THREE MACWILLIAMS TYPE IDENTITIES AND QUANTUM ERROR-CORRECTING G-CODES

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Abstract. In this paper we prove three MacWilliams type identities for irreducible projective representations of finite groups. As an application, we deduce MacWilliams identities of weight enumerators, double weight enumerators and complete weight enumerators for quantum error-correcting G-codes and obtain the Singleton bounds.

1. Introduction

In this paper, we prove three MacWilliams type identities for irreducible projective representations of finite groups via a detailed study of the matrix coefficients. Besides the interests in representation theory, we apply the identities in the study of quantum error-correcting codes (abbreviated as QECCs) and deduce MacWillams identities of weight enumerators, double weight enumerators and complete weight enumerators for quantum error-correcting G-codes. These identities unifies earlier results of this type (cf. [3, 4, 15]). Moreover we obtain the Singleton bounds for quantum error-correcting G-codes.

Let G be a finite group of order $\mathfrak{g}$. Let m and n be positive integers and let (ρ_i, V_i) $(1 \leq i \leq n)$ be m-dimensional irreducible projective representations of G with multiplier α_i. Let $P_1, P_2 \in \mathrm{End}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$. For each $1 \leq i \leq n$, fix $G_i \in G$ a system of representatives of the quotient group $G/\mathrm{Ker}\rho_i$ such that the identity $1_G \in G$ is in G_i. Define $S_n = \{\rho_1(g_1)\otimes\rho_2(g_2)\otimes\cdots\otimes\rho_n(g_n) | g_i \in G_i, 1 \leq i \leq n\}$. An element $E \in S_n$ has *weight* t if $E = e_1 \otimes e_2 \otimes \cdots \otimes e_n$ and $|\{j : \overline{e}_j \neq id\}| = t$. Here for an element $e \in \mathbf{U}(V)$, we denote by $\overline{e}$ the image of e in $\mathbf{PU}(V)$ under the natural projection $\mathbf{U}(V) \to \mathbf{PU}(V)$. We denote the weight of E by $w(E)$.

Definition 1.1. The *weight distributions* B_i and $B_i^{\perp}$ with respect to $\{(\rho_i, V_i)_{1\leq i\leq n},\ P_1,\ P_2\}$ are defined by

$$B_i := \sum_{E\in S_n, w(E)=i} \mathrm{Tr}(E^{-1}P_1)\mathrm{Tr}(EP_2),$$
$$B_i^{\perp} := \sum_{E\in S_n, w(E)=i} \mathrm{Tr}(E^{-1}P_1EP_2),$$

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and the *weight enumerators* with respect to $\{(\rho_i, V_i)_{1\leq i\leq n},\ P_1,\ P_2\}$ are defined by

$$f(x,y) := \sum_{i=0}^{n} B_i x^{n-i} y^i,$$

$$f^{\perp}(x,y) := \sum_{i=0}^{n} B_i^{\perp} x^{n-i} y^i.$$

We then have the following result.

Theorem 1.2. *With the notation as above, we have*

$$f(x,y) = (\prod_{i=1}^{n} s_i)\cdot f^{\perp}(\frac{m^2x + (\mathfrak{g}^2 - m^2)y}{m\mathfrak{g}}, \frac{m(x-y))}{\mathfrak{g}}), \tag{1.1}$$

where $s_i = \#\mathrm{Ker}\rho_i, 1 \leq i \leq n$.

Assume now that the subgroups $\mathrm{Ker}\rho_i$ have the same size, say s. Fix a bijection o_i from G_i to the subset $\{1, 2, ..., \frac{\mathfrak{g}}{s}\}$ of $\mathbb{Z}$ such that the identity 1_G corresponds to the integer 1. Let $\rho_{1i} := o_i^{-1} \circ o_1$ be the induced bijection from G_1 to G_i and denote its inverse by ρ_{i1}. For simplicity, we write o for o_1. Let $\mathrm{IND}(n)$ be the set $\{J = (j_i) \in \mathbb{Z}_{\geq 0}^{\frac{\mathfrak{g}}{s}} | \sum_{i=1}^{\frac{\mathfrak{g}}{s}} j_i = n\}$. For $E = \rho_1(g_1) \otimes \rho_2(g_2) \otimes \cdots \otimes \rho_n(g_n) \in S_n$ and $g \in G_1$, let $N_g(E)$ be the number $\#\{i \mid 1 \leq i \leq n,\ g_i = \rho_{1i}(g)\}$. We define an error set $E[J]$ associated to an index matrix $J = (j_{o(g)}) \in \mathrm{IND}(n)$ by

$$E[J] := \{E \in S_n \mid N_g(E) = j_{o(g)}, \forall g \in G_1\}.$$

Definition 1.3. The *complete weight distributions* with respect to $\{(\rho_i, V_i)_{1\leq i\leq n},\ P_1,\ P_2\}$ are defined by

$$D_J := \sum_{E\in S_n,\ E\in E[J]} \mathrm{Tr}(E^{-1}P_1)\mathrm{Tr}(EP_2),$$

$$D_J^{\perp} := \sum_{E\in S_n,\ E\in E[J]} \mathrm{Tr}(E^{-1}P_1EP_2),$$

and the *complete weight enumerators* with respect to $\{(\rho_i, V_i)_{1\leq i\leq n},\ P_1,\ P_2\}$ are defined by

$$D(M) := \sum_{J=(j_o(g))\in \mathrm{IND}(n)} D_J M^J,$$

$$D^{\perp}(M) := \sum_{J=(j_o(g))\in \mathrm{IND}(n)} D_J^{\perp} M^J,$$

where $M = (M_g)_{g\in G_1}$ is a 1-by-$\frac{\mathfrak{g}}{s}$ matrix and $M^J = \prod_{g\in G_1} M_g^{j_{o(g)}}$.

We then have the following result.

Theorem 1.4. *With the notation as above, let G be an abelian group, then*

$$D(M) = D^{\perp}(M^{\perp}) \tag{1.2}$$

where $M_g^{\perp} = \dfrac{sm}{\mathfrak{g}} \sum_{l\in G_1} \alpha^{-1}(l^{-1}, g^{-1})\alpha(g^{-1}, l^{-1})M_l$ *for all* $g \in G_1$ *and* $M^{\perp} = (M_g^{\perp})$.

Finally we consider a particular case where all the ρ_i are the same and given by the Weyl-Heisenberg representation. More precisely, let $(H, +, 0_H)$ be an abelian group with order m and $\hat{H} = \mathrm{Hom}(\mathrm{H}, \mathbb{C}^\times)$ be its dual group with identity. Fix a basis $\{x_h | h \in H\}$ of $\mathbb{C}^m$ indexed by elements of H. Let ρ be the Weyl-Heisenberg representation of $H \times \hat{H}$ defined by

$$\begin{aligned} \rho : H \times \hat{H} &\to \mathbf{U}(\mathbb{C}^m) \\ (a, \chi) &\to (x_h \to \chi(h) x_{a+h}, \forall h \in H). \end{aligned} \tag{1.3}$$

It is well-known that $(\rho, \mathbb{C}^m)$ is a unitary irreducible faithful projective representation of $H \times \hat{H}$ (cf. [1, Exercise 4.1.8, Theorem 4.8.2]). In the following we consider the case that $G = H \times \hat{H}$ and $(\rho_i, V_i) = (\rho, \mathbb{C}^m)$ for all $1 \leq i \leq n$. For $E \in S_n$, let $w_X(E) = \sum_{\substack{(a,\chi)\in G \\ a \neq 0_H}} N_{(a,\chi)}(E)$ and $w_Z(E) = \sum_{\substack{(a,\chi)\in G \\ \chi \neq 1_{\hat{H}}}} N_{(a,\chi)}(E)$, and we call them X *weight* and Z *weight* of E respectively. Let $E[i,j] = \{E \in S_n | w_X(E) = i, w_Z(E) = j\}$.

Definition 1.5. The *double weight distributions* with respect to $\{\rho,\ P_1,\ P_2\}$ are defined by

$$\begin{aligned} C_{i,j} &:= \sum_{E \in E[i,j]} \mathrm{Tr}(E^{-1} P_1) \mathrm{Tr}(E P_2), \\ C_{i,j}^{\perp} &:= \sum_{E \in E[i,j]} \mathrm{Tr}(E^{-1} P_1 E P_2), \end{aligned}$$

and the *double weight enumerators* with respect to $\{\rho,\ P_1,\ P_2\}$ are defined by

$$\begin{aligned} C(X,Y,Z,W) &:= \sum_{i,j=0}^{n} C_{i,j} X^{n-i} Y^i Z^{n-j} W^j, \\ C^{\perp}(X,Y,Z,W) &:= \sum_{i,j=0}^{n} C_{i,j}^{\perp} X^{n-i} Y^i Z^{n-j} W^j. \end{aligned}$$

We then have the following result.

Theorem 1.6. *With the notation as above, we have*

$$C(X,Y,Z,W) = C^{\perp}(X + (m-1)Y, X - Y, \frac{Z + (m-1)W}{m}, \frac{Z - W}{m}).$$

In Section 2.1, we review basic properties of projective representations and prove necessary identities of matrix coefficients. We then prove Theorems 1.2, 1.4 and 1.6 in Sections 2.2, 2.3 and 2.4 respectively.

In section 3, we apply Theorems 1.2, 1.4 and 1.6 in the study of quantum error correcting G-codes. In particular, we deduce three versions of MacWilliams identities and obtain the Singleton bounds for quantum error correcting G-codes. As we shall see, the Singleton bounds depend only on the size of G. On the other hand, as explained in [8], it is meaningful to construct G-codes for various groups G.

1.1. **Notation.** In this paper, $\mathbb{T}$ is the set of complex numbers with modulus one. For a Hilbert space V, denote by $\mathbf{U}(V)$ the space of unitary operators on V, $\mathbf{PU}(V)$ the quotient of $\mathbf{U}(V)$ by $\mathbb{T}$. Given $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, $E_{i,j} \in \mathbb{C}^{m\times m}$ is the matrix whose ij entry is 1 and other entries are zero.

2. The MacWilliams identities for projective representations

In this section, we prove Theorems 1.2, 1.4 and 1.6. The main ingredients are some identities of matrix coefficients of projective representations of finite groups, which we review/ prove in Section 2.1 via Schur's lemma.

2.1. Projective representaitons of finite groups.

We recall the basic properties of projective representations of finite groups (cf. [2, 5]). Let G be a finite group with identity 1_G and let V be a finite dimensional $\mathbb{C}$-vector space.

Definition 2.1. Let $\alpha : G \times G \to \mathbb{C}^\times$ be a multiplier in $Z^2(G, \mathbb{T})$. A map

$$\rho : G \to \mathbf{U}(V)$$

is called a *projective representation of* G *with respect to* α (or an α*-representation*) if $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for all $x, y \in G$.

We denote this projective representation by (π, V, α) or (π, V). Let $\mathrm{PGL}(V) = \mathrm{GL}(V)/\mathbb{C}^\times$. Let $\pi : \mathbf{U}(V) \to \mathbf{PU}(V)$ be the natural homomorphism. If $\rho : G \to \mathbf{U}(V)$ is a projective representation of G, then $\pi \circ \rho$ is a homomorphism. We define the kernel of ρ by

$$\mathrm{Ker}\rho := \{g \in G | \rho(g) \in \mathbb{C}^\times \cdot 1_V\} = \mathrm{Ker}(\pi \circ \rho), \tag{2.1}$$

If $\mathrm{Ker}\rho$ is 1_G then we call ρ a *faithful projective representation* (cf. [5, Chapter 3]).

Definition 2.2. A *subprojective representation* of a projective representation (π, V) is a vector subspace W of V which is stable under G, i.e $\pi(g)W \subset W$ for all $g \in G$. A projective representation is called *irreducible* if there is no proper nonzero G-stable subspace W of V.

As in linear representations case, we have Schur's lemma for projective representations (cf. [2, Lemma 2.1] and [14, Section 2.2]).

Proposition 2.3 (Schur's lemma)**.** *Let* $\rho^1 : G \to \mathrm{GL}(V_1)$ *and* $\rho^2 : G \to \mathrm{GL}(V_2)$ *be two irreducible* α*-representations of* G*, and let* f *be a linear map from* V_1 *to* V_2 *such that* $\rho^1(g) \circ f = f \circ \rho^1(g)$ *for all* $g \in G$*. Then the following statements hold.*

(1) If ρ^1 *and* ρ^2 *are not isomorphic, then* $f = 0$*.*

(2) If $V_1 = V_2$ *and* $\rho^1 = \rho^2$*, then* f *is a homothety.*

Starting with Schur's lemma, one could deduce orthogonality relations of matrix coefficients of α-representations of finite groups. In particular, we have the following two results. Their proofs are similar as in linear representations case (cf. [14, Section 2.2]).

Corollary 2.4. *Let h be a linear mapping from V_1 to V_2, and define*

$$h^0 = \frac{1}{\mathfrak{g}} \sum_{g \in G} (\rho^2(g))^{-1} h \rho^1(g).$$

Then the following two statements hold.

(1) *If ρ^1 and ρ^2 are not isomorphic, then $h^0 = 0$.*

(2) *If $V_1 = V_2$ and $\rho^1 = \rho^2$, let G_1 be a system of representatives of the quotient group $G/\mathrm{Ker}\rho^1$. Then*

$$h^0 = \frac{1}{\mathfrak{g}} \sum_{g \in G} (\rho^1(g))^{-1} h \rho^1(g) = \frac{s}{\mathfrak{g}} \sum_{g \in G_1} (\rho^1(g))^{-1} h \rho^1(g)$$

and $h^0$is a homothety of ratio $(\frac{1}{m})\mathrm{Tr}(h)$, where $m = \dim(V_1)$ and $s = \#\mathrm{Ker}\rho^1$.

Assume that ρ^1, ρ^2 and h are given in matrix form

$$\rho^1(g) = (r_{i_1 j_1}(g)), \rho^2(g) = (r_{i_2 j_2}(g)), h = (x_{j_2 j_1}).$$

Let $h^0 = \frac{1}{\mathfrak{g}} \sum_{g \in G} (\rho^2(g))^{-1} h \rho^1(g)$. If we write $h^0 = (x^0_{i_2 i_1})$, then

$$x^0_{i_2 i_1} = \frac{1}{\mathfrak{g}} \sum_{g \in G} \sum_{j_2, j_1} r^*_{i_2 j_2}(g) x_{j_2 j_1} r_{j_1 i_1}(g) = \frac{s}{\mathfrak{g}} \sum_{g \in G_1} \sum_{j_2, j_1} r^*_{i_2 j_2}(g) x_{j_2 j_1} r_{j_1 i_1}(g),$$

where $(\rho^2(g))^{-1} = (r^*_{i_2 j_2}(g))$. Therefore we have the following result.

Corollary 2.5. *With the notation as above, the following statements hold.*

(1) *In the case of* Proposition 2.3(1), *we have*

$$\frac{1}{\mathfrak{g}} \sum_{g \in G} r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) = 0$$

for arbitrary i_1, i_2, j_1, j_2.

(2) *In the case of* Proposition 2.3(2), *we have*

$$\frac{1}{\mathfrak{g}} \sum_{g \in G} r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) = \frac{s}{\mathfrak{g}} \sum_{g \in G_1} r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) = \begin{cases} \frac{1}{m} & \text{if } j_1 = j_2 \text{ and } i_1 = i_2 , \\ 0 & \text{otherwise}, \end{cases}$$

where m is the dimension of V, $s = \#\mathrm{Ker}\rho^1$ and $\mathfrak{g}$ is the order of G.

The following result is a twisted version of Corollary 2.5, which is trivial in linear representations case.

Corollary 2.6. *With the notation as above, let G be an abelian group and fix an $l \in G$. In the case of* Proposition 2.3(2), *we have*

$$\frac{1}{\mathfrak{g}} \sum_{g \in G} \alpha(g^{-1}, l^{-1}) \alpha^{-1}(l^{-1}, g^{-1}) r^*_{i_2 j_2}(g) r_{j_1 i_1}(g)$$

$$=\frac{s}{\mathfrak{g}}\sum_{g\in G_1}\alpha(g^{-1},l^{-1})\alpha^{-1}(l^{-1},g^{-1})r^*_{i_2j_2}(g)r_{j_1i_1}(g)$$
$$=\frac{1}{m}r^*_{j_1j_2}(l)r_{i_2i_1}(l)$$

for arbitrary i_1, i_2, j_1, j_2.

Proof. Replacing h with $h' = \rho^1(l^{-1})h$ in Proposition 2.3, we have

$$\begin{aligned} h^0 &= \frac{1}{\mathfrak{g}}\sum_{g\in G}(\rho^1(g))^{-1}\rho^1(l^{-1})h\rho^1(g) \\ &= \frac{s}{\mathfrak{g}}\sum_{g\in G_1}(\rho^1(g))^{-1}\rho^1(l^{-1})h\rho^1(g) \\ &= (\frac{1}{m})\mathrm{Tr}(\rho^1(l^{-1})h). \end{aligned}$$

Note that $\rho^1(g)^{-1}\rho^1(l^{-1}) = \alpha(g^{-1},l^{-1})\alpha^{-1}(l^{-1},g^{-1})\rho^1(l^{-1})\rho^1(g)^{-1}$ for any $g \in G$ since G is abelian. Hence

$$\begin{aligned} &\frac{1}{\mathfrak{g}}\sum_{g\in G}\alpha(g^{-1},l^{-1})\alpha^{-1}(l^{-1},g^{-1})(\rho^1(g))^{-1}h\rho^1(g) \\ =&(\frac{1}{m})(\rho^1(l^{-1}))^{-1}\mathrm{Tr}(\rho^1(l^{-1})h) \\ =&(\frac{1}{m})(\rho^1(l)\mathrm{Tr}(\rho^1(l)^{-1}h), \end{aligned}$$

and

$$\begin{aligned} &\frac{s}{\mathfrak{g}}\sum_{g\in G_1}\alpha(g^{-1},l^{-1})\alpha^{-1}(l^{-1},g^{-1})(\rho^1(g))^{-1}h\rho^1(g) \\ =&(\frac{1}{m})(\rho^1(l^{-1}))^{-1}\mathrm{Tr}(\rho^1(l^{-1})h) \\ =&(\frac{1}{m})(\rho^1(l)\mathrm{Tr}(\rho^1(l)^{-1}h). \end{aligned}$$

So

$$\begin{aligned} &\frac{1}{\mathfrak{g}}\sum_{g\in G}\alpha(g^{-1},l^{-1})\alpha^{-1}(l^{-1},g^{-1})(\rho^1(g))^{-1}h\rho^1(g) \\ =&\frac{s}{\mathfrak{g}}\sum_{g\in G_1}\alpha(g^{-1},l^{-1})\alpha^{-1}(l^{-1},g^{-1})(\rho^1(g))^{-1}h\rho^1(g) \\ =&(\frac{1}{m})(\rho^1(l)\mathrm{Tr}(\rho^1(l)^{-1}h)). \end{aligned}$$

Let h go through the matrices $E_{j_1j_2}$, we obtain the expected identity by comparing the entries of the matrices on both sides of the above equation. $\square$

2.2. **Proof of Theorem 1.2.**

Proof. For $E \in S_n$, say $E = \rho_1(g_1) \otimes ... \otimes \rho_n(g_n)$, $g_t \in G_t$ $(1 \leq t \leq n)$, if we fix a basis of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, then we have the matrix form of E and P_1, P_2.

$$E = (e_{ij}), E^{-1} = (e^*_{ij}), P = (p^1_{ij}), P = (p^2_{ij})\ (1 \leq i, j \leq m^n).$$

Via Kronecker product of matrices, we may write

$$e_{ij} = (\rho_1(g_1))_{i_1 j_1} \otimes ... \otimes (\rho_n(g_n))_{i_n j_n} = \prod_{t=1}^{n} (\rho_t(g_t))_{i_t j_t}.$$

In this way, $f(x, y)$ can be written as

$$\begin{aligned} f(x,y) &= \sum_{t=0}^{n} x^{n-t} y^t \sum_{\substack{E \in S_n \\ w(E)=t}} \sum_{i,j,k,l} e^*_{ij} p^1_{ji} e_{kl} p^2_{lk} \\ &= \sum_{i,j,k,l} p^1_{ji} p^2_{lk} \sum_{E \in S_n} e^*_{ij} e_{kl} x^{n-w(E)} y^{w(E)} \\ &= \sum_{i,j,k,l} p^1_{ji} p^2_{lk} \sum_{E \in S_n} \left(\prod_{t=1}^{n} (\rho_t(g_t)^{-1})_{i_t j_t} (\rho_t(g_t))_{k_t l_t} x^{n-w(E)} y^{w(E)} \right). \end{aligned} \tag{2.2}$$

Let

$$b_{i_t j_t k_t l_t}(x, y) = x (I)_{i_t j_t} (I)_{k_t l_t} + y \left(\sum_{\substack{g \in G_t \\ g \neq 1_G}} (\rho_t(g)^{-1})_{i_t j_t} (\rho_t(g))_{k_t l_t} \right) \tag{2.3}$$

for all $1 \leq t \leq n$. Then it is easy to verify that

$$\begin{aligned} &\sum_{E \in S_n} \left(\prod_{t=1}^{n} (\rho_t(g_t)^{-1})_{i_t j_t} (\rho_t(g_t))_{k_t l_t} x^{n-w(E)} y^{w(E)} \right) \\ &= \sum_{E \in S_n} \left(\prod_{\substack{t=1 \\ g_t \neq 1_G}}^{n} (\rho_t(g_t)^{-1})_{i_t j_t} (\rho_t(g_t))_{k_t l_t} y^{w(E)} \cdot \prod_{\substack{t=1 \\ g_t = 1_G}}^{n} (\rho_t(g_t)^{-1})_{i_t j_t} (\rho_t(g_t))_{k_t l_t} x^{n-w(E)} \right) \\ &= \prod_{t=1}^{n} b_{i_t j_t k_t l_t}(x, y). \end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned} f^{\perp}(x,y) &= \sum_{t=0}^{n} x^{n-t} y^t \sum_{\substack{E \in S_n \\ w(E)=t}} \sum_{i,j,k,l} e^*_{kj} p^1_{ji} e_{il} p^2_{lk} \\ &= \sum_{i,j,k,l} p^1_{ji} p^2_{lk} \sum_{E \in S_n} e^*_{kj} e_{il} x^{n-w(E)} y^{w(E)} \qquad . \\ &= \sum_{i,j,k,l} p^1_{ji} p^2_{lk} \sum_{E \in S_n} \left(\prod_{t=1}^{n} (\rho_t(g_t)^{-1})_{k_t j_t} (\rho_t(g_t))_{i_t l_t} x^{n-(E)} y^{w(E)} \right) \end{aligned} \tag{2.5}$$

Let

$$b^{\perp}_{i_tj_tk_tl_t}(x,y) = x(I)_{k_tj_t}(I)_{i_tl_t} + y(\sum_{\substack{g\in G_t\\ g\neq 1_G}} (\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}). \tag{2.6}$$

Then

$$\begin{aligned}
&\sum_{E\in S_n} (\prod_{t=1}^{n}(\rho_t(g_t)^{-1})_{k_tj_t}(\rho_t(g_t))_{i_tl_t} x^{n-w(E)}y^{w(E)})\\
=&\sum_{E\in S_n} (\prod_{\substack{t=1\\ g_t\neq 1_G}}^{n} (\rho_t(g_t)^{-1})_{k_tj_t}(\rho_t(g_t))_{i_tl_t} y^{w(E)} \cdot \prod_{\substack{t=1\\ g_t= 1_G}}^{n} (\rho_t(g_t)^{-1})_{k_tj_t}(\rho_t(g_t))_{i_tl_t} x^{n-w(E)})\\
=&\prod_{t=1}^{n} b^{\perp}_{i_tj_tk_tl_t}(x,y).
\end{aligned} \tag{2.7}$$

Therefore it suffices to show that

$$b_{i_tj_tk_tl_t}(x,y) = b^{\perp}_{i_tj_tk_tl_t}(\frac{s(m^2x+(\mathfrak{g}^2-m^2)y)}{m\mathfrak{g}}, \frac{sm(x-y)}{\mathfrak{g}}), \text{ for all } \quad 1\leq t\leq n. \tag{2.8}$$

Considering the projective representations (ρ_t, V_t) in Corollary 2.5 we have

$$\begin{aligned}
\frac{s}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t} &= \frac{1}{m}(I)_{k_tj_t}(I)_{i_tl_t},\\
\frac{s}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t} &= \frac{1}{m}(I)_{i_tj_t}(I)_{k_tl_t}.
\end{aligned} \tag{2.9}$$

Thus

$$\begin{aligned}
\frac{sx}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t} &= \frac{x}{m}(I)_{k_tj_t}(I)_{i_tl_t},\\
\frac{sy}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t} &= \frac{y}{m}(I)_{i_tj_t}(I)_{k_tl_t}.
\end{aligned} \tag{2.10}$$

We then obtain that

$$\begin{aligned}
&(\frac{sx}{\mathfrak{g}}-\frac{y}{m})(I)_{i_tj_t}(I)_{k_tl_t} + \frac{sx}{\mathfrak{g}}\sum_{\substack{g\in G_t\\ g\neq 1_G}}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}\\
=&(\frac{x}{m}-\frac{sy}{\mathfrak{g}})(I)_{k_tj_t}(I)_{i_tl_t} + \frac{-sy}{\mathfrak{g}}\sum_{\substack{g\in G_t\\ g\neq 1_G}}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t},
\end{aligned} \tag{2.11}$$

Let $\frac{sx}{\mathfrak{g}}-\frac{y}{m}=X, \frac{sx}{\mathfrak{g}}=Y$, we have

$$\begin{aligned}
&X(I)_{i_tj_t}(I)_{k_tl_t} + Y\sum_{\substack{g\in G_t\\ g\neq 1_G}}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}\\
=&\frac{s(m^2X+(\mathfrak{g}^2-m^2)Y)}{m\mathfrak{g}}(I)_{k_tj_t}(I)_{i_tl_t} + \frac{sm(X-Y)}{\mathfrak{g}}\sum_{\substack{g\in G_t\\ g\neq 1_G}}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}.
\end{aligned} \tag{2.12}$$

Therefore

$$b_{i_tj_tk_tl_t}(X,Y)=b^{\perp}_{i_tj_tk_tl_t}\Big(\frac{s(m^2X+(\mathfrak{g}^2-m^2)Y)}{m\mathfrak{g}},\frac{sm(X-Y)}{\mathfrak{g}}\Big),\quad 1\le t\le n. \tag{2.13}$$

This completes the proof. □

2.3. **Proof of Theorem 1.4.**

Proof. We use the notation in Section 2.2. Direct computation shows that

$$\begin{aligned}
D(M) &= \sum_{J=(j_{o(g)})\in \mathrm{IND}(n)} D_J M^J \\
&= \sum_{J=(j_{o(g)})\in \mathrm{IND}(n)} \prod_{g\in G_1} M_g^{j_{o(g)}} \sum_{E\in E[J]} \sum_{i,j,k,l} e^*_{ij}p^1_{ji}e_{kl}p^2_{lk} \\
&= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \sum_{E\in S_n} e^*_{ij}e_{kl} \prod_{g\in G_1} M_g^{N_g(E)} \\
&= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \sum_{E\in S_n} \prod_{t=1}^{n} (\rho_t(g_t)^{-1})_{i_tj_t}(\rho_t(g_t))_{k_tl_t} \prod_{g\in G_1} M_g^{N_g(E)} \\
&= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \sum_{E\in S_n} \prod_{g\in G_1} \prod_{\substack{t=1\\ \rho_{t1}(g_t)=g}}^{n} (\rho_t(g_t)^{-1})_{i_tj_t}(\rho_t(g_t))_{k_tl_t} M_g^{N_g(E)} \\
&= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \prod_{t=1}^{n} d_t(M),
\end{aligned} \tag{2.14}$$

where

$$d_t(M)=\sum_{g\in G_t}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}M_{\rho_{t1}(g)}.$$

Similarly we have

$$D^{\perp}(M^{\perp})=\sum_{i,j,k,l}p^1_{ji}p^2_{lk}\prod_{t=1}^{n}d_t^{\perp}(M^{\perp})$$

where

$$d_t^{\perp}(M^{\perp})=\sum_{g\in G_1}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}M^{\perp}_{\rho_{t1}(g)}.$$

Let $M_g^{\perp}=\frac{sm}{\mathfrak{g}}\sum\limits_{l\in G_1}\alpha(g^{-1},l^{-1})\alpha^{-1}(l^{-1},g^{-1})M_l$ for all $g\in G_1$. By Corollary 2.6, we have $d_t^{\perp}(M^{\perp})=d_t(M)$ for every t, which completes the proof. □

2.4. **Proof of Theorem 1.6.** We adapt the strategy from the proof of [4, Theorem 4] and give the relation between Theorem 1.4 and Theorem 1.6. Then we get the MacWilliams identity for double weight enumerators easily. For convenience, we change the notation a little bit. Let $\mathrm{IND}(n)$ be $\{J=(j_{\lambda,\mu})\in\mathbb{Z}_{\ge 0}^{m\times m} \mid \sum\limits_{\lambda,\mu=1}^{m} j_{\lambda,\mu}=n\}$ and fix two bijections from H, $\hat{H}$ to the subset $\{1,2,...,m\}$ of $\mathbb{Z}$, such that the group units correspond to integer 1. They induce a bijection o from G to

$\{(\lambda,\mu)|\lambda,\mu\in\{1,2,...,m\}\}$. Then variables of the complete weight enumerators can be written as a m-by-m matrix $M=(M_{\lambda,\mu})$.

Lemma 2.7. *Let H be an abelian group. Let $G=H\times\hat{H}$ and ρ be the representation defined by equation* (1.3)*. Let C, $C^\perp$ be the corresponding double weight polynomials, D, $D^\perp$ the corresponding complete weight polynomials. Then the following identities hold:*

$$\begin{aligned} C(X,Y,Z,W) &= D(\Psi(X,Y,Z,W)),\\ C^\perp(X,Y,Z,W) &= D^\perp(\Psi(X,Y,Z,W)), \end{aligned} \tag{2.15}$$

where $\Psi(X,Y,Z,W)$ is the matrix

$$\begin{pmatrix} XZ & XW & XW & \cdots & XW\\ YZ & YW & YW & \cdots & YW\\ YZ & YW & YW & \cdots & YW\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ YZ & YW & YW & \cdots & YW \end{pmatrix}.$$

Proof. For $J=(j_{o(a,\chi)})\in\mathrm{IND}(n)$, let $|J_1|:=\sum_{\substack{(a,\chi)\in G\\ a\neq 0_H}} j_{o(a,\chi)}$ and $|J_2|:=\sum_{\substack{(a,\chi)\in G\\ \chi\neq 1_{\hat{H}}}} j_{o(a,\chi)}$. From the definition of X weight and Z weight, for $0\le i,j\le n$, we have

$$\sum_{\substack{J\in\mathrm{IND}(n)\\ |J_1|=i,|J_2|=j}} D_J = C_{i,j},$$
$$\sum_{\substack{J\in\mathrm{IND}(n)\\ |J_1|=i,|J_2|=j}} D^\perp = C^\perp_{i,j}.$$

Hence the identities hold and the lemma follows. □

Proof of Theorem 1.6. Let $\Psi^\perp(X,Y,Z,W)=(M^\perp_{(a,\chi)}(X,Y,Z,W))$ be the matrix associated with $\Psi(X,Y,Z,W)$, where $M^\perp_{(a,\chi)}(X,Y,Z,W)$ can be calculated from (1.2):

$$\begin{aligned} M^\perp_{(a,\chi)}(X,Y,Z,W) = \frac{1}{m}(XZ &+ \sum_{\substack{\chi_0\in\hat{H}\\ \chi_0\neq 1_{\hat{H}}}}\chi_0(-a)YZ + \sum_{\substack{b\in H\\ b\neq 0_H}}\chi^{-1}(-b)XW\\ &+ \sum_{\substack{b\neq 0_H,\chi_0\neq 1_{\hat{H}}\\ (b,\chi_0)\in G}}\chi^{-1}(-b)\chi_0(-a)YW). \end{aligned} \tag{2.16}$$

It follows from the properties of characters that

$$
\begin{aligned}
M^{\perp}_{(1,1)} &= \frac{1}{m}(X+(m-1)Y)(Z+(m-1)W),\\
M^{\perp}_{(a,1)} &= \frac{1}{m}(X-Y)(Z+(m-1)W), a \neq 0_H,\\
M^{\perp}_{(1,\chi)} &= \frac{1}{m}(X+(m-1)Y)(Z-W), \chi \neq 1_{\hat{H}},\\
M^{\perp}_{(a,\chi)} &= \frac{1}{m}(X-Y)(Z-W), a \neq 0_H, \chi \neq 1_{\hat{H}}.
\end{aligned} \tag{2.17}
$$

From Equations (2.15), (2.16) and (2.17) we see that

$$
\begin{aligned}
C(X,Y,Z,W) &= D(\Psi(X,Y,Z,W)),\\
&= D^{\perp}(\Psi^{\perp}(X,Y,Z,W)),\\
&= C^{\perp}(X+(m-1)Y, X-Y, \frac{Z+(m-1)W}{m}, \frac{Z-W}{m}).
\end{aligned}
$$

This completes the proof. □

3. The MacWilliams identities for quantum error-correcting G-codes

In 1997, Shor and Laflamme [15] proved the quantum MacWilliams identities of weight enumerator for binary QECCs and later Rains [12] proved this in a general setting. These identities are important in deducing certain bounds of QECCs. Recently Hu-Yang-Yau [3, 4] proved the quantum MacWilliams identities for double and complete enumerators for binary and non-binary QECCs. In the following we recall three versions of quantum MacWilliams identities for general errors and explain that these are special cases of Theorems 1.2 1.4 and 1.6. We refer to [7, 9, 11, 15] for more information of quantum codes.

In this section, $|a\rangle$ and $|b\rangle$ denote complex vectors, $E|a\rangle$ denotes the operator E acting on $|a\rangle$ and $\langle a|b\rangle$ the usual inner product between $|a\rangle$ and $|b\rangle$ in complex vector spaces.

Fix positive integers m, n, let $\mathcal{H} = \mathbb{C}^m$ and a QECC of length n is a subspace $\mathcal{Q} \subset \mathcal{H}^{\otimes n}$. Let $\mathcal{E}$ be a set of unitary linear operators. We say that a QECC $\mathcal{Q}$ to be *$\mathcal{E}$-correcting* if for an orthogonal basis $\{|i_L\rangle\}_i$ of $\mathcal{Q}$ and every $A, B \in \mathcal{E}$, we have $\langle i_L|A^*B|j_L\rangle{=}\lambda_{A,B}\delta_{i,j}$ for some $\lambda_{A,B} \in \mathbb{C}$ depending on A and B. Let $P_{\mathcal{Q}}$ be the projection operator on $\mathcal{Q}$. Then the condition can be restated in the form

$$P_{\mathcal{Q}}A^*BP_{\mathcal{Q}} = \alpha_{ij}P_{\mathcal{Q}},$$

for some Hermitian matrix $\alpha = (\alpha_{ij})$.

Let $\mathcal{E}$ be a set of unitary linear operators on $\mathcal{H}$ and $\mathcal{E}_n = \mathcal{E}^{\otimes n}$. We say that $E \in \mathcal{E}_n$ has *weight* t if $E = A_1 \otimes A_2 \otimes ... \otimes A_n$ where $A_j \in \mathcal{E}, 1 \leq j \leq n$, and $|\{j : A_j \notin \mathbb{C}^{\times} \cdot id\}| = t$. Denote by $w_{\mathcal{Q}}(E)$ the weight of E.

Let $\mathcal{Q} \subset \mathcal{H}^{\otimes n}$ be a QECC with dimension K, we say that $\mathcal{Q}$ can detect an error E if for any $c_1, c_2 \in \mathcal{Q}$, the condition $\langle c_1|E|c_2\rangle = \lambda_E\langle c_1|c_2\rangle$ holds for some $\lambda_E \in \mathbb{C}$ just related to E. The *minimal distance* of the QECC $\mathcal{Q}$ is the maximal integer d such that $\mathcal{Q}$ can detect all errors in $\mathcal{E}_n$ of weight less than d. Then we say that $\mathcal{Q}$ has parameters $((n, K, d))_m$.

Let $\mathcal{Q} \subset \mathcal{H}^{\otimes n}$ be a QECC, we say that $\mathcal{Q}$ can correct all errors of weight $\leq l$ $(0 \leq l \leq n)$ if $\mathcal{Q}$ is $\mathcal{E}_n^l$-correcting, where $\mathcal{E}_n^l = \{E \in \mathcal{E}_n, w_{\mathcal{Q}}(E) \leq l\}$.

Definition 3.1. Let G be a finite group and $\rho : G \to \mathbf{U}(\mathcal{H})$ be an irreducible faithful α-representation. Let $\mathcal{E} = \{\rho(g)|g \in G\}$ and $\mathcal{E}_n = \mathcal{E}^{\otimes n}$. A QECC $\mathcal{Q} \subset \mathcal{H}^n$ is called a *quantum error-correcting G-code*.

In the following, we deduce the MacWilliams identities for quantum error-correcting G-codes.

3.1. **The MacWilliams identity for weight enumerates.**

Definition 3.2. Let $(\rho, \mathcal{H})$ be an irreducible faithful unitary projective representation of a finite group G and let $\mathcal{E} = \{\rho(g)|g \in G\}$. We put $\mathcal{E}_n = \mathcal{E}^{\otimes n}$. Let $\mathcal{Q}$ be a quantum G-code of dimension K, and let

$$\begin{aligned}
B_i &= \frac{1}{K^2} \sum_{E \in \mathcal{E}_n, w_{\mathcal{Q}}(E)=i} \mathrm{Tr}(E^{-1}P_{\mathcal{Q}})\mathrm{Tr}(EP_{\mathcal{Q}}),\\
B_i^{\perp} &= \frac{1}{K} \sum_{E \in \mathcal{E}_n, w_{\mathcal{Q}}(E)=i} \mathrm{Tr}(E^{-1}P_{\mathcal{Q}}EP_{\mathcal{Q}}),\\
D_J &= \frac{1}{K^2} \sum_{E \in \mathcal{E}_n, E \in E[J]} \mathrm{Tr}(E^{-1}P_{\mathcal{Q}})\mathrm{Tr}(EP_{\mathcal{Q}}),\\
D_J^{\perp} &= \frac{1}{K} \sum_{E \in \mathcal{E}_n, E \in E[J]} \mathrm{Tr}(E^{-1}P_{\mathcal{Q}}EP_{\mathcal{Q}}).
\end{aligned}$$

The *weight enumerators* of $\mathcal{Q}$ are defined by

$$\begin{aligned}
f_{\mathcal{Q}}(x,y) &= \sum_{i=0}^{n} B_i x^{n-i} y^i,\\
f_{\mathcal{Q}}^{\perp}(x,y) &= \sum_{i=0}^{n} B_i^{\perp} x^{n-i} y^i,
\end{aligned}$$

and the *complete weight enumerators* of $\mathcal{Q}$ are defined by

$$\begin{aligned}
D_{\mathcal{Q}}(M) &= \sum_{J \in \mathrm{IND}(n)} D_J M^J,\\
D_{\mathcal{Q}}^{\perp}(M) &= \sum_{J \in \mathrm{IND}(n)} D_J^{\perp} M^J,
\end{aligned}$$

where $E[J], \mathrm{IND}(n)$ are defined as in Theorem 1.4.

Theorem 3.3 (Weight enumerators)**.** *With the notation as above, we have*

$$f_{\mathcal{Q}}(x,y) = \frac{1}{K} f_{\mathcal{Q}}^{\perp}\Big(\frac{m^2 x + (\mathfrak{g}^2 - m^2)y}{m\mathfrak{g}}, \frac{m(x-y)}{\mathfrak{g}}\Big),$$

where m is the dimension of $\mathcal{H}$, $\mathfrak{g}$ is the order of G.

Proof. This is a special case of Theorem 1.2 with $(\rho_i, V_i) = (\rho, \mathcal{H})$ for $1 \leq i \leq n$ and $P_1 = P_2 = P_{\mathcal{Q}}$. □

By the same argument as in [12, Theorem 2], we have the following result.

Theorem 3.4. *For a quantum G-code $\mathcal{Q}$ and the corresponding real numbers B_i, $B_i^{\perp}$ $(0 \leq i \leq n)$, we have*

(1) $B_0 = B_0^{\perp} = 1$, $B_i^{\perp} \geq B_i \geq 0$ $(0 \leq i \leq n)$.

(2) If there exists $t \leq n-1$, such that $B_i^{\perp} = B_i$ $(0 \leq i \leq t)$, and $B_{t+1}^{\perp} > B_{t+1}$, then the minimal distance d is $t+1$.

For quantum G-codes, the MacWilliams identities give necessary conditions for their existence. Moreover they also give the bound of minimum distance. The binary version of the quantum Singleton bound was first proved by Knill and Laflamme in [10], and later generalized by Rains using the quantum MacWilliams identities in [13, Theorem 2]. By Theorem 3.3, we obtain the Singleton bound for quantum G-codes.

Theorem 3.5 (Quantum Singleton Bound)**.** *: If $\mathcal{Q}$ is a quantum G-code with parameters $((n, K, d))_m$, then*

$$K \leq \left(\frac{\mathfrak{g}}{m}\right)^{n-2d+2}.$$

3.2. **The MacWilliams identity for complete weight enumerators.** In [3, 4], Hu-Yang-Yau proved the MacWilliams identity for double weight enumerators and the weight complete enumerators for binary and non-binary quantum codes. One could generalize them easily to G-codes.

Theorem 3.6 (Complete enumerators)**.** *Let G be an abelian group with order $\mathfrak{g}$. For a quantum G-code $\mathcal{Q}$ with parameters $((n, K, d))_m$ we have*

$$D_{\mathcal{Q}}(M) = \frac{1}{K} D_{\mathcal{Q}}^{\perp}(M^{\perp}),$$

where $M_g^{\perp} = \dfrac{m}{\mathfrak{g}} \sum_{l \in G} \alpha(g^{-1}, l^{-1}) \alpha^{-1}(l^{-1}, g^{-1}) M_l$, for all $g \in G$ and $M^{\perp} = (M_g^{\perp})$.

Proof. This is a special case of Theorem 1.4. □

3.3. **The MacWilliams identity for double weight enumerates.**

Definition 3.7. Let H be an abelian group and (G, ρ) be the Weyl-Heisenberg representation of $G = H \times \hat{H}$ as in equation (1.3). Let $\mathcal{Q}$ be a quantum G-code with parameters $((n, K, d))_m$, and let

$$C_{i,j} := \frac{1}{K^2} \sum_{E \in E[i,j]} \mathrm{Tr}(E^{-1} P_{\mathcal{Q}}) \mathrm{Tr}(E P_{\mathcal{Q}}),$$

$$C_{i,j}^{\perp} := \frac{1}{K} \sum_{E \in E[i,j]} \mathrm{Tr}(E^{-1} P_{\mathcal{Q}} E P_{\mathcal{Q}}).$$

The *double weight enumerators* of $\mathcal{Q}$ are defined by

$$C(X,Y,Z,W) := \sum_{i,j=0}^{n} C_{i,j} X^{n-i} Y^i Z^{n-j} W^j,$$
$$C^{\perp}(X,Y,Z,W) := \sum_{i,j=0}^{n} C^{\perp}_{i,j} X^{n-i} Y^i Z^{n-j} W^j.$$

The following result follows from Theorem 1.6.

Theorem 3.8 (Double weight enumerators)**.** *The relation between the double weight enumerators of $\mathcal{Q}$ is*

$$C(X,Y,Z,W) = \frac{1}{K} C^{\perp}(X+(m-1)Y, X-Y, \frac{Z+(m-1)W}{m}, \frac{Z-W}{m}).$$

As in [4], we can also define the asymmetric quantum G-code.

Definition 3.9. Let H be an abelian group and (G,ρ) be the Weyl-Heisenberg representation of $G = H \times \hat{H}$ as in equation (1.3). Let d_X and d_Z be the maximum integers such that each error $E \in E[i,j]$ with $i < d_X, j < d_Z$ is detectable, then we call Q an *asymmetric quantum G-code* with parameters $((n,K,d_Z/d_X))_m$.

The following theorems can be deduced in the same way as in [4, Theorems 1, 2, 6].

Theorem 3.10. *Let $\mathcal{Q}$ be a asymmetric quantum G-code with double weight distribution $C_{i,j}$, $C^{\perp}_{i,j}$ and parameters $((n,K,d_Z/d_X))_m$, then*

(1) $C^{\perp}_{i,j} \geq C_{i,j} \geq 0$ for $0 \leq i,j \leq n$, and $C_{0,0} = C^{\perp}_{0,0} = 1$.

(2) If t_X, t_Z are the two largest integers such that $C_{i,j} = C^{\perp}_{i,j}$ for $i < t_X$ and $j < t_Z$, then $d_X = t_X$ and $d_Z = t_Z$.

(3) (Singleton bound) $K \leq m^{n+2-d_X-d_Z}$.

(4) (Hamming bound) $K \leq m^{n(1-H(\frac{\delta_X}{2})-H(\frac{\delta_Z}{2})+o(1))}$. Here $\delta_X = \frac{d_X}{n}$ and $\delta_Z = \frac{d_Z}{n}$ satisfying $0 \leq \delta_X \leq \frac{1}{5}$ and $0 \leq \delta_Z \leq \frac{1}{5}$ respectively, $H(x)$ is the m-ary entropy function defined by

$$H(x) = xlog_m(m-1) - xlog_m x - (1-x)log_m(1-x),\ 0 \leq x \leq 1.$$

Remark 3.11. The set $\mathcal{E} = \{\rho(g), g \in G\}$ forms a *nice error bases* (cf. [7]) if and only if ρ is an irreducible faithful projective representation of G of degree $|G|^{1/2}$. In [8], Knill discussed the construction of quantum codes based on nice error bases and some equivalent characterizations for nice error bases. There are examples where the nice error bases occur with nonablian group G (cf. the list in [7]). The above results show that the Singleton bound and the Hamming bound of quantum G-codes depend only on the size of G. This is closely related to the question motivated by [8, Theorem 3.4].

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