in \mathbb{C}^n$ such that v is phase retrievable for $(\pi, G, \mathbb{C}^n)$ but v is not maximal spanning.*

Proof. It suffices to show that the vectors $v \in \mathbb{C}^n$ given by Lemma 3.4 are phase retrievable. Using similar argument as in case $n = 4$, we prove by negation.

Keep the notation in Lemma 3.4. Suppose that there exists such a $v \in \mathbb{C}^n$ that is not phase retrievable. Then by Corollary 2.4, there exist $x, y \in \mathbb{C}^n$ and $a, b \in \mathbb{C}$ such that

$$x \otimes x - y \otimes y = a\pi(h) + b\pi(h)^{-1} \neq 0.$$

Note that $x \otimes x - y \otimes y$ has rank at most 2, while $a\pi(h) + b\pi(h)^{-1}$ is of the form

$$D = \mathrm{diag}(a + b,\ a\xi^{a_0} + b\xi^{-a_0}, a\xi^{2a_0} + b\xi^{-2a_0},\ \cdots,\ a\xi^{(n-1)a_0} + b\xi^{-(n-1)a_0}). \tag{3.4}$$

which is required to be non-zero. If $h = h^{-1}$, then D must be invertible, which yields a contradiction. Suppose that h has order at least 3. We claim that when a and b vary and are not vanishing simultaneously, at least three of the entries of the matrix D are non-zero. Hence, $a\pi(h) + b\pi(h)^{-1}$ must be of rank at least 3, which also yields a contradiction.

Indeed, for any $i \neq j$, consider the linear equations with respect to a and b

$$\begin{cases} a\xi^{a_0 i} + b\xi^{-a_0 i} = 0, \\ a\xi^{a_0 j} + b\xi^{-a_0 j} = 0. \end{cases}$$

It has non-trivial solutions if and only if the discriminant

$$\Delta = \begin{vmatrix} \xi^{a_0 i} & \xi^{-a_0 i} \\ \xi^{a_0 j} & \xi^{-a_0 j} \end{vmatrix} = \xi^{a_0(i-j)} - \xi^{a_0(j-i)}$$

is zero, or equivalently,

$$n \mid 2a_0(i - j). \tag{3.5}$$

However, the assumption that h has order at least 3 implies that $n \nmid 2a_0$. Hence, fixing arbitrary i, there are at most $\lfloor \frac{n}{2} \rfloor$ integer j's with $0 \leq j \leq n - 1$ satisfying equation (3.5). Therefore, when a and b vary and are not vanishing simultaneously, at least $n - \lfloor \frac{n}{2} \rfloor$ entries of the matrix D are non-zero. Since $n - \lfloor \frac{n}{2} \rfloor \geq 3$ when $n \geq 5$, this proves the claim and the proposition. □

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