ON THE PHASE RETRIEVABILITY OF IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS

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Abstract. Let G be a finite group and $\pi : G \to \mathbf{U}(V)$ be an irreducible representation of G on a complex Hilbert space V. In this paper we study the phase retrieval property of π and the existence of maximal spanning vectors for (π, G, V). By translating the existence of maximal spanning vectors into the existence of cyclic vectors of the form $v \otimes v$ for the representation $(\pi \otimes \pi^*, G, V \otimes V)$, we show that if π is unramified, in the sense that each irreducible component of $\pi \otimes \pi^*$ has multiplicity one, then π admits maximal spanning vectors and hence does phase retrieval. Moreover, if $\mathrm{GA}(1, q)$ is the one-dimensional affine group over the finite field $\mathbb{F}_q$ and $\pi : \mathrm{GA}(1, q) \to \mathbf{U}(\mathbb{C}^{q-1})$ is the unique $(q-1)$-dimensional irreducible representation of $\mathrm{GA}(1, q)$ (which is ramified), we give a characterization of maximal spanning vectors for $(\pi, \mathrm{GA}(1, q), \mathbb{C}^{q-1})$ by a detailed study of the adjoint representation of $\mathrm{GA}(1, q)$ on $L^2(\mathrm{GA}(1, q))$. In particular, we show that the set of maximal spanning vectors are open dense in $\mathbb{C}^{q-1}$ and the representation $(\pi, \mathrm{GA}(1, q), \mathbb{C}^{q-1})$ does phase retrieval. Furthermore, we show that the special representations and the cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$ admit maximal spanning vectors and do phase retrieval.

1. Introduction

The phase retrieval property of a frame is a significant research topic in applied mathematics and engineering (cf. [10, 13]). While studying the phase retrieval property of group frames, mathematicians have discovered deep questions in number theory, representation theory, and harmonic analysis etc., and obtained meaningful results. The motivation of this paper is [17, Conjecture], which conjectured that every irreducible projective representation of a finite group admits maximal spanning vectors, hence does phase retrieval. Li-Han-etc. [17] verified the conjecture for finite abelian groups. In [8, 7, 9], the authors generalized the conjecture to locally compact groups and verified the conjecture for compact abelian groups and several types of locally compact abelian groups via Fourier analysis. In [12] Führ-Oussa showed that irreducible representations of nilpotent Lie groups do phase retrieval via tools from Lie algebras and studied the phase retrieval property of representations of p-groups. In [1] Bartusel-Führ-Oussa studied the problem for the affine group $\mathrm{GA}(1, p)$, where p is a prime number.

In this paper, we study the problem for an arbitrary finite group G and translate the existence of maximal spanning vectors for the representation (π, G, V) into the existence of cyclic vectors of the form $v \otimes v$ for the representation $\pi \otimes \pi^*$, the tensor product representation of π and its dual π^*. One the representation theory side, we have more

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tools, which are elementary, but really powerful for our purpose. In particular, we show that if (π, G, V) is unramified, in the sense that each irreducible component of $\pi \otimes \pi^*$ has multiplicity one, then the set of maximal spanning vectors for (π, G, V) is open dense in V. The method also applies to ramified representations once we have an explicit decomposition of $\pi \otimes \pi^*$. Moreover, in this case we may provide a characterization of maximal spanning vectors. We illustrate this idea in the case where $G = \mathrm{GA}(1, q)$ is the one-dimensional general affine group over the finite field $\mathbb{F}_q$ and $\pi : \mathrm{GA}(1, q) \to \mathbf{U}(\mathbb{C}^{q-1})$ is the unique $(q-1)$-dimensional irreducible representation. Combining these two results, we show that the special representations and the cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$ admit maximal spanning vectors and do phase retrieval.

We explain our motivation and results in the following. We refer to [4] for basics of frames, [6, 19] for basics of group frames and twisted group frames. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle. Let G be a finite group and $\alpha \in Z^2(G, \mathbb{T})$ be a Schur multiplier of G. Let $\pi : G \to \mathbf{U}(V)$ be a finite dimensional irreducible projective representation with respect to α, i.e. $\pi(g)\pi(h) = \alpha(g, h)\pi(gh)$ for all $g, h \in G$. Let $v \in V$ be a nontrivial vector. Then $\Phi_v := \{\pi(g)v \mid g \in G\}$ is a frame for V. If the frame Φ_v is phase retrievable, i.e. the map

$$\begin{aligned} t_v : V/\mathbb{T} &\to \mathbb{R}^{|G|} \\ v &\mapsto (|\langle v, \pi(g)v\rangle|)_{g \in G} \end{aligned}$$

is injective, then we call the vector v *phase retrievable.* If (π, G, V) admits a phase retrievable vector, we say that the representation π *does phase retrieval.*

In general it is not easy to verify directly whether $v \in V$ is phase retrievable. Yet it is easier to verify whether $v \in V$ has maximal span, a stronger property than being phase retrievable. In the above situation, $v \in V$ has *maximal span* (or is *maximal spanning*) if $\mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = \mathrm{HS}(V)$. Here for any $x,\ y \in V$, $x \otimes y$ is the projection

$$\begin{aligned} V &\to V \\ v &\mapsto \langle v, y\rangle x. \end{aligned}$$

In other words, $v \in V$ has maximal span if and only if $\dim \mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = (\dim V)^2$. It is well known and easy to see that if v has maximal span, then $v \in V$ is phase retrievable (cf. [2]). For $u, v \in V$, let $\mathfrak{c}_{u,v}$ be the matrix coefficient

$$\begin{aligned} \mathfrak{c}_{u,v} : G &\to \mathbb{C} \\ g &\mapsto \langle \pi(g)u, v\rangle. \end{aligned}$$

Denoted by $C_\pi \subset L^2(G)$ the subspace spanned by the matrix coefficients $\mathfrak{c}_{u,v}$ $(u, v \in V)$. The inverse of the Fourier transform gives us an isomorphism $\mathcal{E} : V \otimes V \to C_\pi$ $(u \otimes v \mapsto \mathfrak{c}_{u,v})$, we see that $v \in V$ has maximal span if and only if

$$\mathrm{Span}\{\mathfrak{c}_{\pi(g)v, \pi(g)v} \mid g \in G\} = C_\pi,$$

or equivalently,

$$\dim \mathrm{Span}\{\mathfrak{c}_{\pi(g)v, \pi(g)v} \mid g \in G\} = (\dim V)^2.$$

Computation shows that $\mathfrak{c}_{\pi(g)v,\pi(g)v}$ and $\mathfrak{c}_{v,v}$ have the following relation

$$\begin{aligned}\mathfrak{c}_{\pi(g)v,\pi(g)v}(h) = \langle \pi(h)\pi(g)v, \pi(g)v\rangle &= \langle \pi(g)^{-1}\pi(h)\pi(g)v, v\rangle \\ &= \frac{\alpha(g^{-1},h)\alpha(g^{-1}h,g)}{\alpha(g,g^{-1})}\langle \pi(g^{-1}hg)v, v\rangle \\ &= \frac{\alpha(g^{-1},h)\alpha(g^{-1}h,g)}{\alpha(g,g^{-1})}\mathfrak{c}_{v,v}(g^{-1}hg).\end{aligned} \tag{1.1}$$

While computing the dimension of $\mathrm{Span}\{\mathfrak{c}_{\pi(g)v,\pi(g)v} \mid g \in G\}$, there are two types of difficulties

- the complication from the multiplier α;
- the structure of the conjugation $h \mapsto g^{-1}hg$.

In case G is an abelian group, the second difficulty disappears. Furthermore, in abelian case

$$\frac{\alpha(g^{-1},h)\alpha(g^{-1}h,g)}{\alpha(g,g^{-1})} = \frac{\alpha(g^{-1},h)}{\alpha(h,g^{-1})}$$

is a bihomomorphism on $G \times G$. Then one could show that $v \in V$ has maximal span if and only if $\langle \pi(g)v, v\rangle \neq 0$ for all $g \in G$ (cf. [6, Proposition 3.11] and [17, Theorem 1.7]). This property is the key ingredient in [17] and based on abstract harmonic analysis (cf. [14, 15, 16]), has been generalized to locally compact abelian groups and continuous frames case (cf. [5]).

The method in [5, 6, 17] does not apply to nonabelian groups. Since α-representations of G corresponds to special linear representations of the representation group $G(\alpha)$, in this paper we consider linear representations of arbitrary finite groups. Recall that we identified the second V in $V \otimes V$ with its dual space, the map $(g \mapsto (u \otimes v \mapsto \pi(g)u \otimes \pi(g)v))$ defines a representation of G on $V \otimes V$, which is nothing but the tensor product of π and its dual representation π^*. Let $\mathrm{ad} : G \to \mathbf{U}(C_\pi)$ be the adjoint representation defined by $\mathrm{ad}(g)(f)(x) = f(g^{-1}xg)$ for $g \in G$ and $f \in C_\pi$. Then $\mathcal{E}$ is an isomorphism of representations between $(\pi \otimes \pi^*, G, V \otimes V)$ and (ad, G, C_π). One sees that the following three conditions are equivalent.

(1) $v \in V$ is a maximal spanning vector for (π, G, V);
(2) $v \otimes v \in V \otimes V$ is a cyclic vector for the representation $(\pi \otimes \pi^*, G, V \otimes V)$;
(3) $\mathfrak{c}_{v,v} \in C_\pi$ is a cyclic vector for the representation (ad, G, C_π).

These equivalences are the key idea of this paper and they enable us to utilize tools from representation theory to tackle [17, Conjecture] for representations of general finite groups. We note that this idea has appeared before, e.g. in [1, 6]. By a characterization of cyclic vectors for representations that are direct sums of non-isomorphic irreducible representations (cf. Proposition 2.3), we prove the following result.

Theorem 1.1. *Let $\pi : G \to \mathbf{U}(V)$ be an irreducible representation of G. Assume that each irreducible component of $\pi \otimes \pi^*$ has multiplicity one. Then $\pi \otimes \pi^*$ admits cyclic vectors of the form $v \otimes v$. Hence π admits maximal spanning vectors and does phase retrieval.*

Although for some groups G, there is no nontrivial representation that satisfies the condition in Theorem 1.1, this result is the first result in this direction that could be said for all finite groups and the multiplicity condition is relatively easy to check from the character table. The phase retrievability of projective representations of finite abelian groups is just a special case, because for any irreducible projective representation π of a finite abelian group, the tensor product $\pi \otimes \pi^*$ is a direct sum of non-isomorphic one-dimensional linear representations (cf. [6, Section 3.2], Section 5.4). If some irreducible components of $\pi \otimes \pi^*$ have multiplicity greater or equal to two, the situation is more subtle (cf. Proposition 2.5). Nevertheless, if we could obtain an explicit description of the decomposition of $(\pi \otimes \pi^*, G, V \otimes V)$ or of (ad, G, C_π), we may apply Proposition 2.5 to find maximal spanning vectors. As an illustration, we prove the following result.

Theorem 1.2. *Every irreducible representation of* $\mathrm{GA}(1, q)$ *admits maximal spanning vectors and does phase retrieval.*

If $q = 2$, the case is trivial. If $q = 3$, then $G \cong D_6$, the dihedral group with 6 elements, and the case follows by direct computation (cf. [7, Section 4.3.2]). In the following we may and do assume that $q \geq 4$ and the representation is $(q-1)$-dimensional. Moreover, we provide an explicit characterization of the maximal spanning vectors (cf. Remark 4.4).

Although we could not be as explicit as in $\mathrm{GA}(1, q)$ case, the ideas in the proof of Theorem 1.2 apply to general metabelian groups and we explain the strategy in Section 5. As an application of the theorems, we also prove the following result for the group $\mathrm{GL}_2(\mathbb{F}_q)$, which is not easy to check directly and provides strong evidence for [17, Conjecture]. The verification of the phase retrievability of irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ is one of the motivations of this paper.

Corollary 1.3. *The special representations and the cuspidal representations of* $\mathrm{GL}_2(\mathbb{F}_q)$ *admit maximal spanning vectors and do phase retrieval.*

The contents are organised as follows. In Section 2, we study cyclic vectors for representations of finite groups in detail and give a characterization of them (cf. Propositions 2.3 and 2.5). Applying Proposition 2.3, we prove Theorem 1.1. In Section 3, we review basic properties of the affine group $\mathrm{GA}(1, q)$ and translate the statement about maximal spanning vectors of group frames into a statement about cyclic vectors for the adjoint representation. Then in Section 4, by combining the results in Sections 2 and 3, we prove Theorem 1.2 and Corollary 1.3. In Section 5, we explain how we may apply the strategy to prove phase retrievability and characterize maximal spanning vectors for certain representations of metabelian groups and give a new proof for the Weyl-Heisenberg representations. The method in this paper does provide an approach to check the phase retrievability of representations of metabelian groups. We explain this in detail for split metacyclic groups (cf. Remark 5.3), which is the most common dilation-translation situation in engineering and physics.

Notation and convention. In this paper, G is a finite group. Let $L^2(G)$ be the space of functions on G with inner product given by

$$\langle f, f' \rangle = \sum_{g \in G} f(g)\overline{f'(g)}.$$

Let $r : G \to \mathbf{U}(L^2(G))$ be the right regular representation of G, i.e.

$$(r(g)f)(h) = f(hg), \text{ for all } g, h \in G.$$

Let $\mathrm{ad} : G \to \mathbf{U}(L^2(G))$ be the adjoint representation of G, i.e.

$$(\mathrm{ad}(g)f)(h) = f(g^{-1}hg), \text{ for all } g, h \in G.$$

In this paper, all vector spaces are finite dimensional $\mathbb{C}$-spaces. For a Hilbert space V, $\mathbf{U}(V)$ denotes the set of unitary operators on V, $\mathrm{HS}(V)$ denotes the set of Hilbert-Schmidt operators on V. For a matrix M, we denote its transpose by M' and transpose conjugation by M^*.

If $\pi : G \to \mathbf{U}(V)$ is a representation of G on V, we denote it by (π, V) or (π, G, V). We denote the dual representation of π by π^*. A representation π is *multiplicity free* if each irreducible component of π has multiplicity one. A representation π is *unramified* if the tensor product representation $\pi \otimes \pi^*$ is multiplicity free.

2. On cyclic vectors

In this section we study cyclic vectors by adapting the strategy in [6, 19] in the study of tight group frames. We remark that one could also deduce the same results via more algebraic (possibly simpler) argument via the equivalence between $\mathbb{C}[G]$-modules and G-representations. There are two reasons for the choice of our argument as follows. Firstly it is explicit and also reveals properties of the associated frames. Secondly, it is easy to generalize to the case where G is compact or locally compact. The representation theoretic results are certainly known to the experts, what we do in this paper is to connect them to the study of group frames.

Let G be a finite group and $\pi : G \to \mathbf{U}(V)$ be a finite dimensional representation of G. Let $v \in V$ be a nontrivial vector. Define

$$\begin{aligned} S_v : V &\to V \\ u &\mapsto \sum_{g \in G} \langle u, \pi(g)v \rangle \pi(g)v. \end{aligned} \tag{2.1}$$

Then S_v is G-equivariant. The following conditions are equivalent.

(1) v is a cyclic vector for (π, G, V);
(2) $\Phi_v = \{\pi(g)v \mid g \in G\}$ is a frame for V;
(3) S_v is an automorphism.

Moreover, if the above conditions are satisfied, then S_v is the frame operator of the frame Φ_v (cf. [4, 19]).

If π is irreducible, then every nontrivial vector $v \in V$ is a cyclic vector for (π, G, V). Fix such a v. Then $\mathrm{Span}\{\pi(g)v \mid g \in G\} = V$ and $\Phi_v := \{\pi(g)v \mid g \in G\}$ is a tight frame for V. As S_v is G-equivariant, $S_v = \lambda_v \,\mathrm{id}_V$ by Schur's lemma. Moreover, as S_v is the frame operator of a frame, $\lambda_v \in \mathbb{R}_{>0}$. Let $\{w_i \mid 1 \leq i \leq \dim V\}$ be an orthonormal basis of V. For any j, $S_v(w_j) = \lambda_v w_j$. Hence $\lambda_v = \sum_{g \in G} |\langle w_j, \pi(g)v \rangle|^2$, we have

$$\lambda_v \dim V = \sum_j \sum_{g \in G} |\langle w_j, \pi(g)v \rangle|^2 = \sum_{g \in G} ||\pi(g)v||^2 = |G| \cdot ||v||^2.$$

Therefore $S_v = \frac{|G|}{\dim V} ||v||^2 \,\mathrm{id}_V$.

Let $\pi_i : G \to \mathbf{U}(V_i)$ $(i = 1, 2)$ be two representations of G. Let $v_i \in V_i$ be a nontrivial vector. Define

$$\begin{aligned} S_{v_1,v_2} : V_1 &\to V_2 \\ u &\mapsto \sum_{g\in G} \langle u, \pi_1(g)v_1\rangle \pi_2(g)v_2. \end{aligned}$$

Then for any $h \in G$, we have

$$\begin{aligned} S_{v_1,v_2}(\pi_1(h)u) &= \sum_{g\in G} \langle \pi_1(h)u, \pi_1(g)v_1\rangle \pi_2(g)v_2 \\ &= \sum_{g\in G} \langle u, \pi_1(h^{-1}g)v_1\rangle \pi_2(h)\pi_2(h^{-1}g)v_2 \\ &= \pi_2(h) \sum_{g\in G} \langle u, \pi_1(h^{-1}g)v_1\rangle \pi_2(h^{-1}g)v_2 \\ &= \pi_2(h)S_{v_1,v_2}(u). \end{aligned}$$

Hence S_{v_1,v_2} is G-equivariant. In particular, if $\mathrm{Hom}_G(V_1, V_2) = \{0\}$, then $S_{v_1,v_2} = 0$.

Lemma 2.1. *With the notation as above, if π_1 and π_2 are isomorphic irreducible representations and $\sigma : V_1 \to V_2$ is an isomorphism of G-representations, then*

$$S_{v_1,v_2}(u) = \frac{|G| \cdot ||v_1||^2}{(\dim V_1)||\sigma v_1||^2} \langle v_2, \sigma v_1\rangle \sigma(u) \text{ for all } u \in V_1.$$

In particular, if $V_1 = V_2$, then

$$S_{v_1,v_2}(u) = \frac{|G|}{\dim V_1} \langle v_2, v_1\rangle u \text{ for all } u \in V_1.$$

Proof. By Schur's lemma, $S_{v_1,v_2} = \lambda\sigma$ for a constant number λ. Let $u = v_1$, we have

$$\begin{aligned} \langle S_{v_1,v_2}(v_1), \sigma v_1\rangle &= \langle \sum_{g\in G} \langle v_1, \pi_1(g)v_1\rangle \pi_2(g)v_2, \sigma v_1\rangle \\ &= \sum_{g\in G} \langle v_1, \pi_1(g)v_1\rangle \langle \pi_2(g)v_2, \sigma v_1\rangle \\ &= \sum_{g\in G} \langle \pi_1(g^{-1})v_1, v_1\rangle \langle v_2, \sigma(\pi_1(g^{-1})v_1)\rangle \\ &= \langle v_2, \sigma(\sum_{g\in G} \langle v_1, \pi_1(g)v_1\rangle \pi_1(g)v_1)\rangle \\ &= \langle v_2, \sigma(\frac{|G|}{\dim V_1} ||v_1||^2 v_1)\rangle. \end{aligned}$$

The lemma follows. □

Lemma 2.2. *Assume that $(\pi, V) = (\pi_1, V_1) \oplus (\pi_2, V_2)$ is a direct sum of two representations with $\mathrm{Hom}_G(V_1, V_2) = \{0\}$. Let $v \in V$ be a nontrivial vector and we write $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. Then v is a cyclic vector for π if and only if v_i is a cyclic vector for π_i $(i = 1, 2)$.*

Proof. If v is a cyclic vector for π, it is obvious that v_i is a cyclic vector for π_i. We prove the converse. Assume that v_i is a cyclic vector for π_i $(i = 1,\ 2)$, we show that

$$\operatorname{Span}\{\pi(g)v \mid g \in G\} = V. \tag{2.2}$$

Define $S_v : V \to V$ as in equation (2.1). For any $u = u_1 + u_2 \in V$ with $u_i \in V_i$ $(i = 1,\ 2)$, we have

$$\begin{aligned} S_v(u) &= \sum_{g\in G} \langle u, \pi(g)v\rangle \pi(g)(v) \\ &= \sum_{g\in G} \langle u_1 + u_2, \pi_1(g)v_1 + \pi_2(g)v_2\rangle (\pi_1(g)v_1 + \pi_2(g)v_2) \\ &= \sum_{g\in G} \langle u_1, \pi_1(g)v_1\rangle \pi_1(g)v_1 + \sum_{g\in G} \langle u_2, \pi_2(g)v_2\rangle \pi_2(g)v_2 + S_{v_1,v_2}(u_1) + S_{v_2,v_1}(u_2) \\ &= \sum_{g\in G} \langle u_1, \pi_1(g)v_1\rangle \pi_1(g)v_1 + \sum_{g\in G} \langle u_2, \pi_2(g)v_2\rangle \pi_2(g)v_2. \end{aligned}$$

As v_i is a cyclic vector for π_i $(i = 1,\ 2)$, $\Phi_{v_i} = \{\pi_i(g)v_i \mid g \in G\}$ is a frame for V_i and the map

$$u_i \mapsto \sum_{g\in G} \langle u_i, \pi_i(g)v_i\rangle \pi_i(g)v_i$$

is the frame operator of Φ_i, hence induces an automorphism of V_i. Therefore $S_v : V \to V$ is an automorphism and equation (2.2) holds. The lemma follows. □

Applying the lemma repeatedly, we obtain the following result.

Proposition 2.3. *Let I be a finite index set and $(\pi, V) = \oplus_{i\in I}(\pi_i, V_i)$ be a decomposition of π. Let $v \in V$ and write $v = \sum_{i\in I} v_i$ with $v_i \in V_i$. Assume that for any $i \neq j$ in I, $\operatorname{Hom}_G(V_i, V_j) = \{0\}$. Then v is a cyclic vector for (π, V) if and only if v_i is a cyclic vector for (π_i, V_i) for all $i \in I$.*

To understand cyclic vectors, we still have to treat the case where $\operatorname{Hom}_G(V_i, V_j)$ are nontrivial. We start with the following lemma.

Lemma 2.4. *Let $\rho : G \to \mathbf{U}(W)$ be an irreducible representation of G. Assume that $(\pi, V) = \oplus_{i=1}^n(\pi_i, V_i)$ such that $(\pi_i, V_i) = (\rho, W)$ for $i = 1,\ \dots,\ n$. Let $v \in V$ be a nontrivial vector and we write $v = v_1 \oplus \cdots \oplus v_n$ with $v_i \in V_i = W$. Then v is a cyclic vector for π if and only if $v_1,\ v_2,\ \dots,\ v_n$ are linearly independent. In particular, V is a cyclic representation if and only if $n \le \dim W$.*

Proof. For $1 \le i, j \le n$, let $\lambda_{ji} = \frac{|G|}{\dim W}\langle v_j, v_i\rangle$. Define S_v as in equation (2.1). For any $u_i \in V_i = W$, let $\tilde{u}_i$ be the element of V with j-th component 0 for $j \neq i$ and i-th component u_i. We then have

$$\begin{aligned} S_v(\tilde{u}_i) &= \sum_{g\in G} \langle \tilde{u}_i, \pi(g)v\rangle \pi(g)v \\ &= \oplus_{j=1}^n \sum_{g\in G} \langle u_i, \pi_i(g)v_i\rangle \pi_j(g)v_j \\ &= \oplus_{j=1}^n \sum_{g\in G} \langle u_i, \rho(g)v_i\rangle \rho(g)v_j \\ &= \oplus_{j=1}^n \lambda_{ji} u_i. \end{aligned}$$

The last identity follows from Lemma 2.1. Therefore, the operator $S_v : V \to V$ is given by the matrix

$$\begin{pmatrix} \lambda_{11}\,\mathrm{id}_W & \lambda_{12}\,\mathrm{id}_W & \cdots & \lambda_{1n}\,\mathrm{id}_W \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1}\,\mathrm{id}_W & \lambda_{n2}\,\mathrm{id}_W & \cdots & \lambda_{nn}\,\mathrm{id}_W \end{pmatrix} = \left(\lambda_{ij}\right)_{1\le i,j\le n} \otimes \mathrm{id}_W\,.$$

Therefore v is a cyclic vector for (π, V) if and only if $\det\left(\lambda_{ij}\right)_{1\le i,j\le n} \neq 0$, hence if and only if $v_1,\ v_2,\ \ldots,\ v_n$ are linearly independent by the definition of λ_{ij}. The lemma follows. □

Proposition 2.5. *Let $\rho : G \to \mathbf{U}(W)$ be an irreducible representation of G. Assume that $(\pi, V) = \oplus_{i=1}^n(\pi_i, V_i)$ such that $(\pi_i, V_i) \cong (\rho, W)$ for $i = 1,\ \ldots,\ n$. For each $1 \le i \le n$, fix an isomorphism of G-representations $\sigma_i : V_i \to W$. Let $v \in V$ be a nontrivial vector and we write $v = v_1 + \cdots + v_n$ with $v_i \in V_i$. Then v is a cyclic vector for π if and only if $\sigma_1 v_1,\ \sigma_2 v_2,\ \ldots,\ \sigma_n v_n \in W$ are linearly independent.*

Proof. If $r_k : G \to \mathbf{U}(W_i)$ $(i = 1, 2)$ are two equivalent representations of G and $\tau : W_1 \to W_2$ is a G-equivariant isomorphism, it is easy to see that $w_1 \in W_1$ is a cyclic vector for (r_1, W_1) if and only if $\tau(w_1) \in W_2$ is a cyclic vector for (r_2, W_2). Applying this observation to the two representations (π, V) and $(\rho^n, W^{\oplus n})$ with $\tau = \sigma_1 \oplus \cdots \oplus \sigma_n$, the proposition follows from Lemma 2.4. □

From Propositions 2.3 and 2.5, we have the following result.

Corollary 2.6. *A representation of G is cyclic if and only if it is a sub representation of the regular representation.*

Now we apply Proposition 2.3 to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $\dim V = n$. Fix a basis of V, we may identify $V \otimes V$ with $M_n(\mathbb{C})$ equipped with inner product

$$\langle A, B\rangle = \mathrm{Tr}\, B^* A, \text{ for } A,\ B \in M_n(\mathbb{C}).$$

By Proposition 2.3, to prove Theorem 1.1, it suffices to show that for any orthogonal decomposition $M_n(\mathbb{C}) = M_1 \oplus \cdots \oplus M_s$, there exists a vector $v = (v_1,\ \ldots,\ v_n)' \in \mathbb{C}^n$, such that the projection of vv^* in M_i is nonzero for all $1 \le i \le s$.

For each $1 \le i \le s$, fix a basis $A_{i,1},\ \ldots,\ A_{i,r_i}$ of M_i. Then vv^* projects to the zero matrix in M_i if and only if for all $A \in \{A_{i,1},\ \ldots,\ A_{i,r_i}\}$,

$$0 = \langle vv^*, A\rangle = \mathrm{Tr}(A^* vv^*) = \mathrm{Tr}(v^* A^* v) = \langle v, Av\rangle.$$

Note that A is not the zero matrix, $\langle v, Av\rangle$ is a nontrivial polynomial of the real and imaginary parts of the coordinates of v. One sees that the set of $v \in \mathbb{C}^n$ such that v^*v

projects to a nonzero matrix in M_i for all $1 \le i \le s$ is open dense in V. The theorem then follows. □

Theorem 1.1 leaves us the ramified case. From Proposition 2.5, if we have enough information on the decomposition of $V \otimes V$, it is possible to find a maximal spanning vector for (π, G, V). We illustrate the idea in Section 4 for the general affine group $\mathrm{GA}(1, q)$. Moreover, from the proof of Theorem 1.1, in the unramified case, if we have enough information on the decomposition of $V \otimes V$, e.g. we have a nice basis for each M_i, we may find all maximal spanning vectors, which is important for constructing phase retrievable frames. We illustrate this idea in Section 5 for certain metabelian groups. We note that to make the decomposition explicit, it is better to work with the representation (ad, G, C_π) via the isomorphism $\mathcal{E}$, and this is what we do in the rest of this paper.

3. Basics of the affine group $\mathrm{GA}(1, q)$

In this section, we review the basic properties of $\mathrm{GA}(1, q)$. These properties are elementary and scattered in textbooks and online notes. We collect them for the convenience of the readers and fix the notation along the way.

3.1. **The group structure.** Let $\mathbb{F}_q$ be a finite field of characteristic p. Let $\mathrm{GA}(1, q) = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$ be the affine group over the finite field $\mathbb{F}_q$. We denote the elements of $\mathrm{GA}(1, q)$ by (a, b) with $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q^\times$. We have the following identities.

(1) $(a, b)(c, d) = (a + bc, bd)$.
(2) $(a, b)^{-1} = (-ab^{-1}, b^{-1})$.
(3) $(a, b)^{-1}(c, d)(a, b) = (-ab^{-1} + ab^{-1}d + b^{-1}c, d)$.

3.2. **The characters of $\mathbb{F}_q^\times$ and $\mathbb{F}_q$.** The multiplicative group $\mathbb{F}_q^\times$ is cyclic of order $(q-1)$. Fix $\gamma \in \mathbb{F}_q^\times$ a generator of $\mathbb{F}_q^\times$. Let $\chi_1, \dots, \chi_{q-1}$ be the characters (one-dimensional linear representations) of $\mathbb{F}_q^\times$. Assume that χ_1 is the trivial character of $\mathbb{F}_q^\times$.

Fix $\psi : \mathbb{F}_q \to \mathbb{C}$ a nontrivial character of the additive group $\mathbb{F}_q$, e.g.

$$\psi(a) = \exp \frac{2\pi\sqrt{-1}\,\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p}.$$

Then the characters of $\mathbb{F}_q$ are $\{\psi_c \mid c \in \mathbb{F}_q\}$, where $\psi_c(a) = \psi(ca)$ for all $a \in \mathbb{F}_q$. The map $c \mapsto \psi_c$ is an isomorphism between $\mathbb{F}_q$ and the dual group of $\mathbb{F}_q$. In particular, $\{\psi_{\gamma^m} \mid 0 \le m \le q-2\}$ is the set of nontrivial characters of $\mathbb{F}_q$.

3.3. **Conjugacy classes of $\mathrm{GA}(1, q)$.** There are q conjugacy classes of $\mathrm{GA}(1, q)$. Their representatives are given in the following table.

Table 1. Conjugacy classes of $\mathrm{GA}(1, q)$

Representatives	No. of elements in class	No. of classes
$(0, 1)$	1	1
$(1, 1)$	$q-1$	1
$(1, d \neq 1)$	q	$q-2$

3.4. **Representations of** $\mathrm{GA}(1,q)$**.** There are $(q-1)$ one-dimensional representations of $G=\mathrm{GA}(1,q)$ given by $G\to\mathbb{F}_q^\times\xrightarrow{\chi_i}\mathbb{C}^\times$. Denote these representations by $\chi_1,\cdots,\chi_{q-1}$ as well.

There is a $(q-1)$-dimensional irreducible representation of G given by $\mathrm{Ind}_{\mathbb{F}_q}^G\psi$. Denote this representation by π. Then

$$\pi|_{\mathbb{F}_q}=(\mathrm{Ind}_{\mathbb{F}_q}^G\psi)|_{\mathbb{F}_q}\cong\oplus_{c\neq 0}\psi_c=\oplus_{i=0}^{q-2}\psi_{\gamma^i}.$$

Under this identification and fix a basis for each ψ_{γ^i}, the representation π is given by

$$\pi(c,1)=\mathrm{diag}(\psi_1(c),\ \psi_\gamma(c),\ \ldots,\ \psi_{\gamma^{q-2}}(c)),$$

$$\pi(0,\gamma)=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&1\\1&0&0&\cdots&0\end{pmatrix}. \tag{3.1}$$

The character table of $\mathrm{GA}(1,q)$ is as follows.

TABLE 2. The character table of $\mathrm{GA}(1,q)$

Representations	χ_1	$\cdots$	χ_{q-1}	π
$(0,1)$	1	$\cdots$	1	$q-1$
$(1,1)$	1	$\cdots$	1	-1
$(1,d\neq 1)$	$\chi_1(d)$	$\cdots$	$\chi_{q-1}(d)$	0

Let χ_{ad} be the character of the adjoint representation $(\mathrm{ad},L^2(G))$. Then $\chi_{\mathrm{ad}}(g)=\sharp\{h\in G\mid g^{-1}hg=h\}$. We then have

$$\chi_{\mathrm{ad}}(0,1)=q(q-1),$$
$$\chi_{\mathrm{ad}}(1,1)=q,$$
$$\chi_{\mathrm{ad}}(1,d\neq 1)=q-1.$$

Therefore $\chi_{\mathrm{ad}}=(q-2)\chi_\pi+(q-1)\chi_1+\sum_{i=1}^{q-1}\chi_i$, and

$$(\mathrm{ad},L^2(G))\cong\pi^{\oplus(q-2)}\oplus\chi_1^{\oplus(q-1)}\oplus(\oplus_{i=1}^{q-1}\chi_i).$$

3.5. **Decomposition of** $L^2(G)$**.** Let $r:G\to\mathbf{U}(L^2(G))$ be the right regular representation. For $(c,d)\in G$, let $\delta_{(c,d)}$ be the characteristic function of the point set $\{(c,d)\}$. Let $\delta_d=\sum_{c\in\mathbb{F}_q}\delta_{(c,d)}$. Define

$$L^2(G)^\circ:=\{f\in L^2(G)\mid f(c,d)=f(0,d)\text{ for all }c\in\mathbb{F}_q\}=\mathrm{Span}\{\delta_d\mid d\in\mathbb{F}_q^\times\}.$$

Let $L^2(G)'$ be the orthogonal complement of $L^2(G)^\circ$ in $L^2(G)$. As

$$r(a,b)\delta_{(c,d)}=\delta_{(c-b^{-1}da,b^{-1}d)},$$

we have $r(a,b)\delta_d=\delta_{b^{-1}d}$, and

$$(r,L^2(G))\cong\oplus_{i=1}^{q-1}\chi_i\oplus\pi^{\oplus(q-1)},$$
$$(r,L^2(G)^\circ)\cong\oplus_{i=1}^{q-1}\chi_i, \tag{3.2}$$
$$(r,L^2(G)')\cong\pi^{\oplus(q-1)}.$$

For $d \in \mathbb{F}_q^\times$, define $L^2(G)_d = \mathrm{Span}\{\delta_{(c,d)} \mid c \in \mathbb{F}_q\}$, $L^2(G)_d^\circ = \mathbb{C}\langle \delta_d \rangle$, $L^2(G)'_d$ the orthogonal complement of $L^2(G)_d^\circ$ in $L^2(G)_d$.

Lemma 3.1. *With the notation as above, we have*

$$(\mathrm{ad}, L^2(G)_d^\circ) \cong \chi_1, \ \textit{for } d \in \mathbb{F}_q^\times,$$
$$(\mathrm{ad}, L^2(G)'_1) \cong \oplus_{i=1}^{q-1} \chi_i,$$
$$(\mathrm{ad}, L^2(G)'_d) \cong \pi, \ \textit{for } d \neq 1.$$

Proof. By direct computation, we have

$$(\mathrm{ad}(a,b)\delta_{(c,d)})(x,y) = \delta_{(c,d)}(ab^{-1}(y-1) + b^{-1}x, y).$$

Hence $\mathrm{ad}(a,b)\delta_{(c,d)} = \delta_{(bc+a-ad,d)}$. In particular, $\mathrm{ad}(a,b)\chi_i = \chi_i$ and $\mathrm{ad}(a,b)\delta_{(c,1)} = \delta_{(bc,1)}$. The first isomorphism is clear. By checking the character of $(\mathrm{ad}, L^2(G)_d)$, the other two isomorphisms follow easily. □

3.6. **The coefficient $\mathfrak{c}_{v,v}$.** Let $\pi : G \to \mathbf{U}(V)$ be the $(q-1)$-dimensional irreducible representation and $v \in V$ be a nontrivial vector. Let $\mathfrak{c}_{v,v} : G \to \mathbb{C}$ be the matrix coefficient $g \mapsto \langle \pi(g)v, v\rangle$. As $\mathfrak{c}_{v,v} \in L^2(G)$ is a matrix coefficient of π, by equation (3.2), we have $\mathfrak{c}_{v,v} \in L^2(G)'$.

Let $\mathfrak{c}_{v,v;d}$ be the projection of $\mathfrak{c}_{v,v}$ in $L^2(G)_d$. Then

$$\mathfrak{c}_{v,v;d} = \sum_{c \in \mathbb{F}_q} \mathfrak{c}_{v,v}(c,d)\delta_{(c,d)}.$$

As $\mathfrak{c}_{v,v} \in L^2(G)'$, it has trivial component in $\mathbb{C}\langle\delta_d\rangle$. Hence $\mathfrak{c}_{v,v;d} \in L^2(G)'_d$.

From equation (1.1), we have

$$\mathfrak{c}_{\pi(a,b)v,\pi(a,b)v} = \mathrm{ad}(a,b)\mathfrak{c}_{v,v}.$$

We then obtain the following lemma. It also says that π is ramified when $q \geq 4$.

Lemma 3.2. *With the notation as above, v is a maximal spanning vector for (π, G, V) if and only if $\mathfrak{c}_{v,v}$ is a cyclic vector for the representation $(\mathrm{ad}, L^2(G)') \cong \pi^{\oplus(q-2)} \oplus \chi_1 \oplus \cdots \oplus \chi_{q-1}$.*

4. The maximal spanning vectors for $(\pi, \mathrm{GA}(1,q), \mathbb{C}^{q-1})$

In this section, we continue with the notation in Section 3. In particular, $G = \mathrm{GA}(1,q)$ and $\pi : G \to \mathbf{U}(V)$ is the representation of G given by equation (3.1). In $L^2(G)_1$, the $(q-1)$ elements

$$\sum_{c \in \mathbb{F}_q} \psi_d(c)\delta_{(c,1)}, \quad d \in \mathbb{F}_q^\times,$$

form a basis of $L^2(G)'_1$. For each $1 \leq i \leq q-1$, let K_i be the one-dimensional space with basis

$$\sum_{d \in \mathbb{F}_q^\times} \chi_i(d) \sum_{c \in \mathbb{F}_q} \psi_d(c)\delta_{(c,1)}.$$

We then have

$$\begin{aligned}\operatorname{ad}(a,b)(\sum_{d\in\mathbb{F}_q^\times}\chi_i(d)\sum_{c\in\mathbb{F}_q}\psi_d(c)\delta_{(c,1)}) &= \sum_{d\in\mathbb{F}_q^\times}\chi_i(d)\sum_{c\in\mathbb{F}_q}\psi_d(c)\delta_{(bc,1)}\\ &= \sum_{d\in\mathbb{F}_q^\times}\chi_i(d)\sum_{c\in\mathbb{F}_q}\psi_{db^{-1}}(c)\delta_{(c,1)}\\ &= \chi_i(b)\sum_{d\in\mathbb{F}_q^\times}\chi_i(db^{-1})\sum_{c\in\mathbb{F}_q}\psi_{db^{-1}}(c)\delta_{(c,1)}\\ &= \chi_i(b)\sum_{d\in\mathbb{F}_q^\times}\chi_i(d)\sum_{c\in\mathbb{F}_q}\psi_d(c)\delta_{(c,1)}.\end{aligned}$$

Therefore, $(\operatorname{ad}, K_i) = \chi_i$. We have the following decomposition

$$\operatorname{ad}|_{L^2(G)'} = \oplus_{\substack{d\in\mathbb{F}_q^\times\\ d\neq 1}}(\operatorname{ad}, L^2(G)'_d)\oplus(\chi_1, K_1)\oplus\cdots\oplus(\chi_{q-1}, K_{q-1})$$

and $(\operatorname{ad}, L^2(G)'_d)\cong\pi$ for all $d\neq 1$. For $d,\ d'\neq 1$, define

$$\begin{aligned}\sigma_{d,d'} : L^2(G)_d &\to L^2(G)_{d'}\\ \delta_{(c,d)} &\mapsto \delta_{(\frac{1-d'}{1-d}c,d')}.\end{aligned}$$

Direct computation shows that

$$\sigma_{d,d'}(\operatorname{ad}(a,b)\delta_{(c,d)}) = \delta_{(\frac{1-d'}{1-d}(bc+a-ad),d')} = \operatorname{ad}(a,b)(\sigma_{d,d'}(\delta_{(c,d)})).$$

Hence it induces an isomorphism of irreducible G-representations

$$\sigma_{d,d'} : (\operatorname{ad}, L^2(G)'_d)\to(\operatorname{ad}, L^2(G)'_{d'}).$$

Combining Lemma 3.2, Proposition 2.3, Proposition 2.5, we have the following lemma.

Lemma 4.1. *Let $v\in V$ be a vector. Then v is maximal spanning for (π, G, V) if and only if the following two conditions are satisfied:*

1. *the projection of $\mathfrak{c}_{v,v}$ in K_i is nonzero for all $1\leq i\leq q-1$;*
2. *the vectors $\sigma_{\gamma^m,\gamma}(\mathfrak{c}_{v,v;\gamma^m})$ $(1\leq m\leq q-2)$ are linearly independent in $L^2(G)'_\gamma$, where $\mathfrak{c}_{v,v;\gamma^m}$ is the projection of $\mathfrak{c}_{v,v}$ in $L^2(G)'_{\gamma^m}$.*

The reason we use $\{\gamma^m\}$ as the index set will be clear soon. For $1\leq i\leq q-1$, define

$$N_i = \{v\in V\mid \mathfrak{c}_{v,v}\text{ projects to zero in }K_i\},$$

and define

$$N_0 = \{v\in V\mid \sigma_{\gamma^m,\gamma}(\mathfrak{c}_{v,v;\gamma^m})\ (1\leq m\leq q-2)\text{ are linearly dependent in }L^2(G)'_\gamma\}.$$

Lemma 4.2. *For all $1\leq i\leq q-1$, $V-N_i$ is open dense in V.*

Proof. From the construction, $v \in N_i$ if and only if

$$\begin{aligned} 0 &= \langle \mathfrak{c}_{v,v}, \sum_{d\in\mathbb{F}_q^\times} \chi_i(d) \sum_{c\in\mathbb{F}_q} \psi_d(c)\delta_{(c,1)}\rangle \\ &= \sum_{d\in\mathbb{F}_q^\times} \overline{\chi_i(d)} \sum_{c\in\mathbb{F}_q} \langle \mathfrak{c}_{v,v}, \psi_d(c)\delta_{(c,1)}\rangle \\ &= \sum_{d\in\mathbb{F}_q^\times} \overline{\chi_i(d)} \sum_{c\in\mathbb{F}_q} \overline{\psi_d(c)}\mathfrak{c}_{v,v}(c,1) \\ &= \langle \sum_{d\in\mathbb{F}_q^\times} \overline{\chi_i(d)} \sum_{c\in\mathbb{F}_q} \overline{\psi_d(c)}\pi(c,1)v, v\rangle. \end{aligned}$$

From the matrix form of π given by equation (3.1), $\pi(c,1) = \mathrm{diag}(\psi_1(c), \ldots, \psi_{\gamma^{q-2}}(c))$. As the characters ψ_d are orthogonal, we have

$$\sum_{d\in\mathbb{F}_q^\times} \overline{\chi_i(d)} \sum_{c\in\mathbb{F}_q} \overline{\psi_d(c)}\pi(c,1) = q\,\mathrm{diag}(\overline{\chi_i(1)}, \ldots, \overline{\chi_i(\gamma^{q-2})}).$$

Write $v = (v_0, \ldots, v_{q-2})' \in \mathbb{C}^{q-1}$, then N_i is the zero set of nontrivial polynomials on the real and imaginary parts of v_i, hence $V - N_i$ is open dense in V. The lemma follows. ☐

Lemma 4.3. *The set $V - N_0$ is open dense in V.*

Proof. For $1 \le m \le q-2$, the $(q-1)$ elements

$$\frac{1}{q}\sum_{c\in\mathbb{F}_q} \psi(c)\delta_{(c,\gamma^m)}, \ \frac{1}{q}\sum_{c\in\mathbb{F}_q} \psi_\gamma(c)\delta_{(c,\gamma^m)}, \ \ldots, \ \frac{1}{q}\sum_{c\in\mathbb{F}_q} \psi_{\gamma^{q-2}}(c)\delta_{(c,\gamma^m)} \tag{4.1}$$

form an orthonormal basis of $L^2(G)'_{\gamma^m}$. The inner product of $\mathfrak{c}_{v,v}$ and $\sum_{c\in\mathbb{F}_q}\psi_{\gamma^i}(c)\delta_{(c,\gamma^m)}$ is

$$\begin{aligned} \langle \mathfrak{c}_{v,v}, \sum_{c\in\mathbb{F}_q} \psi_{\gamma^i}(c)\delta_{(c,\gamma^m)}\rangle &= \sum_{c\in\mathbb{F}_q} \overline{\psi_{\gamma^i}(c)}\mathfrak{c}_{v,v}(c,\gamma^m) \\ &= \langle \sum_{c\in\mathbb{F}_q} \overline{\psi_{\gamma^i}(c)}\pi(c,\gamma^m)v, v\rangle \\ &= \langle (\sum_{c\in\mathbb{F}_q} \overline{\psi_{\gamma^i}(c)}\pi(c,1))\pi(0,\gamma^m)v, v\rangle. \end{aligned}$$

Via equation (3.1) and the orthogonal relations of characters, $\sum_{c\in\mathbb{F}_q}\overline{\psi_{\gamma^i}(c)}\pi(c,1)$ is the $(q-1)\times(q-1)$ matrix with (i,i)-entry q and other entries 0. Write $v = (v_0, \ldots, v_{q-2})' \in \mathbb{C}^{q-1}$, we then have

$$\langle \mathfrak{c}_{v,v}, \sum_{c\in\mathbb{F}_q} \psi_{\gamma^i}(c)\delta_{(c,\gamma^m)}\rangle = q\overline{v_i}v_{i+m}.$$

Here the subscripts are understood modulo $(q-1)$. Therefore, under the basis given by equation (4.1), the projection of $\mathfrak{c}_{v,v}$ in $L^2(G)'_{\gamma^m}$ has coordinate

$$(\overline{v_0}v_m, \ \overline{v_1}v_{1+m}, \ \overline{v_2}v_{2+m}, \ \ldots, \ \overline{v_{q-2}}v_{q-2+m}).$$

Moreover, we have

$$\sigma_{\gamma^m,\gamma}(\sum_{c\in\mathbb{F}_q}\psi_{\gamma^i}(c)\delta_{(c,\gamma^m)}) = \sum_{c\in\mathbb{F}_q}\psi_{\gamma^i}(c)\delta_{(\frac{1-\gamma}{1-\gamma^m}c,\gamma)}$$
$$= \sum_{c\in\mathbb{F}_q}\psi_{\gamma^i}(\frac{1-\gamma^m}{1-\gamma}c)\delta_{(c,\gamma)} = \sum_{c\in\mathbb{F}_q}\psi_{\gamma^{i+j(m)}}(c)\delta_{(c,\gamma)},$$

where $j(m)\in\{0,\ 1,\ 2,\ \dots,\ q-2\}$ with $\frac{1-\gamma^m}{1-\gamma}=\gamma^{j(m)}$. Hence under the basis given by equation (4.1), $\sigma_{\gamma^m,\gamma}(\mathfrak{c}_{v,v;\gamma^m})$ has coordinate

$$(\overline{v_{q-2-j(m)+1}}v_{q-2-j(m)+1+m},\ \overline{v_{q-2-j(m)+2}}v_{q-2-j(m)+2+m},\ \dots,$$
$$\overline{v_{q-2}}v_{q-2+m},\ \overline{v_0}v_m,\ \dots,\ \overline{v_{q-2-j(m)}}v_{q-2-j(m)+m}).$$

Denote this row vector as $\mathfrak{p}_m\in\mathbb{C}^{q-1}$. From the construction, $v\in V-N_0$ if and only if the $(q-2)\times(q-1)$ matrix

$$\mathfrak{P} := \big(\mathfrak{p}_m\big)_{1\le m\le q-2}$$

has rank $q-2$. Note that $j(1)=0$ and $j(m)\neq j(m')$ for $m\neq m'$, $j(m)$ runs through the set $\{0,\ 1,\ \dots,q-2\}$ except one nonzero element. Let J be the element not in the image of j. Let $\mathfrak{P}_J$ be the sub-matrix of $\mathfrak{P}$ by deleting the J-th column of $\mathfrak{P}$. We claim that $\det(P_J)$ is a nonzero polynomial with respect to the real and imaginary parts of v_0, …, v_{q-2}. Indeed, by expanding $\det(P_J)$, there are two terms involving v_0 and $\overline{v_0}$ with highest degree

$$\pm\overline{v_0}v_1\overline{v_0}v_2\cdots\overline{v_0}v_{q-2} \text{ and } \pm\overline{v_{q-2}}v_0\overline{v_{q-3}}v_0\cdots\overline{v_1}v_0,$$

the claim follows easily. Therefore the set of $v\in V$ with $\mathrm{rank}(\mathfrak{P})=q-2$ is open dense in V and the lemma follows. □

Now it is easy to prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. By Lemma 4.1, the set of maximal spanning vectors for $(\pi,\mathrm{GA}(1,q),V)$ is

$$\bigcap_{i=0}^{q-1}(V-N_i).$$

By Lemmas 4.2 and 4.3, $V-N_i$ is open dense in V for all $0\le i\le q-1$, therefore the set of maximal spanning vectors for $(\pi,\mathrm{GA}(1,q),V)$ is open dense in V and the theorem follows. □

Remark 4.4. From the proof of Lemmas 4.2 and 4.3, write $v=(v_0,\ \dots,\ v_{q-2})'\in\mathbb{C}^{q-1}$, then v is maximal spanning for $(\pi,G,\mathbb{C}^{q-1})$ if and only if $\mathrm{rank}(\mathfrak{P})=q-2$ and $|v_0|^2+\zeta|v_2|^2+\cdots+\zeta^{q-2}|v_{q-2}|^2\neq 0$, for any $(q-1)$-th root of unity ζ. See [1, Theorem 6.5] for the special case where q is a prime number.

Proof of Corollary 1.3. The $q=2$ case is trivial. Assume that $q\ge 3$. Let $\rho:\mathrm{GL}_2(\mathbb{F}_q)\to\mathbf{U}(\mathbb{C}^{q-1})$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$. We identify $\mathrm{GA}(1,q)$ with a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ via

$$\begin{aligned} \mathrm{GA}(1,q) &\to \{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)\} \leq \mathrm{GL}_2(\mathbb{F}_q) \\ (a,b) &\mapsto \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then $\rho|_{\mathrm{GA}(1,q)} \cong \pi$ (cf. [3, Section 6] and [11, Section 5.2]). Hence the set of maximal spanning vectors for $(\rho, \mathrm{GL}_2(\mathbb{F}_q), \mathbb{C}^{q-1})$ is open dense and ρ does phase retrieval.

Let $\rho : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbf{U}(\mathbb{C}^q)$ be a special representation of $\mathrm{GL}_2(\mathbb{F}_q)$. From the character table for $\mathrm{GL}_2(\mathbb{F}_q)$ (cf. [11, Page 70]), we claim that $(\rho, \mathrm{GL}_2(\mathbb{F}_q), \mathbb{C}^q)$ is unramified. We use the notation as in [11, Section 5.2] in the following. In particular, the special representations are denoted by V_α, where $\alpha : \mathbb{F}_q^\times \to \mathbb{T}$ is a character. Since $V_\alpha \otimes V_\alpha^* = V \otimes V^*$, we may assume that α is trivial for simplicity. On the four conjugacy classes a_x, b_x, $c_{x,y}$, $d_{x,y}$, the character of $V \otimes V^*$ takes value

$$q^2,\ 0,\ 1,\ 1,\ \text{respectively.}$$

Then

$$\begin{aligned} &\langle \chi_{V\otimes V^*}, \chi_{V\otimes V^*} \rangle \\ =& \sum_{x\in\mathbb{F}_q^\times} q^2 \cdot q^2 + 0 + \sum_{\substack{x,y\in\mathbb{F}_q^\times \\ x\neq y}} (1\cdot 1)(q^2+q) + \sum_{\substack{x,y\mathbb{F}_q \\ y\neq 0}} (1\cdot 1)(q^2-q) \\ =& q^4(q-1) + 0 + q(q+1)\frac{(q-1)(q-2)}{2} + q(q-1)\frac{q(q-1)}{2} \\ =& |\mathrm{GL}_2(\mathbb{F}_q)|(q+1). \end{aligned}$$

It suffices to find $(q+1)$ irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ that appear in $V \otimes V^*$. We explain the details if q is odd and state the minor difference if q is even.

(1) The trivial character appears in $V \otimes V^*$ with multiplicity one since $V \cong V^*$.

(2) If $\alpha : \mathbb{F}_q^\times \to \mathbb{T}$ is not the trivial character, by direct computation, the representation $W_{\alpha,\alpha^{-1}}$ appears in $V \otimes V^*$ with multiplicity one. If $\alpha \neq \alpha^{-1}$, then $W_{\alpha,\alpha^{-1}}$ is irreducible. Note that $W_{\alpha,\beta} = W_{\beta,\alpha}$, there are $\frac{q-3}{2}$ $(q+1)$-dimensional irreducible representations in $V \otimes V^*$. (If q is even, there are $\frac{q-2}{2}$ $(q+1)$-dimensional irreducible representations in $V \otimes V^*$.)

If $\alpha = \alpha^{-1}$, i.e. $\alpha^2 = 1$, then $W_{\alpha,\alpha^{-1}} = W_{\alpha,\alpha} = U_\alpha \oplus V_\alpha$. Direct computation shows that U_α, the one-dimensional representation associated with α, does not appear in $V \otimes V^*$. Hence in this case V_α $(\alpha^2 = 1)$ appears in $V \otimes V^*$. (If q is even, this case does not happen.)

(3) If $\alpha : \mathbb{F}_q^\times \to \mathbb{T}$ is the trivial character, by direct computation, $\langle \chi_{W_{\alpha,\alpha^{-1}}}, \chi_{V\otimes V^*} \rangle = 2|\mathrm{GL}_2(\mathbb{F}_q)|$. Note that in this case $W_{\alpha,\alpha^{-1}}$ is the direct sum of the trivial character and V. Therefore V appears in $V \otimes V^*$ with multiplicity one.

(4) Let $\varphi : \mathbb{F}_{q^2}^\times \to \mathbb{T}$ be a character of the multiplicative group of the quadratic extension of $\mathbb{F}_q$. Then one checks that $\langle \chi_{X_\varphi}, \chi_{V\otimes V^*} \rangle = 1$ if and only if φ^{q+1} is trivial, or equivalently, $\varphi(\mathbb{F}_q^\times) = 1$. As $X_\varphi = X_{\varphi^q}$, there are $\frac{q-1}{2}$ $(q-1)$-dimensional irreducible representations in $V \otimes V^*$. (If q is even, there are $\frac{q}{2}$ $(q-1)$-dimensional irreducible representations in $V \otimes V^*$.)

The claim holds and the corollary then follows from Theorem 1.1. □

Remark 4.5. The principal series representations $W_{\alpha,\beta}$ ($\alpha \neq \beta$) of $\mathrm{GL}_2(\mathbb{F}_q)$ are ramified and $W_{\alpha,\beta}|_{\mathrm{GA}(1,q)}$ is just $(q+1)$-copies of the trivial character (cf. [3, Section 6]). The above method does not apply to these representations. For special representations of $\mathrm{GL}_2(\mathbb{F}_q)$, although there are plenty of maximal spanning vectors, it is not easy to write one explicitly.

5. On other metabelian groups

Let $\pi : G \to \mathbf{U}(V)$ be an irreducible representation of G and $C_\pi \subset L^2(G)$ be the space spanned by the matrix coefficients of π. We know that $v \in V$ is a maximal spanning vector for (π, G, V) if and only if $\mathfrak{c}_{v,v}$ is a cyclic vector of the representation (ad, G, C_π). Once we have an explicit description of C_π and (ad, G, C_π), we may proceed as in Section 4 to find maximal spanning vectors. In the following, we explain this strategy for certain representations of metabelian groups.

5.1. **The group G.** Let $(N, +, 0_N)$ and $(H, \cdot, 1_H)$ be two finite abelian groups. Assume that H acts on N via morphism $\iota : H \to \mathrm{Aut}(N)$. For $h \in H$ and $n \in N$, write ${}^h n$ for $\iota(h)(n)$. Let $G = N \rtimes H$ be the semidirect product of N and H associated with ι. Let $\widehat{N}$ be the dual group of N. The action of H on N induces an action of H on $\widehat{N}$ given by

$$ {}^h\psi(n) = \psi({}^{h^{-1}}n), \text{ for } \psi \in \widehat{N},\ h \in H,\ n \in N. $$

If $\psi \in \widehat{N}$ and ${}^h\psi \ncong \psi$ for all $h \neq 1_H$, then by Markey's criterion, $\pi := \mathrm{Ind}_N^G \psi$ is an irreducible representation of G. Let $O_0,\ O_1,\ \dots,\ O_s$ be the orbits of N under the action of H with $O_0 = \{0_N\}$. Then there are three types of conjugacy classes of $G = N \rtimes H$, given by

$$ \{(0_N, 1_H)\},\ (O_i, 1_H)\ (1 \leq i \leq s),\ (X, h)\ (X \subset N,\ h \neq 1_H). $$

On these conjugacy classes, the character of $\mathrm{Ind}_N^G \psi$ takes value

$$ |H|,\ \sum_{h \in H} {}^h\psi,\ 0,\ \text{repectively.} $$

The one-dimensional representations of H induces one-dimensional representations of G, we denote them as $\chi_1,\ \dots,\ \chi_{|H|}$.

We use similar notation as in Section 3. Let r and ad be the right regular representation and adjoint representation of G on $L^2(G)$ respectively. We write elements of G as $(a, b) \in G$ with $a \in N$ and $b \in H$. Let $\delta_{(c,d)}$ be the characteristic function of the point set $\{(c, d)\}$ for all $(c, d) \in G$. For the convenience of the readers, we collect the following formulas:

(1) $(a, b)(c, d) = (a + {}^b c, bd)$;
(2) $(a, b)^{-1} = (-{}^{b^{-1}}a, b^{-1})$;
(3) $(a, b)^{-1}(c, d)(a, b) = (-{}^{b^{-1}}a + {}^{b^{-1}}c + {}^{b^{-1}d}a, d)$;
(4) $r(a, b)\delta_{(c,d)} = \delta_{(c - {}^{b^{-1}d}a,\, b^{-1}d)}$;
(5) $\mathrm{ad}(a, b)\delta_{(c,d)} = \delta_{({}^b c + a - {}^d a,\, d)}$.

5.2. **The decomposition of C_π.** For each $d \in H$, let $L^2(G)_d \subset L^2(G)$ be the subspace spanned by $\delta_{(c,d)}$ $(c \in N)$. There is an $|H|$-dimensional subspace of $L^2(G)_d$ spanned by

$$\sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}, \ h \in H.$$

Denote this subspace by $L^2(G)_{d,\psi}$. By the discussion in [18, Section 2.6], we have

$$C_\pi = \oplus_{d\in H} L^2(G)_{d,\psi}.$$

To understand the representation (ad, C_π), we compute

$$\begin{aligned} \mathrm{ad}(a,b)(\sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}) &= \sum_{c\in N} {}^h\psi(c)\delta_{({}^bc+a-{}^da,d)} \\ &= \sum_{c\in N} {}^h\psi({}^{b^{-1}}(c-a+{}^da))\delta_{(c,d)} \\ &= \sum_{c\in N} {}^{bh}\psi(c-a+{}^da)\delta_{(c,d)} \\ &= {}^{bh}\psi(-a+{}^da)\sum_{c\in N} {}^{bh}\psi(c)\delta_{(c,d)}. \end{aligned} \tag{5.1}$$

If $d = 1_H$, then

$$\mathrm{ad}(a,b)(\sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}) = \sum_{c\in N} {}^{bh}\psi(c)\delta_{(c,d)}.$$

Therefore $(\mathrm{ad}, L^2(G)_{1_H,\psi}) \cong \chi_1 \oplus \cdots \oplus \chi_{|H|}$. More precisely, for $1 \le i \le |H|$, let K_i be the one-dimensional space generated by

$$\sum_{d\in H} \overline{\chi_i(d)} \sum_{c\in N} {}^d\psi(c)\delta_{(c,1_H)},$$

then $(\mathrm{ad}, K_i) = \chi_i$.

If $d \ne 1_H$, equation (5.1) shows that $L^2(G)_{d,\psi}$ is a G-stable $|H|$-dimensional space. Therefore $\pi_{d,\psi} := (\mathrm{ad}, L^2(G)_{d,\psi})$ is an $|H|$-dimensional representation of G. Moreover, the character of this representation is given by

$$\begin{aligned} (0_N, 1_H) &\mapsto |H|, \\ (a \ne 0, 1_H) &\mapsto \sum_{h\in H} {}^h\psi(-a+{}^da) = \sum_{h\in H} ({}^{d^{-1}h}\psi \cdot {}^h\psi^{-1})(a), \\ (N, b \ne 1_H) &\mapsto 0. \end{aligned} \tag{5.2}$$

Note that in general the set

$$\{{}^{d^{-1}h}\psi \cdot {}^h\psi^{-1} \mid h \in H\}$$

is not an orbit of $\widehat{N}$ under the action of H (cf. Section 5.4). Assume that it is an orbit with $|H|$-elements, then $\pi_{d,\psi}$ is the irreducible representation $\mathrm{Ind}_N^G({}^{d^{-1}}\psi \cdot \psi^{-1})$ of G. In the case of $G = \mathrm{GA}(1,q)$ as in Section 4, all these $\pi_{d,\psi}$ are isomorphic to π. For simplicity, we assume that π satisfies the following condition.

(F): The representations $\pi_{d,\psi}$ $(d \in H,\ d \ne 1_H)$ are irreducible and pairwise non-isomorphic.

Therefore, the representation (ad, C_π) is multiplicity free and we have

$$(\mathrm{ad}, C_\pi) \cong (\bigoplus_{\substack{d\in H\\ d\neq 1_H}} \pi_{d,\psi}) \bigoplus \chi_1 \bigoplus \cdots \bigoplus \chi_{|H|}.$$

By Theorem 1.1, π admits maximal spanning vectors and does phase retrieval. Next we find all maximal spanning vectors. First by Proposition 2.3, we have the following lemma.

Lemma 5.1. *Let $\psi \in \widehat{N}$ be a character of N such that $\psi \ncong {}^h\psi$ for all $h \neq 1_H$. Let (π, G, V) be the induced representation $\mathrm{Ind}_N^G \psi$ and assume that π satisfies condition **(F)**. Then $v \in V$ is a maximal spanning vector for (π, G, V) if and only if the following two conditions are satisfied*

1. *the projections of $\mathfrak{c}_{v,v}$ to K_i $(1 \leq i \leq |H|)$ are nonzero;*
2. *the projections of $\mathfrak{c}_{v,v}$ to $L^2(G)_{d,\psi}$ $(d \in H,\ d \neq 1_H)$ are nonzero.*

5.3. **The characterization of maximal spanning vectors.** Assume that we are in the case as in Lemma 5.1. As $(\mathrm{Ind}_N^G \psi)|_N \cong \oplus_{h\in H} {}^h\psi$, if we fix a basis for each ${}^h\psi$, then $\pi(0_N, d)$ $(d \in H)$ is a permutation on the coordinates and

$$\pi(c, 1_H) = \mathrm{diag}({}^h\psi(c))_{h\in H}.$$

For $1 \leq i \leq |H|$, the projection of $\mathfrak{c}_{v,v}$ to K_i is zero if and only if

$$\begin{aligned}
0 &= \langle \mathfrak{c}_{v,v}, \sum_{d\in H}\sum_{c\in N} \overline{\chi_i(d)}{}^d\psi(c)\delta_{(c,1_H)}\rangle \\
&= \sum_{d\in H}\sum_{c\in N} \chi_i(d)\overline{{}^d\psi(c)}\mathfrak{c}_{v,v}(c, 1_H) \\
&= \langle \sum_{d\in H}\sum_{c\in N} \chi_i(d)\overline{{}^d\psi(c)}\,\mathrm{diag}({}^b\psi(c))_{b\in H} v, v\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{d\in H}\sum_{c\in N} \chi_i(d)\overline{{}^d\psi(c)}{}^b\psi(c) &= \sum_{d\in H} \chi_i(d) \sum_{c\in N} \overline{{}^d\psi(c)}{}^b\psi(c) \\
&= |N|\chi_i(b),
\end{aligned}$$

we have

$$\sum_{d\in H}\sum_{c\in N} \chi_i(d)\overline{{}^d\psi(c)}\,\mathrm{diag}({}^b\psi(c))_{b\in H} = |N|\,\mathrm{diag}(\chi_i(b))_{b\in H}.$$

Therefore, the set of $v \in V$ with nonzero projection in K_i is open dense in V.

For the projection of $\mathfrak{c}_{v,v}$ in $L^2(G)_{d,\psi}$, we claim that the set of $v \in V$ with nonzero projection in the one-dimensional space $K_{h,d} := \mathbb{C}\langle \sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}\rangle$ is open dense in V, for all $h \in H$, $d \in H$ and $d \neq 1_H$.

Indeed, the projection of $\mathfrak{c}_{v,v}$ in $K_{h,d}$ is zero if and only if

$$\begin{aligned}0 &= \langle \mathfrak{c}_{v,v}, \sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}\rangle \\ &= \sum_{c\in N} \overline{{}^h\psi(c)}\mathfrak{c}_{v,v}(c,d) \\ &= \langle \sum_{c\in N} \overline{{}^h\psi(c)}\pi(c,d)v, v\rangle.\end{aligned}$$

Note that

$$\sum_{c\in N} \overline{{}^h\psi(c)}\pi(c,d) = \sum_{c\in N} \overline{{}^h\psi(c)}\pi(c,1_H)\pi(0_N,d) = [\sum_{c\in N} \overline{{}^h\psi(c)}\pi(c,1_H)]\pi(0_N,d),$$

and $\sum_{c\in N} \overline{{}^h\psi(c)}\pi(c,1_H)$ is not the zero matrix, the claim follows.

Write $v = (v_h)'_{h\in H} \in \mathbb{C}^{|H|}$. Then v is a maximal spanning vector for (π, G, V) if and only if the following two conditions hold

1. $\sum_{h\in H} \chi(h)|v_h|^2 \neq 0$ for all $\chi \in \widehat{H}$;
2. for each $d \neq 1_H$, there exists at least one h such that the projection of $\mathfrak{c}_{v,v}$ in $K_{h,d}$ is nonzero.

From the above computation, the second condition holds if all the v_i are nonzero. Hence it is easy to write down a maximal spanning vector for (π, G, V).

Example 5.2. It is not difficult to find representations that satisfy condition **(F)**. Let H be the cyclic group $\mathbb{Z}/m\mathbb{Z}$, N be the cyclic group $\mathbb{Z}/n\mathbb{Z}$, where $m \geq 3$, $n = 2^m - 1$. Let α and β be generators of H and N respectively and the action of H on N is give by ${}^\alpha\beta = 2\beta$. Then the action of H on $\widehat{N}$ is free, the irreducible representations of the metacyclic group $G = N \rtimes H$ is either one-dimensional from the characters of H, or is the induced representation $\operatorname{Ind}_N^G \psi$ for a nontrivial character $\psi \in \widehat{N}$. Moreover, assume that $n = 2^m - 1$ is a prime number (e.g. $(m, n) = (3, 7),\ (5, 31)...$), then every $\operatorname{Ind}_N^G \psi$ satisfies condition **(F)**. Therefore, every irreducible representation of $G = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z}$ admits maximal spanning vectors, hence does phase retrieval.

Remark 5.3. The general split metacyclic group case is similar. Let $r \geq 2$ be a positive integer. Let H be the cyclic group $\mathbb{Z}/m\mathbb{Z}$, N be the cyclic group $\mathbb{Z}/n\mathbb{Z}$, such that $r^m \equiv 1 \pmod{n}$. Let α and β be generators of H and N respectively and the action of H on N is give by ${}^\alpha\beta = r\beta$. Let π be an irreducible representation of $G = N \rtimes H$. If π is trivial on N, then it is induced from a character of H and the situation is trivial. If π is not trivial on N, let $\psi : N \to \mathbb{T}$ be a character of N that appears in $\pi|_N$. Assume that ψ is of conductor d, i.e. it is a composition of the form $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \xrightarrow{\sim} \mu_d \subset \mathbb{T}$. By Frobenius reciprocity, π appears in $\Pi := \operatorname{Ind}_N^G \psi$. Note that $\Pi|_N = \oplus_{h\in H} {}^h\psi$. In general, $\{{}^h\psi\}_{h\in H}$ breaks into several orbits. Let s be the smallest positive integer with $r^s \equiv 1 \pmod{d}$. Then the subspace $\oplus_{h\in H/\langle s\alpha\rangle} {}^h\psi$ is an irreducible representation of G. By Frobenius reciprocity again, this is π and $\Pi \cong \pi^{\oplus(m/s)}$. One then sees that the subgroup $\langle d\beta\rangle \rtimes \langle s\alpha\rangle \subset N \rtimes H$ is in the kernel of π and π factors through $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \rtimes \mathbb{Z}/s\mathbb{Z}$. Therefore, we may assume at the beginning that ψ is of conductor n and $n \nmid r^{m'} - 1$ for any positive integer $m' < m$.

With the above assumption and the notation in Section 5.2, the representations $\pi_{d,\psi}$ ($d \neq 1_H$) are irreducible. On the other hand, it is possible that when d varies, the orbits $\{{}^{d^{-1}h}\psi \cdot {}^{h}\psi^{-1} \mid h \in H\}$ coincide, therefore the corresponding $\pi_{d,\psi}$ are isomorphic. If this happens, by identifying the orbit elements, it is easy to write down the isomorphism between $\pi_{d,\psi}$ and $\pi_{d',\psi}$ (just as we did for $\mathrm{GA}(1,q)$). Assume that $H - \{1_H\} = \cup_{i\in I} D_i$, where $\pi_{d,\psi} \cong \pi_{d',\psi}$ if and only if $d,\ d'$ are in the same D_i. If $|D_i| = 1$, the linear independent condition is trivial. If $|D_i| \geq 2$, the linear dependent property is a closed condition and the set of vectors with linear independent condition is open dense. Since I is a finite index set, the set of maximal spanning vectors for π is open dense. In particular, *the irreducible representations of split metacyclic groups does phase retrieval.*

5.4. **The Weyl-Heisenberg case revisit.** We have seen that the condition **(F)** is not essential. In the $G = \mathrm{GA}(1,q)$ case, if some of the $\pi_{d,\psi}$ are isomorphic, we may apply Proposition 2.5 to find maximal spanning vectors by writing down the isomorphisms explicitly (cf. Lemma 4.3 and Remark 5.3). In the following, we explain another extreme example where the representations $\pi_{d,\psi}$ are direct sums of one dimensional representations.

We keep the notation as above. Let H be a finite abelian group of order n and $\widehat{H}$ be the dual group of H. Let μ_n be the group of n-th roots of unity and let $N = \mu_n \times \widehat{H}$. Define the action of H on N by

$$^{h}(\zeta,\chi) = (\zeta\cdot\chi(h),\chi), \qquad \text{for } h \in H,\ \zeta\in\mu_n,\ \chi\in\widehat{H}.$$

Let $G = N \rtimes H$ be the associated semidirect product of N and H. Let $\psi : N = \mu_n \times \widehat{H} \to \mathbb{C}^\times$ be the character $(\zeta,\chi)\mapsto\zeta$. Then

$$^{h}\psi(\zeta,\chi) = \psi({}^{h^{-1}}(\zeta,\chi)) = \chi(h^{-1})\zeta.$$

Therefore $\pi := \mathrm{Ind}_N^G \psi : G \to \mathbf{U}(\mathbb{C}^n)$ is an n-dimensional irreducible representation of G. We claim that in this case the representations $\pi_{d,\psi}$ on $L^2(G)_{d,\psi}$ is a direct sum of one dimensional representations. Indeed, the characters appear in equation (5.2) satisfies

$$^{h}({}^{d^{-1}}\psi\cdot\psi^{-1}) = {}^{d^{-1}}\psi\cdot\psi^{-1}.$$

Let $\chi_1, \ldots, \chi_n$ be the characters of H. For $1 \leq i \leq n$ and $d \in H$, let $K_{i,d} \subset L^2(G)_{d,\psi}$ be the one dimensional subspace of $L^2(G)$ generated by

$$\sum_{h\in H}\overline{\chi_i(h)}\sum_{c\in N}{}^{h}\psi(c)\delta_{(c,d)}.$$

For $(a,b) \in N \rtimes H$ with $a = (\zeta, \chi) \in N = \mu_n \times \widehat{H}$ and $b \in H$, by equation (5.1), we have

$$\begin{aligned} &\mathrm{ad}(a,b)(\sum_{h\in H} \overline{\chi_i(h)} \sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}) \\ =& \sum_{h\in H} \overline{\chi_i(h)}\, \mathrm{ad}(a,b)(\sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}) \\ =& \sum_{h\in H} \overline{\chi_i(h)} \sum_{c\in N} \chi(d)^{bh}\psi(c)\delta_{(c,d)} \\ =& \chi(d)\chi_i(b) \sum_{h\in H} \overline{\chi_i(bh)} \sum_{c\in N} {}^{bh}\psi(c)\delta_{(c,d)} \\ =& \chi(d)\chi_i(b)(\sum_{h\in H} \overline{\chi_i(h)} \sum_{c\in N} {}^h\psi(c)\delta_{(c,d)}). \end{aligned}$$

In other words, $(\mathrm{ad}, K_{i,d}) = \chi_{i,d}$, where $\chi_{i,d} : G \to \mathbb{T}$ is the character given by

$$\chi_{i,d}(\zeta, \chi, h) = \chi(d)\chi_i(h), \quad \text{for } \zeta \in \mu_n,\ \chi \in \widehat{H},\ h \in H.$$

Hence $(\mathrm{ad}, L^2(G)_{d,\psi}) \cong \oplus_{1\le i\le n}\chi_{i,d}$ and the claim follows. Note that for $1 \le i \le n$, $d \in H$, the characters $\chi_{i,d}$ are pairwise non-isomorphic and they are just all the characters of the abelian group $\widehat{H} \times H$. By Proposition 2.3, $v \in \mathbb{C}^n$ is a maximal spanning vector for $(\pi, G, \mathbb{C}^n)$ if and only if the projection of $\mathfrak{c}_{v,v}$ in $K_{i,d}$ is nonzero for each $1 \le i \le n$, $d \in H$. As in Section 5.3, $\mathfrak{c}_{v,v}$ projects to zero in $K_{i,d}$ if and only if

$$\begin{aligned} 0 &= \langle \mathfrak{c}_{v,v}, \sum_{h\in H}\sum_{c\in N} \overline{\chi_i(h)}^h\psi(c)\delta_{(c,d)}\rangle \\ &= \sum_{h\in H}\sum_{c\in N} \chi_i(h)\overline{{}^h\psi(c)}\mathfrak{c}_{v,v}(c,d) \\ &= \langle \sum_{h\in H} \chi_i(h) \sum_{c\in N} \overline{{}^h\psi(c)}\pi(c, 1_H)\pi(0,d)v, v\rangle \\ &= \langle |N|\,\mathrm{diag}(\chi_i(h))_{h\in H} \cdot \pi(0,d)v, v\rangle. \end{aligned}$$

Therefore, v is maximal spanning if and only if $\langle \pi(g)v, v\rangle \neq 0$ for all $g \in G$.

Moreover, let $\alpha \in Z^2(\widehat{H} \times H, \mathbb{T})$ be the cocyle $\alpha((\chi, h), (\chi', h')) = \chi'(h)$ and define $\tilde{\pi} : \widehat{H} \times H \to \mathbf{U}(\mathbb{C}^n)$ by

$$\tilde{\pi}(\chi, h) = \pi(1, \chi, h), \quad \text{for } \chi \in \widehat{H},\ h \in H.$$

Then $\tilde{\pi}$ is the unique projective representation of $\widehat{H} \times H$ with multiplier α (the Weyl-Heisenberg representation of $\widehat{H} \times H$). A vector $v \in \mathbb{C}^n$ is a maximal spanning vector for $(\pi, G, \mathbb{C}^n)$ if and only if v is a maximal spanning vector for $(\tilde{\pi}, \widehat{H} \times H, \mathbb{C}^n)$. Note that $\pi(\zeta, \chi, h) = \zeta\tilde{\pi}(\chi, h)$, we obtain [17, Theorem 1.7] for the Weyl-Heisenberg representations. The general abelian case is similar.

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