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ON π-DIVISIBLE $\mathcal{O}$-MODULES OVER FIELDS OF CHARACTERISTIC p*

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Abstract. In this paper, we construct a Dieudonné theory for π-divisible $\mathcal{O}$-modules over a perfect field of characteristic p. Applying this theory, we prove the existence of slope filtration of π-divisible $\mathcal{O}$-modules over an integral normal Noetherian base. We also study minimal π-divisible $\mathcal{O}$-modules over an algebraically closed field of characteristic p and prove that their isomorphism classes are determined by their π-torsion parts by introducing Oort's filtration. Moreover, after a detailed study of deformations of π-divisible $\mathcal{O}$-modules via displays, we prove the generalized Traverso's isogeny conjecture.

Key words. $\mathcal{O}$-isocrystal, $\mathcal{O}$-crystal, Dieudonné $\mathcal{O}$-module, π-divisible $\mathcal{O}$-module, completely slope divisible $\mathcal{O}$-module, slope filtration, Oort filtration, Traverso's isogeny conjecture.

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1. Introduction.

Let p be a prime number. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and π be a uniformizer of $\mathcal{O}$. Let S be an $\mathcal{O}$-scheme such that π is locally nilpotent in $\mathcal{O}_S$. A *formal $\mathcal{O}$-module* over S is a commutative and smooth formal group X over S with an action of $\mathcal{O}$ given by $\iota : \mathcal{O} \to \mathrm{End}(X)$, such that the induced action of $\mathcal{O}$ on $\mathrm{Lie}(X)$ coincides with the natural action via $\mathcal{O} \to \mathcal{O}_S$. A formal $\mathcal{O}$-module X is called *π-divisible* if the endomorphism $\pi : X \to X$ is an isogeny. In [8], Drinfeld studied formal $\mathcal{O}$-modules via Cartier modules and defined a certain moduli problem for formal $\mathcal{O}$-modules with an action of $\mathcal{O}_D$ that extends the action of $\mathcal{O}$, where $\mathcal{O}_D$ is the maximal order of a central division algebra D over $\mathrm{Frac}(\mathcal{O})$. Drinfeld's result has many applications (cf. [14, 18, 26, 33, 34, 35, 36]), in particular to the p-adic uniformization of Shimura varieties and to arithmetic, e.g. Drinfeld himself [8] deduced the p-adic uniformization of Shimura curves over $\mathbb{Q}$.

Displays are important tools in the study of p-divisible groups and one advantage is that they classify p-divisible groups over a general base. By the work of Zink and Lau, for any ring R in which p is nilpotent, we know that the category of formal p-divisible groups over R is equivalent to the category of nilpotent displays over R (cf. [19, Theorem 1.1]). Combining the ideas of Drinfeld [8], Zink [44, 45], and Lau [19, 21], Ashendorff [1] obtained an equivalence between the category of π-divisible formal $\mathcal{O}$-modules over R and the category of nilpotent $\mathcal{O}$-displays over R, where R is an $\mathcal{O}$-algebra in which π is nilpotent. If moreover one defines π-divisible $\mathcal{O}$-modules similarly, R is a complete Noetherian local $\mathcal{O}$-algebra with perfect residue field such that $\pi R = 0$ if $p = 2$, the above equivalence extends to an equivalence between the category of π-divisible $\mathcal{O}$-modules over R and the category of Dieudonné $\mathcal{O}$-displays over R. We remark that the assumption for $p = 2$ is from [44, Lemma 2] (cf. Remark 3.12). Moreover, we could adapt Lau's construction of Dieudonné displays [22] to remove the assumption for $p = 2$ (cf. Remark 3.13). These equivalences are summarized in [2] and recalled in Section 3.1. This paper is a continuation and a complement as well of [2]: we study π-divisible $\mathcal{O}$-modules over certain special base and study deformations of π-divisible $\mathcal{O}$-modules via displays.

More specifically, in this paper, we obtain the classification of π-divisible $\mathcal{O}$-

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modules over perfect fields of characteristic p via Dieudonné $\mathcal{O}$-modules, prove the existence of slope filtration of π-divisible $\mathcal{O}$-modules over an integral normal Noetherian base, explain the determination of minimal π-divisible $\mathcal{O}$-modules over algebraically closed fields of characteristic p by their π-torsion parts, and prove Traverso's isogeny conjecture etc. The reason we put together these results in one paper is that they are related to each other and the main tools of all their proofs are Dieudonné modules and displays. As we shall see, these results are generalizations of the corresponding results on p-divisible groups and their proofs are adapted from those for p-divisible groups. Some of the results are known to the experts, we include them here to fill the gap in the literature and provide a reference for future studies in moduli problems and of PEL-type Shimura varieties.

In the following, we explain the main results more precisely and outline the contents of the paper. First, we fix some notation.

1.1. Notation. In this paper, p is a prime number, $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$, π is a uniformizer of $\mathcal{O}$, $\mathbb{F} = \mathbb{F}_q = \mathcal{O}/\pi\mathcal{O}$. Denote by $\mathrm{Fr} : \mathbb{F} \to \mathbb{F}$ the Frobenius morphism given by $x \mapsto x^q$. Denote by $W_{\mathcal{O}}(A)$ the ring of ramified Witt vectors for $A \in \mathrm{Alg}_{\mathcal{O}}$ (cf. [11, Section 1.2] and [13]). Here for any ring R, Alg_R denotes the category of R-algebras.

In this paper, $k \in \mathrm{Alg}_{\mathcal{O}}$ is a field of characteristic p. Write $A_k = \bigcup_{n\in\mathbb{Z}_{\geq 1}} W_{\mathcal{O}}(k^{1/q^n})$. Then $A_k/\pi A_k \cong k^{\mathrm{perf}}$, the perfect closure of k. Denote by $F_k = A_k[\pi^{-1}] = \mathrm{Frac}(A_k)$. Note that A_k is a discrete valuation ring and F_k is a field (cf. [37, Part 1]). The *Frobenius* and *Verschiebung* on A_k or F_k are denoted by ^{F} and ^{V} respectively.

Let S be a scheme. Denote by Sch_S the category of schemes over S. Let T be a scheme over S. Denote by X_T the base change of X with respect to $T \to S$, where X is an appropriate object over S (e.g. a scheme, a functor, a sheaf etc.).

1.2. $\mathcal{O}$-crystals and $\mathcal{O}$-isocrystals. Let f be an integer. The notions of f-$\mathcal{O}$-isocrystal and f-$\mathcal{O}$-crystal are defined at the beginning of Section 2. An *f-$\mathcal{O}$-isocrystal* (resp. *f-$\mathcal{O}$-crystal*) over k is a pair (N, V) (resp. (M, V)), where N (resp. M) is a finitely generated free F_k-module (resp. A_k-module) and $V : N \to N$ (resp. $V : M \to M$) is an injective ${}^{F^f}$-linear map.

For an f-$\mathcal{O}$-isocrystal, there is an invariant called *the first Newton slope* (Definition 2.8). Using this invariant, we have a decomposition result for f-$\mathcal{O}$-isocrystals over k. Consequently, we may define the Newton polygon attached to an f-$\mathcal{O}$-isocrystal for k perfect (Section 2.2).

THEOREM 1.1. *Assume that k is perfect. Let (N, V) be an f-$\mathcal{O}$-isocrystal over k with first Newton slope λ. There exist uniquely determined sub-$\mathcal{O}$-isocrystals (N_i, V_i) with* $\mathrm{Newton}(N_i, V_i) = \lambda_i$, *such that $(N, V) = \oplus_{i=1}^r (N_i, V_i)$ and $\lambda = \lambda_1 < \lambda_2 < \cdots < \lambda_r$.*

In the case where k is algebraically closed, we have a complete classification of f-$\mathcal{O}$-isocrystals over k, which is more explicit than Theorem 1.1. More precisely, let $r, s \in \mathbb{Z}$ with $s > 0$ and $(r, s) = 1$. Let $N_{r,s}$ be an F_k-vector space with a basis $e_1, \cdots, e_s$. We define on $N_{r,s}$ a structure of f-$\mathcal{O}$-isocrystal by

$$Ve_i = \begin{cases} e_{i+1} & \text{if } i < s, \\ \pi^r e_1 & \text{if } i = s. \end{cases}$$

THEOREM 1.2. *Assume that k is algebraically closed. Let (N, V) be an f-$\mathcal{O}$-isocystal over k. Then there exists a direct sum decomposition*

$$(N, V) = \oplus_{i=1}^{u} N_{r_i,s_i}^{t_i},$$

where u, r_i, s_i, $t_i \in \mathbb{Z}$, u, t_i, $s_i > 0$, $(s_i, r_i) = 1$, $r_1/s_1 < \cdots < r_u/s_u$. Moreover, the numbers u, r_i, s_i, t_i are uniquely determined by (N, V).

Theorem 1.1 and Theorem 1.2 are proved in Section 2.2 and Section 2.3 respectively. These results are well-known (cf. [24, 47]). We give detailed proofs for two reasons. First, along the proof, we introduce several notions, for which we could not find appropriate references. Second, we make some computations and prove certain lemmas, which are needed in other parts of the paper.

1.3. π-divisible $\mathcal{O}$-modules. By the main result of [2], over a *good* base, we could use $\mathcal{O}$-displays (cf. Definitions 3.7, 3.8, and 3.10) to classify π-divisible $\mathcal{O}$-modules. The classification results of [2] are recalled in Section 3.1.

A *Dieudonné $\mathcal{O}$-module* over k is a finitely generated free A_k-module M, together with an ^{F}-linear map $F : M \to M$ and an ${}^{F^{-1}}$-linear map $V : M \to M$, such that $FV = \pi$. A Dieudonné $\mathcal{O}$-module is *reduced* if the operator V on $M/\pi M$ is nilpotent.

By studying the relation between $\mathcal{O}$-crystals, Dieudonné $\mathcal{O}$-modules, and $\mathcal{O}$-displays, we obtain the following result in Section 3.2.

THEOREM 1.3. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a field of characteristic p.*

(1) *The category of potential formal π-divisible $\mathcal{O}$-modules over k (cf. Definition 3.21) is equivalent to the category of reduced Dieudonné $\mathcal{O}$-modules over k.*
(2) *Assume that k is perfect, then the following categories are equivalent.*
   - *The category of (formal) π-divisible $\mathcal{O}$-modules over k.*
   - *The category of (reduced) Dieudonné $\mathcal{O}$-modules over k.*
   - *The category of (-1)-$\mathcal{O}$-crystals with slopes in $[0, 1]$ (in $(0, 1]$).*

REMARK 1.4. In this paper, we choose the equivalences so that they are *covariant*. For k perfect, the Newton polygon of a π-divisible $\mathcal{O}$-modules over k is defined to be the Newton polygon of the associated (-1)-$\mathcal{O}$-crystal and we denote it by $\mathbb{N}(X)$.

Let X be a π-divisible $\mathcal{O}$-module over an $\mathbb{F}$-scheme S. A *slope filtration of X* is a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_m = X$$

consisting of π-divisible sub-$\mathcal{O}$-modules of X, such that there exist rational numbers $1 \geq \lambda_1 > \cdots > \lambda_m \geq 0$ and the subquotient X_i/X_{i-1} is isoclinic of slope λ_i for $1 \leq i \leq m$ (cf. Definition 3.39).

Inside the (-1)-$\mathcal{O}$-isocrystal $N_{r,s}$ with $s \geq r \geq 0$ and $(r, s) = 1$, there is a special (-1)-$\mathcal{O}$-crystal $M_{r,s}$ (Section 3.2.1). If k is perfect, let $G_{r,s}$ be the π-divisible $\mathcal{O}$-module associated with $M_{r,s}$ via Theorem 1.3. Combining Theorem 1.2 and Theorem 1.3, in the case where k is algebraically closed, every π-divisible $\mathcal{O}$-module over k is isogenous to a direct sum of some $G_{r,s}$. Hence in this case the slope filtration exists up to isogeny. In general, a decomposition as above does not exist. Nevertheless, we have the following result, which is proved in Section 3.3.4.

THEOREM 1.5. *A π-divisible $\mathcal{O}$-module over an integral, normal, Noetherian $\mathbb{F}$-scheme S with constant Newton polygon is isogenous to a π-divisible $\mathcal{O}$-module over S that admits a slope filtration.*

Theorem 1.5 is proved by a detailed study of completely slope divisible π-divisible $\mathcal{O}$-modules, which is the content of Section 3.3.3.

1.4. Minimal π-divisible $\mathcal{O}$-modules. Each Newton polygon β with slopes in $[0,1]$ corresponds to a (-1)-$\mathcal{O}$-isocrystal (N,V) over $\mathbb{F}$, hence corresponds to an isogeny class of π-divisible $\mathcal{O}$-modules over $\mathbb{F}$. Let $\oplus_i G^{r_i}_{m_i,m_i+n_i}$ be a representative of this isogeny class as mentioned above. Define $H(\beta) := \oplus_i H^{r_i}_{m_i,m_i+n_i}$. Here $H_{m,m+n}$ is a π-divisible $\mathcal{O}$-module whose associated Dieudonné $\mathcal{O}$-module $M(H_{m,m+n})$ is as follows. It is a free $W_{\mathcal{O}}(\mathbb{F})$-module of rank $m+n$ with basis $e_0,\ e_1,\ \ldots,\ e_{m+n-1}$. For $j \in \mathbb{Z}_{\geq 0}$, we write $e_j = \pi^a e_i$ if $j = i + a(m+n)$. The actions of F and V on $M(H_{m,m+n})$ are given by $F(e_i) = e_{i+n}$ and $V(e_i) = e_{i+m}$. It is obvious that $H(\beta)$ is a π-divisible $\mathcal{O}$-module determined by the Newton polygon β.

A π-divisible $\mathcal{O}$-module X is called *minimal* if there exists a Newton polygon β and an isomorphism $X_k \cong H(\beta)_k$. The minimal π-divisible $\mathcal{O}$-modules are special. In the case of p-divisible groups, the minimal objects have interesting properties and have applications in deformation theory (cf. [6, 31]).

In Section 4, we prove the following result, which generalizes the main result of [31].

THEOREM 1.6. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p and X be a minimal π-divisible $\mathcal{O}$-module over k. If Y is another π-divisible $\mathcal{O}$-module over k such that $X[\pi] \cong Y[\pi]$, then $X \cong Y$.*

In order to prove Theorem 1.6, we adapt the argument in [31] and prove the following two results, which are contents of Section 4.2 and Section 4.3 respectively.

(1) Let $H = \oplus_{i=1}^{t} (H_{m_i,m_i+n_i})^{r_i}$ be a minimal π-divisible $\mathcal{O}$-module over k. Suppose that X is a π-divisible $\mathcal{O}$-module over k such that $X[\pi] \cong H[\pi]$. Suppose that $\lambda_1 = n_1/(n_1+m_1) \leq 1/2$. Then there exists a π-divisible sub-$\mathcal{O}$-module $X_1 \subset X$ such that
$$X_1 \cong (H_{m_1,m_1+n_1})^{r_1}, \quad (X/X_1)[\pi] \cong \prod_{2 \leq i \leq t} (H_{m_i,m_i+n_i}[\pi])^{r_i}.$$

(2) Let (m,n) and (d,e) be pairs of pairwise coprime positive integers. Suppose that $\frac{n}{m+n} < \frac{e}{d+e}$. Let
$$0 \to Z := H_{m,m+n} \to T \to Y := H_{d,d+e} \to 0$$
be an exact sequence of π-divisible $\mathcal{O}$-modules. Then this sequence splits and $T \cong Z \oplus Y$ if the induced sequence of π-torsions
$$0 \to Z[\pi] \to T[\pi] \to Y[\pi] \to 0$$
splits.

Both results are proved by translating the corresponding questions into questions of Dieudonné $\mathcal{O}$-modules.

1.5. Traverso's isogeny conjecture. By Theorem 1.6, if $k \in \mathrm{Alg}_{\mathcal{O}}$ is an algebraically closed field of characteristic p, the isomorphism class of a minimal π-divisible $\mathcal{O}$-module over k is determined by its π-torsion. For general π-divisible $\mathcal{O}$-modules, we have the following result, which is proved in Section 5.1.

THEOREM 1.7. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Let X be a π-divisible $\mathcal{O}$-module over k. There exists a minimal number $n_X \in \mathbb{Z}_{>0}$*

such that X is uniquely determined up to isomorphism by $X[\pi^{n_X}]$, i.e. if X' is a π-divisible $\mathcal{O}$-module over k such that $X'[\pi^{n_X}] \cong X[\pi^{n_X}]$, then $X' \cong X$.

By Theorem 1.7, there exists a minimal natural number $b_X \in \mathbb{Z}_{>0}$ such that the isogeny class of X is determined by $X[\pi^{b_X}]$. We call b_X the *isogeny cutoff* of X. We prove the following result, which is a generalization of [41, Conjecture 5]. Denote by $\lceil x \rceil$ the smallest integer greater or equal to x for $x \in \mathbb{R}$.

THEOREM 1.8 (Traverso's isogeny conjecture). *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Let X be a π-divisible $\mathcal{O}$-module over k. Assume that X has height h and dimension d, then $b_X \leq \lceil \frac{d(h-d)}{h} \rceil$.*

Certainly, the theorem holds for minimal π-divisible $\mathcal{O}$-modules. As a first step towards a proof of the general case, we prove the following result in Section 5.2.

THEOREM 1.9. *If $a(X) \leq 1$, then Traverso's isogeny conjecture holds for X. Here $a(X) = \dim_k M/(FM + VM)$ and (M, F, V) is the covariant Dieudonné $\mathcal{O}$-module associated with X.*

The general case of Traverso's isogeny conjecture follows from Theorem 1.9 and Theorem 1.10. The detailed argument is given in Section 5.3.1 and Theorem 1.10 is proved in Section 6.2 using deformation theory. In Section 5.3.2, we study the relation between minimal height and isogeny cutoff. In Section 5.3.3 we show that the bound is sharp in Traverso's isogeny conjecture by constructing explicit examples.

1.6. Deformations of π-divisible $\mathcal{O}$-modules. To complete the proof of Traverso's isogeny conjecture, we prove the following result in Section 6.2.

THEOREM 1.10. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Every π-divisible $\mathcal{O}$-module over k satisfies Oort's condition (cf. Definition 5.18).*

Roughly speaking, Oort's condition says that a π-divisible $\mathcal{O}$-module over k has a deformation over $k[[t]]$ with certain special properties. To study the deformations of π-divisible $\mathcal{O}$-modules, we use the equivalence between π-divisible $\mathcal{O}$-modules and $\mathcal{O}$-displays and transfer our problem into (semi)-linear problem. The precise correspondence is given in Section 6.1. Put simply, by Theorem 3.11, deforming a π-divisible $\mathcal{O}$-module is equivalent to deforming the structure equation of the associated $\mathcal{O}$-display.

Another ingredient of the proof of Theorem 1.10 is the catalogue of a minimal simple π-divisible $\mathcal{O}$-module. Recall that a *catalogue* for a moduli problem $\mathcal{MP}$ is a family $\mathbb{X} \to T$ in $\mathcal{MP}$ such that every object of $\mathcal{MP}$ over an algebraically closed field k appears as the fiber of the family $\mathbb{X}$ over a point $\operatorname{Spec} k \to T$. See [6, Section 5] for more details on this notion. For our purpose we prove the following result (cf. Section 6.1.6).

THEOREM 1.11. *The catalogue associated with a simple minimal π-divisible $\mathcal{O}$-module over $\mathbb{F}$ with height $d + c$ and dimension c is geometrically irreducible of dimension $(c-1)(d-1)/2$ over $\mathbb{F}$.*

An immediate consequence of Theorem 1.11 and the purity result on Newton polygons of π-divisible $\mathcal{O}$-modules (cf. Theorem 6.11) is Corollary 6.12, which says that a simple π-divisible $\mathcal{O}$-module over k has a deformation over $k[[t]]$ such that the Newton polygon is constant and the a-number of the generic fiber is one. Theorem 1.10 then follows from an explicit construction (cf. Propositions 6.14 and 6.15).

2. $\mathcal{O}$-crystals and $\mathcal{O}$-isocrystals. We use the notation introduced in Section 1.1. Note that if k is perfect, then $A_k = W_{\mathcal{O}}(k)$ is the ring of ramified Witt vectors and $F_k = \mathrm{Frac}(W_{\mathcal{O}}(k))$. In fact, this is the case for most parts of this section. We refer to [17] for more details on isocrystals over varieties over k. The following definition is slightly different from the classical case.

Definition 2.1. Let $f \in \mathbb{Z}$. An *f-$\mathcal{O}$-isocrystal* over k is a pair (N, V), where N is a finitely generated F_k-vector space, $V : N \to N$ is an injective ${}^{F^f}$-linear map. A morphism $(N, V) \to (N', V')$ of f-$\mathcal{O}$-isocrystals is a homomorphism $h : N \to N'$ of F_k-modules which satisfies $V' \circ h = h \circ V$.

An *f-$\mathcal{O}$-crystal* over k is a pair (M, V), where M is a finitely generated free A_k-module, $V : M \to M$ is an injective ${}^{F^f}$-linear map. A morphism $(M, V) \to (M', V')$ of f-$\mathcal{O}$-crystals is a homomorphism $h : M \to M'$ of A_k-modules which satisfies $V' \circ h = h \circ V$.

Let N be a finitely generated F_k-vector space. A finitely generated A_k-module $M \subset N$ is called a *lattice* if $M \otimes_{A_k} F_k = N$. Let (N, V) be an f-$\mathcal{O}$-isocrystal. If there exists a lattice M of N such that $VM \subset M$, then $(M, V|_M)$ is an f-$\mathcal{O}$-crystal. In this case, we say that (N, V) is *effective*.

On the other hand, if (M, V) is an f-$\mathcal{O}$-crystal, then $(M \otimes_{A_k} F_k, V \otimes \mathrm{id})$ is an f-$\mathcal{O}$-isocrystal.

In the following, we study f-$\mathcal{O}$-crystals and f-$\mathcal{O}$-isocrystals. Many results are similar to those in the classical case (cf. [24, 47]). For completeness and later use, we give detailed proofs.

2.1. Basic properties. Let M and M' be two lattices of N. Then $\pi^s M' \subset M$ for some $s \in \mathbb{Z}_{\geq 1}$. It is clear that $M/\pi^s M'$ is an A_k-module of finite length. Define

$$[M : M'] = \mathrm{Length}_{A_k} M/\pi^s M' - \mathrm{Length}_{A_k} M'/\pi^s M'.$$

The following lemma is clear.

Lemma 2.2. *Let M, M', M'' be lattices of N.*
(1) $[M : M''] = [M : M'] + [M' : M'']$.
(2) *The number $[M : VM]$ is independent of the choice of M, i.e.,*

$$[M : VM] = [M' : VM'].$$

Definition 2.3. Let (N, V) be an f-$\mathcal{O}$-isocrystal. The number $\dim_{F_k} N$ is called the *height* of N. The number $[M : VM]$ is called the *dimension* of N, where $M \subset N$ is an arbitrary lattice. If h is the height of (N, V) and d is the dimension of (N, V), we call the pair (h, d) the *type* of (N, V).

Definition 2.4. Let (N, V) be an f-$\mathcal{O}$-isocrystal, $M \subset N$ be a lattice, $m \in N$. Define

$$\mathrm{ord}_M V = \max\{t \in \mathbb{Z} : \ VM \subset \pi^t M\};$$
$$\mathrm{ord}_M m = \max\{t \in \mathbb{Z} : \ m \in \pi^t M\}.$$

Lemma 2.5. *Let M and M' be lattices in N. Let c and c' be integers such that $\pi^c M \subset M'$ and $\pi^{c'} M' \subset M$. Then*

$$|\,\mathrm{ord}_M V - \mathrm{ord}_{M'} V| \leq c + c', \quad |\,\mathrm{ord}_M m - \mathrm{ord}_{M'} m| \leq \max\{c, c'\}.$$

Proof. Let $x = \operatorname{ord}_M m$, $y = \operatorname{ord}_M V$. Then

$$m \in \pi^x M \subset \pi^{x-c} M',$$
$$VM' \subset \pi^{-c'} VM \subset \pi^{y-c'} M \subset \pi^{y-c-c'} M'.$$

Hence $\operatorname{ord}_{M'} m \geq x - c$, $\operatorname{ord}_{M'} V \geq y - c - c'$. The lemma follows easily. □

Lemma 2.6. *Let (N, V) be an f-$\mathcal{O}$-isocrystal of type (h, d). Then for any lattice $M \subset N$ and $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\operatorname{ord}_M V \leq \frac{1}{n} \operatorname{ord}_M V^n \leq \frac{d}{h}. \tag{2.1.1}$$

If there exists n such that $\operatorname{ord}_M V \neq \frac{1}{n} \operatorname{ord}_M V^n$, then

$$\operatorname{ord}_M V + \frac{1}{h} \leq \frac{1}{h} \operatorname{ord}_M V^h. \tag{2.1.2}$$

Proof. Let $x = \operatorname{ord}_M V$, $y = \operatorname{ord}_M V^n$. It is easy to see that $nx \leq y$. Moreover,

$$nd = n[M : VM] = [M : V^n M] \geq [M : \pi^y M] = y[M : \pi M] = yh.$$

We obtain inequality (2.1.1).

Assume that $V^n M \subset \pi^{nx+1} M$ for some $n \geq 2$. Set $M_i = \{m \in M : \ V^i m \in \pi^{ix+1} M\}$. Then we obtain a chain

$$\pi M = M_0 \subset M_1 \subset \cdots \subset M_n = M.$$

It is easy to see that if $M_i = M_{i+1}$, then $M_i = M_{i+j}$ for any $j \geq 0$. Note that $\operatorname{Length} M/\pi M = h$, we must have $M_h = M$. The inequality (2.1.2) then follows. □

Lemma 2.7. *Let (N, V) be an f-$\mathcal{O}$-isocrystal over k with height h. Then (N, V) is effective if there exists a lattice $M \subset N$ and $V^{h+1} M \subset \pi^{-1} M$.*

Proof. We need to find a lattice that is stable under V. Let $M' = \sum_{j=0}^{h} V^j M$. Then

$$\sum_{j=0}^{h+1} V^j M' = \sum_{j=0}^{2h+1} V^j M = M' + \sum_{j=0}^{h} V^j (V^{h+1} M) \subset \pi^{-1} M'.$$

Consider the chain

$$M' \subset M' + VM' \subset \cdots \subset \sum_{j=0}^{h+1} V^j M' \subset \pi^{-1} M'.$$

Note that $\operatorname{Length} \pi^{-1} M'/M' = h$, there exists n $(0 \leq n \leq h)$, such that $\sum_{j=0}^{n} V^j M' = \sum_{j=0}^{n+1} V^j M'$. Let $M'' = \sum_{j=0}^{n} V^j M'$, then $VM'' \subset M''$. The lemma follows. □

Definition 2.8. Let (N, V) be an f-$\mathcal{O}$-isocrystal. We call

$$\operatorname{Newton}(N, V) = \sup\{\frac{1}{n} \operatorname{ord}_M V^n : \ n \in \mathbb{Z}_{\geq 1}, M \subset N \text{ lattice}\}$$

the *first Newton slope* of (N, V).

Lemma 2.9. *Let (N, V) be an f-$\mathcal{O}$-isocrystal, $M \subset N$ a lattice. Then*

$$\begin{aligned}\mathrm{Newton}(N, V) &= \lim_{n\to\infty} \frac{1}{n} \operatorname{ord}_M V^n,\\ \mathrm{Newton}(N, \pi^s V^r) &= r\,\mathrm{Newton}(N, V) + s.\end{aligned}$$

Proof. Let $\lambda = \mathrm{Newton}(N, V)$. Let $M' \subset N$ be another lattice and $x = \operatorname{ord}_{M'} V^n$ ($n \in \mathbb{Z}_{\geq 1}$). Let c and c' be integers such that $\pi^c M \subset M'$ and $\pi^{c'} M' \subset M$. Then

$$\begin{aligned}\sup_m \frac{1}{m} \operatorname{ord}_M V^m &\geq \sup_l \frac{1}{ln} \operatorname{ord}_M V^{ln}\\ &\geq \sup_l \frac{1}{ln}(\operatorname{ord}_{M'} V^{ln} - c - c') \text{ (by Lemma 2.5)}\\ &\geq \sup_l (\frac{x}{n} - \frac{c + c'}{ln}) = \frac{x}{n}.\end{aligned}$$

So $\mathrm{Newton}(N, V) = \sup_n \frac{1}{n} \operatorname{ord}_M V^n$. Let $\epsilon \in \mathbb{R}_{>0}$, $y = \lambda - \epsilon/2$. Then there exists $m \in \mathbb{Z}_{>0}$, such that $\frac{1}{m} \operatorname{ord}_M V^m > x$. Let r and s be non-negative integers such that $0 \leq s \leq m$ and $\operatorname{ord}_M V^{mr+s} > mry + s \operatorname{ord}_M V$. Choose r_0 in such a way that for $r \geq r_0$ and $0 \leq s < m$,

$$s(y - \operatorname{ord}_M V) < \frac{\epsilon}{2}(mr + s).$$

Then for $n > mr_0$, $n = mr + s$ with $r \geq r_0$ and $0 \leq s < m$, we have

$$\lambda \geq \frac{1}{n} \operatorname{ord}_M V^n > y + \frac{s}{mr + s}(\operatorname{ord}_M V - y) > y - \frac{\epsilon}{2} = \lambda - \epsilon.$$

The lemma follows. □

Lemma 2.10. *Let (N, V) be an f-$\mathcal{O}$-isocrystal. Let $s \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geq 0}$ be integers such that* $\mathrm{Newton}(N, V) \geq \frac{r}{s}$. *Then there exists a lattice $M \subset N$ such that $V^s M \subset \pi^r M$.*

Proof. Let h be the height of N. Define $V' = \pi^{1-r(h+1)} V^{s(h+1)}$. Then (N, V') is an $(fs(h + 1))$-$\mathcal{O}$-isocrystal and $\mathrm{Newton}(N, V') \geq 1$. Therefore, there exists a lattice $M \subset N$ such that $V'^n M \subset M$ for some positive integer n. Let $M' = M + V'M + \cdots + V'^{n-1} M$. Then $V'M' \subset M'$, i.e., $(\pi^{-r} V^s)^{h+1} M' \subset \pi^{-1} M'$. The lemma follows from Lemma 2.7. □

Lemma 2.11. *Let x be a real number, $R \in \mathbb{Z}_{\geq 2}$. Then there exist integers s and r such that $1 \leq s \leq R - 1$ and $|x - \frac{r}{s}| \leq \frac{1}{Rs}$.*

Proof. For any $s \in \mathbb{Z}$, there exists $t_s \in \mathbb{R}$ such that $sx - t_s \in \mathbb{Z}$ and $-\frac{1}{R} \leq t_s < 1 - \frac{1}{R}$. Suppose for all $s \in \{1, 2, \dots, R - 1\}$, $t_s > \frac{1}{R}$, then there exist s_1 and s_2 in the set with $s_1 > s_2$ and $|t_{s_1} - t_{s_2}| \leq \frac{1}{R}$. Then $(s_1 - s_2)x - (t_{s_1} - t_{s_2}) \in \mathbb{Z}$. This is a contradiction. The lemma follows. □

Theorem 2.12. *Let (N, V) be an f-$\mathcal{O}$-isocrystal of type (h, d) and $\lambda =$ $\mathrm{Newton}(N, V)$. Then $\lambda \in \mathbb{Q}$ and there exist integers r and s with $0 < s \leq h$ and $r \leq d$, such that $\lambda = \frac{r}{s}$.*

Moreover, there exists a lattice $M \subset N$ such that $\operatorname{ord}_M V^s = r$.

Proof. By Lemma 2.11, there exist integers s and r with $1 \leq s \leq h$ such that $|\lambda - \frac{r}{s}| \leq \frac{1}{s(h+1)}$. Let $V' = \pi^{-r}V^s$, $\lambda' = \mathrm{Newton}(N, V')$. Then by Lemma 2.9,

$$|\lambda'| = |s\lambda - r| \leq \frac{1}{h+1}.$$

Thus there exists a lattice $M' \subset N$ such that $(V')^{h+1}M' \subset \pi^{-1}M'$. By Lemma 2.7, (N, V') is effective, i.e., there exists a lattice $M \subset N$ such that $V'M \subset M$. Hence

$$\mathrm{ord}_M V' \geq 0 > \lambda' - \frac{1}{h} \geq \frac{1}{h}\,\mathrm{ord}_M V'^h - \frac{1}{h}.$$

By Lemma 2.6, $\mathrm{ord}_M V' = \frac{1}{n}\,\mathrm{ord}_M V'^n$ for any $n \geq 1$ and $\lambda' = \mathrm{ord}_M V' \in \mathbb{Z}$. Hence $\lambda' = 0$. Therefore, $\lambda = \frac{r}{s}$ and $\mathrm{ord}_M V^s = r$. The theorem follows. □

2.2. Newton polygon.

DEFINITION 2.13. An f-$\mathcal{O}$-isocrystal (N, V) of type (h, d) is *isoclinic* if $\mathrm{Newton}(N, V) = \frac{d}{h}$.

Assume that k is perfect in the rest of Section 2.2. The following result is a consequence of Fitting's Lemma.

LEMMA 2.14. *Let (M, V) be an f-$\mathcal{O}$-crystal. There is a decomposition*

$$(M, V) = (M_{\mathrm{et}}, V) \oplus (M_{\mathrm{nil}}, V)$$

such that $V : M_{\mathrm{et}} \to M_{\mathrm{et}}$ is bijective and $V^n M_{\mathrm{nil}} \subset \pi M_{\mathrm{nil}}$ for $n \gg 0$.

Proof. The map V induces a map $V : M/\pi^n M \to M/\pi^n M$. Define

$$M_{n,\mathrm{nil}} = \cup_l \mathrm{Ker}(V^l : M/\pi^n M \to M/\pi^n M); \quad M_{n,\mathrm{et}} = \cap_l \mathrm{Im}(V^l : M/\pi^n M \to M/\pi^n M).$$

It is easy to see that for large r, we have $M_{n,\mathrm{nil}} = \mathrm{Ker}\, V^r$ and $M_{n,\mathrm{et}} = \mathrm{Im}\, V^r$. Moreover, $M/\pi^n M = M_{n,\mathrm{et}} \oplus M_{n,\mathrm{nil}}$, V is bijective on $M_{n,\mathrm{et}}$ and is nilpotent on $M_{n,\mathrm{nil}}$. Let $M_{\mathrm{et}} = \varprojlim M_{n,\mathrm{et}}$ and $M_{\mathrm{nil}} = \varprojlim M_{n,\mathrm{nil}}$. The claim follows. □

LEMMA 2.15. *Let (N, V) be an f-$\mathcal{O}$-isocrystal of type (h, d). The following conditions are equivalent.*

(1) *(N, V) is isoclinic.*
(2) *There exists a lattice $M \subset N$ such that $V^h M = \pi^d M$.*
(3) *There exist integers $s \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}$ and a lattice $M \subset N$ such that $V^s M = \pi^r M$.*
(4) *Let $M \subset N$ be a lattice. Then* $\mathrm{Newton}(N, V) = \lim_{n\to\infty} \frac{1}{n}\,\mathrm{ord}_M V^n m$ *for any $m \in N - \{0\}$.*

Proof. The equivalences between the first three conditions are easy. We show that they are equivalent to (4).

(2) ⇒ (4). Since $V^h M = \pi^d M$, we have

$$\mathrm{ord}_M V^{hn} m = nd + \mathrm{ord}_M m$$

for any $m \in N - \{0\}$. Choose integer c such that $|\,\mathrm{ord}_M V^a m| \leq c$ for any $0 \leq a < h$. Then

$$\Big|\frac{1}{hn+a}\,\mathrm{ord}_M V^{hn+a} m - \frac{nd}{hn+a}\Big| \leq \frac{c}{hn+a}.$$

Let $n \to \infty$, we obtain (4).

(4) $\Rightarrow$ (3). Let $\lambda = \frac{r}{s}$ be the first Newton slope of (N, V). Then there exists a lattice $M \subset N$ such that $V^s M \subset \pi^r M$. Let $V' = \pi^{-r} V^s$. Then (M, V') is an $\mathcal{O}$-crystal. Hence (M, V') has a decomposition as in Lemma 2.14,

$$(M, V') = (M_{\text{et}}, V') \oplus (M_{\text{nil}}, V').$$

Let l be a large integer so that $V'^l M_{\text{nil}} \subset \pi M_{\text{nil}}$. Suppose that $M_{\text{nil}} \neq \{0\}$. Let m be a nontrivial element of M_{nil}. Then $\operatorname{ord}_M V'^{ln} m \geq n$ for any $n \geq 1$. Hence

$$\frac{1}{sln} \operatorname{ord}_M V^{sln} m \geq \frac{n}{sln} + \frac{rln}{sln} = \frac{1}{sl} + \lambda.$$

Taking $n \to \infty$, we obtain a contradiction. Hence $M = M_{\text{et}}$ and (3) holds. The lemma follows. □

The following result is clear.

LEMMA 2.16. *Let (N, V) be an f-$\mathcal{O}$-isocrystal.*

(1) *Let $(N_1, V_1) \subset (N, V)$ be a sub-object and $(N, V) \to (N_2, V_2)$ be a quotient object. Then* $\operatorname{Newton}(N_i, V_i) \geq \operatorname{Newton}(N, V)$ *for $i = 1, 2$. The equality holds if (N, V) is isoclinic.*

(2) *Let (N', V') be an f-$\mathcal{O}$-isocrystal and* $\operatorname{Newton}(N', V') > \operatorname{Newton}(N, V)$. *If (N, V) is isoclinic, then*

$$\operatorname{Hom}((N, V), (N', V')) = \operatorname{Hom}((N', V'), (N, V)) = \{0\}.$$

LEMMA 2.17. *Let (N, V) be an f-$\mathcal{O}$-isocrystal with first Newton slope $\lambda = r/s$. Then (N, V) has a unique decomposition*

$$(N, V) = (N_1, V_1) \oplus (N_2, V_2),$$

where (N_1, V_1) is isoclinic with first Newton slope λ and $\operatorname{Newton}(N_2, V_2) > \lambda$.

Proof. Consider a lattice $M \subset N$ with $V^s M \subset \pi^r M$. Let $V' = \pi^{-r} V^s$. Then M has a decomposition as in Lemma 2.14: $(M, V') = (M_{\text{et}}, V') \oplus (M_{\text{nil}}, V')$. Tensoring with F_k, we obtain a decomposition $(N, V') = (N_{\text{et}}, V') \oplus (N_{\text{nil}}, V')$. Here (N_{et}, V') is isoclinic with first Newton slope 0 and $\operatorname{Newton}(N_{\text{nil}}, V') > 0$. This decomposition is unique by Lemma 2.16. The induced decomposition $(N, V) = (N_{\text{et}}, V) \oplus (N_{\text{nil}}, V)$ is the desired one. The lemma follows. □

By successive applications of Lemma 2.17, the following result is clear.

THEOREM 2.18. *Assume that k is perfect. Let (N, V) be an f-$\mathcal{O}$-isocrystal over k with first Newton slope λ. There exist uniquely determined sub-$\mathcal{O}$-isocrystals (N_i, V_i) with* $\operatorname{Newton}(N_i, V_i) = \lambda_i$, *such that $(N, V) = \oplus_{i=1}^r (N_i, V_i)$ and $\lambda = \lambda_1 < \lambda_2 < \cdots < \lambda_r$.*

The decomposition in the theorem is called the *Newton decomposition* of (N, V). We call the numbers λ_i the *Newton slopes* of (N, V). If (N_i, V_i) is of type (h_i, d_i), then (N, V) is of type (h, d), where $h = \sum_i h_i$ and $d = \sum_i d_i$. Note that $\lambda_i = d_i/h_i$. We obtain the sequence of Newton slopes, in which each λ_i is repeated h_i times:

$$(\mu_1, \cdots, \mu_h) = (\lambda_1, \cdots, \lambda_1, \cdots, \lambda_r, \cdots, \lambda_r).$$

Define $\operatorname{Newton}_{(N,V)}(i) = \sum_{j=1}^i \mu_j$ for $1 \leq i \leq h$ and $\operatorname{Newton}_{(N,V)}(0) = 0$. The graph of the function $\operatorname{Newton}_{(N,V)}$ is called the *Newton polygon* of (N, V). Note that the starting point is $(0, 0)$ and the ending point is (h, d).

2.3. Classification of f-$\mathcal{O}$-isocrystals over algebraically closed k. Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Let $r, s \in \mathbb{Z}$ with $s > 0$ and $(r, s) = 1$. Let $N_{r,s}$ be an F_k-vector space with a basis $e_1, \cdots, e_s$. We define on $N_{r,s}$ a structure of f-$\mathcal{O}$-isocrystal by

$$Ve_i = \begin{cases} e_{i+1} & \text{if } i < s, \\ \pi^r e_1 & \text{if } i = s. \end{cases}$$

LEMMA 2.19. *The f-$\mathcal{O}$-isocrystal $N_{r,s}$ is isoclinic of slope r/s and it contains no proper sub-$\mathcal{O}$-isocrystal.*

Proof. The first claim is clear. Assume that (N, V) with type (h, d) is a sub-$\mathcal{O}$-isocrystal of $N_{r,s}$ such that $h < s$. By Theorem 2.18, (N, V) is isoclinic of slope r/s. Hence $d/h = r/s$. The lemma follows since r and s are coprime. □

THEOREM 2.20. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Let (N, V) be an f-$\mathcal{O}$-isocystal over k. Then there exists a direct sum decomposition*

$$(N, V) = \oplus_{i=1}^{u} N_{r_i,s_i}^{t_i},$$

where $u,\ r_i,\ s_i,\ t_i \in \mathbb{Z}$, $u,\ t_i,\ s_i > 0$, $(s_i, r_i) = 1$, $r_1/s_1 < \cdots < r_u/s_u$. Moreover, the numbers $u,\ r_i,\ s_i,\ t_i$ are uniquely determined by (N, V).

The argument is similar to the argument for isocrystals (cf. [47, Chapter 6, Section 4]). It follows from the following three lemmas.

LEMMA 2.21. *Let V be a nonzero finite dimensional k-vector space, a be a positive integer, $\phi : V \to V$ be a $\mathbb{Z}$-linear isomorphism such that $\phi(xv) = x^{q^a}\phi(v)$ for all $x \in k$ and $v \in V$. Then there exists a basis $e_1,\ \ldots,\ e_n$ of V such that $\phi(e_i) = e_i$.*

Proof. Let $v \in V$ be a nonzero vector. Let r be the greatest integer such that the vectors $v,\ \phi(v),\ \ldots,\ \phi^{r-1}(v)$ are linearly independent. Assume that

$$\phi^r(v) = \sum_{i=0}^{r-1} h_i\phi^i(v),$$

where $h_i \in k$ and at least one of them is not zero. We claim that there exists a vector $w = \sum_{i=0}^{r-1} x_i\phi^i(v)$ such that $\phi(w) = w$. Indeed, the equation $\phi(w) = w$ is equivalent to

$$\sum_{i=0}^{r-2} x_i^{q^a}\phi^{i+1}(v) + \sum_{i=0}^{r-1} x_{r-1}^{q^a} h_i\phi^i(v) = \sum_{i=0}^{r-1} x_i\phi^i(v).$$

Comparing the coefficients, we deduce that

$$x_{r-1} = h_0^{q^{a(r-1)}} x_{r-1}^{q^{ar}} + \cdots + h_{r-1}x_{r-1}^{q^a}.$$

The above equation has a nonzero solution since k is algebraically closed. Hence we obtain a nonzero $w \in V$ with $\phi(w) = w$.

Let $e_1, \cdots, e_r$ be a system of linearly independent ϕ-invariant vectors. Let W be the subspace generated by these elements. If $W = V$, we are done. If $W \neq V$, then applying the argument above to the space V/W, there exists an element $e_{r+1} \in V$ such that

$$\phi(e_{r+1}) = e_{r+1} + \sum_{i=1}^{r} a_i e_i.$$

Let y_i $(1 \leq i \leq r)$ be elements of k, such that $a_i - y_i + y_i^{q^a} = 0$ $(1 \leq i \leq r)$. Let $e'_{r+1} = e_{r+1} + \sum_{i=1}^{r} y_i e_i$. Then $\phi(e'_{r+1}) = e'_{r+1}$. The lemma follows since V is finite dimensional. □

LEMMA 2.22. *Let (M, V) be an f-$\mathcal{O}$-crystal over k such that $VM = M$. Then there exists a basis $e_1, \ldots, e_r$ of M such that $Ve_i = e_i$ for $1 \leq i \leq r$.*

Proof. Without loss of generality, we may assume that V is ${}^{F^a}$-linear and a is positive. It suffices to construct a basis $e_1^{(n)}, \ldots, e_r^{(n)}$ such that

$$Ve_i^{(n)} \equiv e_i^{(n)} \pmod{\pi^n}, \quad e_i^{(n)} \equiv e_i^{(n-1)} \pmod{\pi^{n-1}},$$

since we may then take limits to obtain a basis with the required properties.

For $n = 1$, the construction follows from Lemma 2.21. The induction step goes as follows. Let $f_i = e_i^{(n)}$. We may write

$$Vf_i - f_i = \pi^n \sum_{j=1}^{r} a_{ij} f_j, \quad a_{ij} \in A_k.$$

Choose $x_{ij} \in A_k$ such that

$$\overline{a}_{ij} + \overline{x}_{ij}^{q^a} - \overline{x}_{ij} = 0.$$

Here $\overline{x} \in k$ is the image of x under the natural projection $A_k \to k$. Define $e_i^{(n+1)} = f_i + \pi^n \sum_{j=1}^{r} x_{ij} f_j$ $(1 \leq i \leq r)$. It is easy to check that these elements have the expected properties. The lemma follows. □

LEMMA 2.23. *Let (N, V) be an isoclinic f-$\mathcal{O}$-isocrystal of slope r/s where $s \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}$, r and s are coprime. Then (N, V) is a direct sum of copies of $N_{r,s}$.*

Proof. Choose a lattice $M \subset N$ such that $V^s M = \pi^r M$. Let $m_1, \ldots, m_h$ be a basis of M such that $V^s m_i = \pi^r m_i$ $(1 \leq i \leq h)$. Let $N_i = (\sum_{j=0}^{s-1} V^j m_i) \otimes F_k$. Then we have a surjection of f-$\mathcal{O}$-isocrystals

$$\begin{aligned} N_{r,s} &\to N_i \\ e_j &\mapsto V^{j-1} m_i. \end{aligned} \tag{2.3.1}$$

We see that $(N, V) = \sum_{i=1}^{h} (N_i, V)$ and for each i we have $(N_i, V) \cong N_{r,s}$. The lemma follows from Lemma 2.19. □

3. π-divisible $\mathcal{O}$-modules. In this section, using Dieudonné modules and displays, we deduce several classification results for π-divisible $\mathcal{O}$-modules over fields of characteristic p. As an application, we prove the existence of slope filtration for π-divisible $\mathcal{O}$-modules under some technical conditions, which is analogous to [43, Theorem 7] and [32, Theorem 2.1].

3.1. π-divisible $\mathcal{O}$-modules and $\mathcal{O}$-displays.

DEFINITION 3.1. Let S be an $\mathcal{O}$-scheme such that π is locally nilpotent in $\mathcal{O}_S$. Let X be a p-divisible group over S (cf. [25, Chap. 1, Definition (2.1)] and [38, (2.1) Definitions]). Assume that there exists an $\mathcal{O}$-action on X given by $\iota : \mathcal{O} \to \mathrm{End}(X)$. We call that X is a *π-divisible $\mathcal{O}$-module* if the action of $\mathcal{O}$ on $\mathrm{Lie}(X)$ induced from ι coincides with the action induced from $\mathcal{O} \to \mathcal{O}_S$.

To describe π-divisible $\mathcal{O}$-modules from other perspectives, we introduce $\mathcal{O}$-group schemes.

DEFINITION 3.2. Let S be an $\mathcal{O}$-scheme such that π is locally nilpotent in $\mathcal{O}_S$. Let G be a finite locally free group scheme over S. We call G an *$\mathcal{O}$-group scheme* if it is the kernel of an isogeny between π-divisible $\mathcal{O}$-modules.

REMARK 3.3. The above definition is different from the definition of *strict $\mathcal{O}$-group scheme* [10, Definition 1]. We take this definition to avoid the complication coming from embedding group schemes into π-divisible $\mathcal{O}$-modules. On the other hand, if G is obtained from an $\mathcal{O}$-module, then it is *strict* in the sense of [10].

REMARK 3.4. Let S be an $\mathcal{O}$-scheme such that π is locally nilpotent in $\mathcal{O}_S$. A *π-divisible $\mathcal{O}$-module* over S is an fppf sheaf X of $\mathcal{O}$-module over S, which satisfies the following conditions:

(1) X is π-divisible, i.e., the homomorphism $\pi : X \to X$ is an epimorphism.
(2) X is π-torsion, i.e., the canonical morphism $\varinjlim X[\pi^n] \to X$ is an isomorphism. Here $X[\pi^n]$ denotes the kernel of the multiplication by $\pi^n : X \to X$.
(3) $X[\pi]$ is representable by a finite locally free $\mathcal{O}$-group scheme over S.

Homomorphisms of π-divisible $\mathcal{O}$-modules over S are homomorphisms of sheaves that are compatible with the $\mathcal{O}$-actions.

The order of $X[\pi]$ is of the form q^h, where $h : S \to \mathbb{Z}_{\geq 0}$ is a locally constant function, called the *height* of X.

The *type* of X is the pair (h, d), where h is the height of X and d is the dimension of X. Note that $h \geq d$ and one may observe this via $\mathcal{O}$-displays. For example, the Lubin-Tate group associated with $(\mathcal{O}, \pi)$ in local class field theory is of type $(1, 1)$ as a π-divisible $\mathcal{O}$-module.

REMARK 3.5. The description in Remark 3.4 is similar to [25, Chap. 1, Definition (2.1)]. As explained in [25, Chap. 1, Remark (2.3)], in terms of Tate's definition (cf. [38, (2.1) Definitions]), a π-divisible $\mathcal{O}$-module over S is an inductive system

$$G = (G_v, i_v), \quad v \geq 0,$$

where

- G_v is a locally free $\mathcal{O}$-group scheme over S of order q^{vh},
- for each $v \geq 0$, the sequence

$$0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{\pi^v} G_{v+1}$$

  is exact.

In this language, a homomorphism $f : G = (G_v, i_v) \to H = (H_v, i_v)$ of π-divisible $\mathcal{O}$-modules is a system of homomorphisms $f_v : G_v \to H_v$ which satisfies $i_v \circ f_v = f_{v+1} \circ i_v$ for all $v \geq 1$.

Remark 3.6. Let X be a π-divisible $\mathcal{O}$-module over S. Let $X^\vee$ be the usual Serre dual of X. The $\mathcal{O}$-action on X induces an $\mathcal{O}$-action on $X^\vee$. But this action in general does not satisfy the condition in Definition 3.1 and the Serre dual of a π-divisible $\mathcal{O}$-module is not necessarily a π-divisible $\mathcal{O}$-module. To fix this, we use Serre $\mathcal{O}$-dual (cf. Definition 3.15).

We review the classification of π-divisible $\mathcal{O}$-modules by $\mathcal{O}$-displays.

Definition 3.7. Let R be an $\mathcal{O}$-algebra. An *$\mathcal{O}$-display* over R is a quadruple $\mathcal{P} = (P, Q, F, F_1)$, where P is a finitely generated projective $W_{\mathcal{O}}(R)$-module, $Q \subset P$ is a submodule, $F : P \to P$ and $F_1 : Q \to P$ are ^{F}-linear maps, such that the following conditions hold:

1. There exists a decomposition $P = L \oplus T$, such that $Q = L \oplus I_{\mathcal{O}}(R)T$, where $I_{\mathcal{O}}(R) = {}^V W_{\mathcal{O}}(R)$. Such a decomposition is called a *normal decomposition* of $\mathcal{P}$.
2. $F_1 : Q \to P$ is an ^{F}-linear epimorphism.
3. For any $x \in P$ and $w \in W_{\mathcal{O}}(R)$, we have

$$F_1({}^V wx) = wFx.$$

Denote by $\operatorname{disp}_{\mathcal{O}}/R$ the category of $\mathcal{O}$-displays over R.

Let R be an $\mathcal{O}$-algebra and $\mathcal{P}$ an $\mathcal{O}$-display over R. There exists a unique $W_{\mathcal{O}}(R)$-linear map

$$V^\sharp : P \to W_{\mathcal{O}}(R) \otimes_{W_{\mathcal{O}}(R), {}^F} P,$$

which satisfies the following equations for all $w \in W_{\mathcal{O}}(R), x \in P$ and $y \in Q$:

$$\begin{aligned} V^\sharp(wFx) &= \pi \cdot w \otimes x, \\ V^\sharp(wF_1 y) &= w \otimes y. \end{aligned}$$

By $V^{n\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{W_{\mathcal{O}}(R), {}^{F^n}} P$ we mean the composite map ${}^{F^{(n-1)}}V^\sharp \circ \ldots \circ {}^F V^\sharp \circ V^\sharp$, where ${}^{F^i}V^\sharp$ is the $W_{\mathcal{O}}(R)$-linear map

$$\mathrm{id} \otimes_{W_{\mathcal{O}}(R), {}^{F^i}} V^\sharp : W_{\mathcal{O}}(R) \otimes_{W_{\mathcal{O}}(R), {}^{F^i}} P \to W_{\mathcal{O}}(R) \otimes_{W_{\mathcal{O}}(R), {}^{F^{i+1}}} P.$$

Definition 3.8. Let $R \in \mathrm{Alg}_{\mathcal{O}}$ with π nilpotent in R and $\mathcal{P}$ be an $\mathcal{O}$-display over R. We call $\mathcal{P}$ *nilpotent*, if there exists a number N such that the composite map

$$\mathrm{pr} \circ V^{N\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{W_{\mathcal{O}}(R), {}^{F^N}} P \to W_{\mathcal{O}}(R)/(I_{\mathcal{O},R} + \pi W_{\mathcal{O}}(R)) \otimes_{W_{\mathcal{O}}(R), {}^{F^N}} P$$

is the zero map.

Denote by $\operatorname{ndisp}_{\mathcal{O}}/R$ the subcategory of $\operatorname{disp}_{\mathcal{O}}/R$ consisting of nilpotent objects.

To give the definition of Dieudonné $\mathcal{O}$-displays, we recall the construction of $\widehat{W}_{\mathcal{O}}$ from [2, Section 1.2.1]. Let R be a local $\mathcal{O}$-algebra. Assume that R is an Artinian local ring with perfect residue field k. Let $\mathfrak{m} \subset R$ be the maximal ideal of R. Then we have the following exact sequence

$$0 \to W_{\mathcal{O}}(\mathfrak{m}) \to W_{\mathcal{O}}(R) \xrightarrow{\tau} W_{\mathcal{O}}(k) \to 0.$$

It admits a canonical section $\delta : W_{\mathcal{O}}(k) \xrightarrow{\Delta} W_{\mathcal{O}}(W_{\mathcal{O}}(k)) \to W_{\mathcal{O}}(R)$, which is a ring homomorphism commuting with ^{F}. Here Δ is the unique natural morphism (Cartier morphism) of $\mathcal{O}$-algebras

$$\Delta : W_{\mathcal{O}}(-) \longrightarrow W_{\mathcal{O}}(W_{\mathcal{O}}(-))$$

such that $\mathcal{W}(\Delta(x)) = [{}^{F^n}x]_{n\geq 0}$, where $\mathcal{W} = (w_0, w_1, \dots)$ is given by the Witt polynomials. The Cartier morphism is the morphism E in [13, Theorem 6.17].

Since $\mathfrak{m}$ is nilpotent, we have a subalgebra of $W_{\mathcal{O}}(\mathfrak{m})$:

$$\widehat{W}_{\mathcal{O}}(\mathfrak{m}) = \{(x_0, x_1, \cdots) \in W_{\mathcal{O}}(\mathfrak{m}) \mid x_i = 0 \text{ for all but finitely many } i\}.$$

Note that $\widehat{W}_{\mathcal{O}}(\mathfrak{m})$ is stable under ^{F} and ^{V}.

Definition 3.9. In the case R is Artinian, we define the subring $\widehat{W}_{\mathcal{O}}(R) \subset W_{\mathcal{O}}(R)$ by

$$\widehat{W}_{\mathcal{O}}(R) = \{\xi \in W_{\mathcal{O}}(R) \mid \xi - \delta\tau(\xi) \in \widehat{W}_{\mathcal{O}}(\mathfrak{m})\}.$$

Again we have an exact sequence

$$0 \to \widehat{W}_{\mathcal{O}}(\mathfrak{m}) \to \widehat{W}_{\mathcal{O}}(R) \xrightarrow{\tau} W_{\mathcal{O}}(k) \to 0$$

with a canonical section δ of τ.

In the case R is a complete Noetherian local $\mathcal{O}$-algebra with perfect residue field k, we define $\widehat{W}_{\mathcal{O}}(R) := \varprojlim \widehat{W}_{\mathcal{O}}(R/\mathfrak{m}_R^n)$, where $\mathfrak{m}_R \subset R$ is the maximal ideal.

We also define $\widehat{I}_{\mathcal{O}}(R) = {}^V(\widehat{W}_{\mathcal{O}}(R))$.

By [2, Lemma 1.8], if $p \geq 3$, $\widehat{W}_{\mathcal{O}}(R)$ is stable under ^{F} and ^{V}. If $p = 2$, the same holds for R with $\pi R = 0$ (cf. [44, Lemma 2]).

Definition 3.10. Let R be a complete Noetherian local $\mathcal{O}$-algebra with perfect residue field k of characteristic p. Assume that $\pi R = 0$ if $p = 2$. A *Dieudonné $\mathcal{O}$-display* over R is a quadruple $\mathcal{P} = (P, Q, F, F_1)$, where P is a finitely generated prjective $\widehat{W}_{\mathcal{O}}(R)$-module, $Q \subset P$ is a submodule, $F : P \to P$ and $F_1 : Q \to P$ are ^{F}-linear maps, such that the following conditions hold:

(1) There exists a decomposition $P = L \oplus T$, such that $Q = L \oplus \widehat{I}_{\mathcal{O}}(R)T$. Such a decomposition is called a *normal decomposition* of $\mathcal{P}$.
(2) $F_1 : Q \to P$ is an ^{F}-linear epimorphism.
(3) For any $x \in P$ and $w \in \widehat{W}_{\mathcal{O}}(R)$, we have

$$F_1({}^V wx) = wFx.$$

Denote by $\mathrm{Ddisp}_{\mathcal{O}}/R$ the category of Dieudonné $\mathcal{O}$-displays over R.

The main result of [2] is the following (cf. [2, Section 1.1]).

Theorem 3.11. *Let $R \in \mathrm{Alg}_{\mathcal{O}}$ with π nilpotent in R. There exists a covariant functor* $\mathrm{BT}_{\mathcal{O}}$

$$\mathrm{BT}_{\mathcal{O}} : \mathrm{ndisp}_{\mathcal{O}}/R \to (\pi\text{-divisible formal } \mathcal{O}\text{-modules}/R),$$

which is an equivalence of categories.

Let R be a Noetherian complete local $\mathcal{O}$-algebra with perfect residue field of characteristic p. Assume that $\pi R = 0$ if $p = 2$. Then the equivalence $\mathrm{BT}_{\mathcal{O}}$ *above extends to an equivalence*

$$\mathrm{BT}_{\mathcal{O}} : \mathrm{Ddisp}_{\mathcal{O}} /R \to (\pi\text{-divisible } \mathcal{O}\text{-modules}/R).$$

REMARK 3.12. In [2, Theorem 1.5], there is the assumption that $p \neq 2$. This is saved by adding the condition $\pi R = 0$ if $p = 2$. In this case, $\widehat{W}_{\mathcal{O}}(R)$ is stable under ^{F} and ^{V} by [2, Lemma 1.8] and [44, Lemma 2], the notion of Dieudonné $\mathcal{O}$-display over R is welled-defined and the proof of [2, Theorem 1.5] works for $p = 2$ if $\pi R = 0$. This condition is vacuous if the base R is an $\mathbb{F}$-algebra.

REMARK 3.13. In [22], Lau modified the Verschiebung of $W(R)$ and used it to define Dieudonné displays. The advantage of Lau's construction is that $\widehat{W}(R)$ (denoted by $\mathbb{W}(R)$ in [22]) is stable under Frobenius and the modified Verschiebung without assumption for $p = 2$ (cf. [22, Lemma 1.7]). By [22, Theorem A], for R a Noetherian complete local ring with perfect residue field of characteristic p, the category of Dieudonné displays over R and the category of p-divisible groups over R are equivalent. One could adapt Lau's idea and modify the definition of Dieudonné $\mathcal{O}$-displays as follows to remove the assumption for $p = 2$ in Theorem 3.11.

Let $[\pi] \in W_{\mathcal{O}}(\mathcal{O})$ be the Teichmüller lift of π. Then $\pi - [\pi]$ is in the image of the Verschiebung ^{V} and $\epsilon := {}^{V^{-1}}(\pi - [\pi]) \in W_{\mathcal{O}}(\mathcal{O})$ is a unit (cf. [2, Lemma 2.24]). This unit is under the notation u_0 in [22]. Let R be a complete Noetherian local $\mathcal{O}$-algebra with perfect residue field k of characteristic p. Denote the image of ϵ in $W_{\mathcal{O}}(R)$ by ϵ as well. Define ${}^{\mathbb{V}} : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ by ${}^{\mathbb{V}}x = {}^V\epsilon x$. Then $\widehat{W}_{\mathcal{O}}(R)$ is stable under ^{F} and ${}^{\mathbb{V}}$. If $p = 2$, replacing ^{V} by ${}^{\mathbb{V}}$ in Definition 3.10, we obtain Lau's definition of Dieudonné $\mathcal{O}$-displays over R.

REMARK 3.14. Let X be a p-divisible formal group over R and let (P, Q, F, F_1) be the associated display over R via the result of Zink-Lau. An $\mathcal{O}$-action on X corresponds to an $\mathcal{O}$-action on (P, Q, F, F_1). In particular, P is a module over $W(R) \otimes \mathcal{O}$. If the $\mathcal{O}$-action is strict in the sense that X is a formal π-divisible $\mathcal{O}$-module (cf. Definition 3.1, [2, Definition 2.4]), from the $W(R) \otimes \mathcal{O}$-module P, one could then construct $W_{\mathcal{O}}(R)$ modules and a well-defined $\mathcal{O}$-display over R (cf. [45, Proposition 29], [2, Section 2.5]). This is crucial for the induction step in the proof of Theorem 3.11.

The equivalences in Theorem 3.11 provide us a powerful tool in the study of π-divisible $\mathcal{O}$-modules, in particular in the study of deformations (cf. Section 6). On the other hand, deformations of p-divisible groups with general $\mathcal{O}$-action are rather difficult to describe. See [3, 9] for some special examples.

DEFINITION 3.15. Let $\mu_{\mathcal{O}}$ be the π-divisible $\mathcal{O}$-module attached to the $\mathcal{O}$-display

$$(W_{\mathcal{O}}(R),\ {}^V W_{\mathcal{O}}(R),\ {}^F,\ {}^{V^{-1}})$$

via Theorem 3.11. Let X be a π-divisible $\mathcal{O}$-module over R. Then $\mathrm{Hom}(X, \mu_{\mathcal{O}})$ is a π-divisible $\mathcal{O}$-module over R and we call it the *Serre $\mathcal{O}$-dual* of X.

We note that there is a notion of dual $\mathcal{O}$-display and [2] showed that the functor $\mathrm{BT}_{\mathcal{O}}$ in Theorem 3.11 is compatible with duality. Theorem 3.11 works for very general base R. In next section, we take R to be a field k of characteristic p and deduce some classification results. In Section 6, we apply Theorem 3.11 to study deformations of π-divisible $\mathcal{O}$-modules.

3.2. π-divisible $\mathcal{O}$-modules over fields of characteristic p.

3.2.1. The perfect case.

DEFINITION 3.16. Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a perfect field of characteristic p. A *Dieudonné $\mathcal{O}$-module* over k is a free $W_{\mathcal{O}}(k)$-module M of finite rank with operators $F,\ V : M \to M$, such that

(1) $F(\xi m) = {}^{F}\xi F(m)$,
(2) $V({}^{F}\xi m) = \xi V(m)$,
(3) $FV = \pi$.

A Dieudonné $\mathcal{O}$-module M is called *reduced* if $V : M \to M$ is nilpotent modulo πM.

If (M, F, V) is a Dieudonné $\mathcal{O}$-module over k, then the pair (M, V) is a (-1)-$\mathcal{O}$-crystal over k. Conversely, if (M, V) is a (-1)-$\mathcal{O}$-crystal over k such that $\pi M \subset VM$, then $(M, \pi V^{-1}, V)$ is a Dieudonné $\mathcal{O}$-module over k. This Dieudonné $\mathcal{O}$-module is reduced if V is nilpotent on $M/\pi M$. Denote by Crys_k the category of (-1)-$\mathcal{O}$-crystals (M, V) over k with $\pi M \subset VM$. Denote by RdCrys_k the subcategory of Crys_k consisting of reduced objects.

PROPOSITION 3.17. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a perfect field of characteristic p.*

(1) *The category RdCrys_k is equivalent to the category of formal π-divisible $\mathcal{O}$-modules over k.*
(2) *The category Crys_k is equivalent to the category of π-divisible $\mathcal{O}$-modules over k.*

Proof. We only prove the first statement, the other one is entirely similar. By the above discussion and Theorem 3.11, it suffices to establish an equivalence between the category of reduced Dieudonné $\mathcal{O}$-modules over k and the category $\mathrm{ndisp}_{\mathcal{O}}/k$. Let (M, F, V) be a Dieudonné $\mathcal{O}$-module over k. Define

$$\mathcal{P} = (M, VM, F : M \to M, F_1 := V^{-1} : VM \to M).$$

Then $\mathcal{P}$ is an $\mathcal{O}$-display over k. Moreover, the map $V^{\sharp}$ is given by

$$\begin{aligned} V^{\sharp} : M &\to W_{\mathcal{O}}(k) \otimes_{W_{\mathcal{O}}(k), F} M \\ m &\mapsto 1 \otimes Vm. \end{aligned}$$

The $\mathcal{O}$-display $\mathcal{P}$ is nilpotent if and only if the Dieudonné $\mathcal{O}$-module M is reduced.

Conversely, let $\mathcal{P} = (P, Q, F, F_1)$ be an $\mathcal{O}$-display over k. Define $V : P \to P$ by

$$\begin{aligned} V : P \xrightarrow{V^{\sharp}} W_{\mathcal{O}}(k) \otimes_{W_{\mathcal{O}}(k), F} P &\to P \\ w \otimes x &\mapsto {}^{F^{-1}}wx. \end{aligned}$$

This makes sense because k is perfect and ^{F} is an automorphism of $W_{\mathcal{O}}(k)$. Thus we obtain a Dieudonné $\mathcal{O}$-module over k. The proposition follows. □

REMARK 3.18. Let (M, V) be a (-1)-$\mathcal{O}$-crystal. Then it is an $\mathcal{O}$-crystal attached to a (formal) π-divisible $\mathcal{O}$-module over k via the above equivalence if and only if its Newton slopes are in the interval $[0, 1]$ ($(0, 1]$).

Let $r, s \in \mathbb{Z}_{\geq 0}$ with $s \geq r$ and $(r, s) = 1$. Define a (-1)-$\mathcal{O}$-crystal $M_{r,s}$ as follows. As a free $W_{\mathcal{O}}(k)$-module, $M_{r,s}$ has rank s. Let $e_1, \dots, e_s$ be a basis of $M_{r,s}$. Define $V : M_{r,s} \to M_{r,s}$ by

$$Ve_i = \begin{cases} \pi e_{i-1} & \text{if } 2 \leq i \leq r+1, \\ e_{i-1} & \text{if } r+2 \leq i \leq s, \\ e_s & \text{if } i = 1. \end{cases}$$

It is a lattice of $N_{r,s}$ (cf. Section 2.3). The corresponding ^{F}-linear morphism $F : M_{r,s} \to M_{r,s}$ is defined by

$$Fe_i = \begin{cases} e_{i+1} & \text{if } 1 \leq i \leq r, \\ \pi e_{i+1} & \text{if } r+1 \leq i \leq s-1, \\ \pi e_1 & \text{if } i = s. \end{cases}$$

Via the above equivalence, attached to $M_{r,s}$, there is a unique π-divisible $\mathcal{O}$-module $G_{r,s}$ defined by the following exact sequence

$$0 \to M_{r,s} \xrightarrow{F^r - V^{s-r}} M_{r,s} \to G_{r,s} \to 0.$$

Here we regard $M_{r,s}$ as a sheaf by base change (cf. [2, Theorem 2.12]). Now Theorem 2.20 gives us the following result.

Theorem 3.19. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Every π-divisible $\mathcal{O}$-module over k is isogenous to a direct product of $\mathcal{O}$-modules $G_{r,s}$.*

Remark 3.20. Sometimes, it is useful to write $N_{r,s} = F_k\langle F\rangle/(F^s - \pi^{s-r})$ and $M_{r,s} = A_k[F, V]/(FV - \pi, F^r - V^{s-r})$. Then we embed $M_{r,s}$ into $N_{r,s}$ by sending V to $\pi^{r+1-s}F^{s-1}$.

3.2.2. The imperfect case. Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a field of characteristic p. Denote by k^{1/q^n} the field extension of k by adjoining the q^n-th roots of elements of k.

Definition 3.21. A *potential formal π-divisible $\mathcal{O}$-module over k* is a pair (Z, n), where $n \in \mathbb{Z}_{\geq 0}$, Z is a formal π-divisible $\mathcal{O}$-module over k^{1/q^n}.

Lemma 3.22. *Let X and Y be formal π-divisible $\mathcal{O}$-modules over k. Then the natural map*

$$\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_{k^{1/q^n}}(X_{k^{1/q^n}}, Y_{k^{1/q^n}})$$

is a bijection.

Proof. We may assume that $n = 1$. Let $R = k^{1/q} \otimes_k k^{1/q}$. Then we have an exact sequence

$$\mathrm{Hom}_k(X, Y) \hookrightarrow \mathrm{Hom}_{k^{1/q}}(X_{k^{1/q^n}}, Y_{k^{1/q^n}}) \rightrightarrows \mathrm{Hom}_R(X_R, Y_R),$$

where the last two arrows are induced from $p_1, p_2 : R = k^{1/q} \otimes_k k^{1/q} \to k^{1/q}$. In order to prove the lemma, it suffices to show that $p_1^*(\phi) = p_2^*(\phi)$ for any $\phi \in \mathrm{Hom}_{k^{1/q}}(X_{k^{1/q^n}}, Y_{k^{1/q^n}})$.

Let $I = \mathrm{Ker}(\mu : R = k^{1/q} \otimes_k k^{1/q} \to k^{1/q})$. Here μ is induced from multiplication. Then $\pi I = 0$ and $I^q = 0$. The claim follows from the rigidity lemma [16, Lemma 1.1.3]. Note that if a formal Lie group H is associated with an $\mathcal{O}$-module, then it is $\mathcal{O}$-special in the sense that the $\mathcal{O}$-action $\mathcal{O} \to \mathrm{End}(H)$ is given by

$$([a]X)_i = aX_i + \text{terms with higher degree}$$

for all $a \in \mathcal{O}$. □

Definition 3.23. Let (Z, n), (T, m) be two potential formal π-divisible $\mathcal{O}$-modules over k. Define

$$\mathrm{Hom}((Z, n), (T, m)) = \mathrm{Hom}_{k^{1/q^l}}(Z_{k^{1/q^l}}, T_{k^{1/q^l}}),$$

where $l = \max\{m, n\}$.

Definition 3.24. A *Dieudonné $\mathcal{O}$-module* over k is a finitely generated free A_k-module M, together with an ^{F}-linear map $F : M \to M$ and an ${}^{F^{-1}}$-linear map $V : M \to M$, such that $FV = \pi$.

A Dieudonné $\mathcal{O}$-module is *reduced* if the operator V on $M/\pi M$ is nilpotent.

Proposition 3.25. *With the notation as above. The category of potential formal π-divisible $\mathcal{O}$-modules over k is equivalent to the category of reduced Dieudonné $\mathcal{O}$-modules over k.*

Proof. Let (X, n) be a potential formal π-divisible $\mathcal{O}$-module. Let (P, Q, F, F_1) be the $\mathcal{O}$-display over k^{perf} associated with $X \otimes_{k^{1/q^n}} k^{\mathrm{perf}}$. Then as in the proof of Proposition 3.17, $P \otimes_{W_{\mathcal{O}}(k^{\mathrm{perf}})} A_k$ is a reduced Dieudonné $\mathcal{O}$-module over k.

Conversely, let (M, F, V) be a reduced Dieudonné $\mathcal{O}$-module over k. We have a short exact sequence

$$0 \to VM/\pi M \to M/\pi M \to M/VM \to 0.$$

Let $(e_1, \ldots, e_h)$ be a basis of M over A_k. Assume that $(e_1, \ldots, e_d)$ induces a basis of M/VM and $(e_{d+1}, \ldots, e_h) \subset VM$. Define an $\mathcal{O}$-display structure on M via the structure equation

$$\begin{cases} Fe_i = \sum_{j=1}^h \xi_{ij} e_j, \text{ for } i = 1, \ldots, d, \\ F_1 e_j = V^{-1} e_j = \sum_{j=1}^h \xi_{ij} e_j, \text{ for } i = d+1, \ldots, h. \end{cases} \tag{3.2.1}$$

Since $\xi_{ij} \in A_k$, there exists a big N, such that $\xi_{ij} \in W_{\mathcal{O}}(k^{1/q^N})$ for all i, j. Hence we obtain an $\mathcal{O}$-display over k^{1/q^N}, which is nilpotent since M is reduced. Therefore we obtain a π-divisible formal $\mathcal{O}$-module over k^{1/q^N}.

The above two constructions are clearly inverse of each other. The proposition follows. □

Remark 3.26. The category of potential formal π-divisible $\mathcal{O}$-modules over k and the category of formal π-divisible $\mathcal{O}$-modules over k are obviously different. Yet they are the same up to isogeny. More precisely, let Isog_k be the category of π-divisible formal $\mathcal{O}$-modules over k up to isogeny, i.e., the objects are π-divisible $\mathcal{O}$-modules over k, and the morphisms are given by

$$\mathrm{Hom}_{\mathrm{Isog}}(X, Y) := \mathrm{Hom}(X, Y) \otimes \mathbb{Q}.$$

Let (Z,n) be a potential π-divisible $\mathcal{O}$-module over k. Note that $\mathrm{Fr}^n : k^{1/q^n} \to k^{1/q^n}$ factors through k, the π-divisible formal $\mathcal{O}$-module $Z^{(n)} = Z \otimes_{k^{1/q^n},\mathrm{Fr}^n} k^{1/q^n}$ is defined over k. Moreover, we have an isogeny $\mathrm{Fr}^n_Z : Z \to Z^{(n)}$, which is an isogeny of potential formal π-divisible $\mathcal{O}$-modules (Z,n) and $(Z^{(n)},1)$.

3.3. π-divisible $\mathcal{O}$-modules over S. Let S be an $\mathbb{F}$-scheme. In this section, we study π-divisible $\mathcal{O}$-modules over S. Such a module is also called *a family of π-divisible $\mathcal{O}$-modules.*

3.3.1. Decomposition of Frobenius $\mathcal{O}$-modules. We first study Frobenius $\mathcal{O}$-modules and obtain a similar decomposition as in Lemma 2.14. The following materials are similar to those in [43, Section 2]. The difference is that we now work on $\mathcal{O}$-algebras. Denote by $\mathrm{Fr}_S : S \to S$ the Frobenius morphism of S.

Definition 3.27. Fix an integer $a \in \mathbb{Z}_{>0}$. A *Frobenius $\mathcal{O}$-module* over S is a finitely generated locally free $\mathcal{O}_S$-module $\mathcal{M}$ together with a Fr^a_S-linear map $\Phi : \mathcal{M} \to \mathcal{M}$.

Lemma 3.28. *Let R be a local $\mathbb{F}$-algebra with maximal ideal $\mathfrak{m}$ such that R is $\mathfrak{m}$-adically separated. Let M be a finitely generated R-module. If there exists a Fr^a_R-linear isomorphism $\Phi : M \to M$, then M is free. In particular, M is a Frobenius $\mathcal{O}$-module over $S = \operatorname{Spec} R$.*

Proof. Choose a minimal resolution of M

$$0 \to U \to P \to M \to 0,$$

where P is a finitely generated free R-module and $U \subset \mathfrak{m}P$. Since R is $\mathfrak{m}$-adically separated, it suffices to show that $U \subset \mathfrak{m}^n P$ for all $n \in \mathbb{Z}_{>0}$. By the freeness of P, the linearization $\Phi^\sharp : M \otimes_{R,\mathrm{Fr}^a} R \to M$ extends to $\Phi^\sharp : P \otimes_{R,\mathrm{Fr}^a} R \to P$ and induces a commutative diagram

$$\begin{array}{ccc} P \otimes_{R,\mathrm{Fr}^a} R & \longrightarrow & M \otimes_{R,\mathrm{Fr}^a} R \\ {\scriptstyle \Phi^\sharp}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi^\sharp} \\ P & \longrightarrow & M. \end{array} \tag{3.3.1}$$

Note that $P/\mathfrak{m}P \cong M/\mathfrak{m}M$. Therefore modulo $\mathfrak{m}$, the arrows in the diagram are all isomorphism. In particular, the left vertical arrow is an isomorphism modulo $\mathfrak{m}$. By Nakayama's lemma, it is surjective, hence it is an isomorphism since the domain and the range are both free R-modules of the same rank. It follows that $\Phi^\sharp(U \otimes_{R,\mathrm{Fr}^a} R) = U$. Hence

$$U = \Phi^\sharp(U \otimes_{R,\mathrm{Fr}^a} R) \subset \Phi^\sharp(\mathfrak{m}P \otimes_{R,\mathrm{Fr}^a} R) \subset \mathfrak{m}^{q^a} P. \tag{3.3.2}$$

The lemma follows by induction. □

Definition 3.29. Let $\mathcal{M}$ be a Frobenius $\mathcal{O}$-module over S. Define a functor $C_{\mathcal{M}}$ on Sch_S by

$$C_{\mathcal{M}}(T) = \{x \in \Gamma(T, \mathcal{M}_T) \mid \Phi x = x\}, \tag{3.3.3}$$

for all $T \in \mathrm{Sch}_S$.

PROPOSITION 3.30. *The functor $C_{\mathcal{M}}$ is represented by a scheme which is etale and affine over S.*

Proof. As explained in the proof of [43, Proposition 3], the functor is a sheaf for the flat topology and the question is local on S, we may assume that $S = \operatorname{Spec}(R)$ is affine and $\mathcal{M}$ is the sheaf associated with a free R-module M. Fix an isomorphism $M \cong R^n$ and write the operator Φ in matrix form

$$\Phi x = U x^{q^a}, \quad x \in R^n.$$

Then the functor $C_{\mathcal{M}}(A) := C_{\mathcal{M}}(\operatorname{Spec} A)$ is the functor of solutions of the equation

$$x = U x^{q^a}, \quad x \in A^n,$$

which is a closed subscheme of the affine space $\mathbb{A}^n_R$.

To prove that $C_{\mathcal{M}}$ is etale, it suffices to show that if $A \to \bar{A}$ is a surjection of R-algebras with kernel $\mathfrak{a}$, such that $\mathfrak{a}^2 = 0$, then the induced map

$$C_{\mathcal{M}}(A) \to C_{\mathcal{M}}(\bar{A})$$

is a bijection. Indeed, let $\bar{x} \in C_{\mathcal{M}}(\bar{A})$ and let $x \in A^n$ be a lifting of x. Set $\rho = \Phi x - x$. Then $\rho \in \mathfrak{a} \otimes_R M$. Since $\Phi(\mathfrak{a} \otimes_R M) = 0$, we have $\Phi(x + \rho) = x + \rho$. Therefore $x + \rho \in C_{\mathcal{M}}(A)$ is the unique lifting of x in $C_{\mathcal{M}}(A)$. The proposition follows. □

Let $\mathbb{F}_a$ be the finite extension of $\mathbb{F}$ with degree a. Assume that S is an $\mathbb{F}_a$-scheme. Then $C_{\mathcal{M}}$ may be considered as a sheaf of $\mathbb{F}_a$-vector spaces. As explained after [43, Proposition 3], because $C_{\mathcal{M}}$ is unramified and separated over S, we have the following lemma.

LEMMA 3.31. *With the notation as above, if S is connected and $\eta \in S$ is a point, then the map*

$$C_{\mathcal{M}}(S) \to C_{\mathcal{M}}(\eta)$$

is injective.

Let $(\mathcal{M}, \Phi)$ be a Frobenius $\mathcal{O}$-module over S. Assume that $S = \operatorname{Spec} K$ with K a field over $\mathbb{F}$. First, assume that K is algebraically closed. By a similar argument as in Lemma 2.14, we have a decomposition

$$\mathcal{M} = \mathcal{M}_{\mathrm{et}} \oplus \mathcal{M}_{\mathrm{nil}},$$

where Φ is bijective on $\mathcal{M}_{\mathrm{et}}$ and nilpotent on $\mathcal{M}_{\mathrm{nil}}$. By Lemma 2.21, we have

$$K \otimes_{\mathbb{F}_a} C_{\mathcal{M}}(\operatorname{Spec} K) \cong \mathcal{M}_{\mathrm{et}}. \tag{3.3.4}$$

Assume that now K is separably closed, then $C_{\mathcal{M}}(\operatorname{Spec} K) = C_{\mathcal{M}}(\operatorname{Spec} \bar{K})$. We define $\mathcal{M}_{\mathrm{et}} \subset \mathcal{M}$ via equation (3.3.4). For a general field K, let K^s be the separable closure of K. Define

$$\mathcal{M}_{\mathrm{et}} := (C_{\mathcal{M}}(\operatorname{Spec} K^s) \otimes_{\mathbb{F}_a} K^s)^{\mathrm{Gal}(K^s/K)}.$$

Then Φ acts bijectively on $\mathcal{M}_{\mathrm{et}}$ and acts nilpotently on $\mathcal{M}/\mathcal{M}_{\mathrm{et}}$. By reducing to the case where K is algebraically closed, one sees that the functor $\mathcal{M} \to \mathcal{M}_{\mathrm{et}}$ is exact and commutes with base change.

The following lemma is entirely similar to [43, Lemma 4].

LEMMA 3.32. *Assume that* $S = \operatorname{Spec} R$ *and* S *is connected. Then the natural map*

$$C_{\mathcal{M}}(\operatorname{Spec} R) \otimes_{\mathbb{F}_a} R \to \mathcal{M} \tag{3.3.5}$$

is an injection onto a direct summand of $\mathcal{M}$*.*

Assume that now $S = \operatorname{Spec} R$, where R is a henselian local ring with maximal ideal $\mathfrak{m}$. Then there is a unique Φ-invariant direct summand $\mathcal{L} \subset \mathcal{M}$, such that Φ is bijective on $\mathcal{L}$ and is nilpotent on $\mathcal{M}/(\mathcal{L} + \mathfrak{m}\mathcal{M})$ (cf. explanation after [43, Lemma 4]).

DEFINITION 3.33. Let S be a scheme over $\mathbb{F}$ and $(\mathcal{M}, \Phi)$ a Frobenius $\mathcal{O}$-module over S. For each point $\eta \in S$, define function $\mu_{(\mathcal{M},\Phi)}$ by

$$\mu_{(\mathcal{M},\Phi)}(\eta) = \dim_{\mathbb{F}_a}(C_{\mathcal{M}})_{\bar{\eta}}.$$

Here $\bar{\eta}$ is some geometric point over η.

REMARK 3.34.
(1) If the function $\mu_{(\mathcal{M},\Phi)}$ is constant on S, there exists a unique Φ-invariant submodule $\mathcal{L} \subset \mathcal{M}$, such that $\mathcal{L}$ is locally a direct summand of $\mathcal{M}$, Φ acts bijectively on $\mathcal{L}$ and nilpotently on $\mathcal{M}/\mathcal{L}$. Indeed, this follows from the argument in [43, Pages 6-7].
(2) If the scheme S is perfect, then the exact sequence

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{M}/\mathcal{L} \to 0$$

splits canonically. Indeed, it suffices to prove this for $S = \operatorname{Spec} R$ with R perfect. Assume that Φ^n is zero on $\mathcal{M}/\mathcal{L}$. Define $\mathcal{M}_{\mathrm{nil}} = \operatorname{Ker} \Phi^n$. Then one checks that $\mathcal{M}_{\mathrm{nil}} \to \mathcal{M}/\mathcal{L}$ is a bijection and $\mathcal{M} = \mathcal{M}_{\mathrm{et}} \oplus \mathcal{M}_{\mathrm{nil}}$.

Now we can state the purity result, whose proof is entirely similar to the proof of [43, Proposition 5].

PROPOSITION 3.35. *Let* $R \in \mathrm{Alg}_{\mathbb{F}}$ *be a Noetherian local ring of dimension* ≥ 2*. Let* $(\mathcal{M}, \Phi)$ *be a Frobenius* $\mathcal{O}$*-module over* $\operatorname{Spec} R$*. Assume that the function* $\mu_{(\mathcal{M},\Phi)}$ *is constant outside the closed point* s *of* $\operatorname{Spec} R$*. Then it is constant on* $\operatorname{Spec} R$*.*

Proof. Without loss of generality, we may assume that R is a complete local ring with algebraically closed residue field. Let $S = \operatorname{Spec} R$ and $U = S - \{s\}$. Since $C_{\mathcal{M}}$ is etale over S it admits a unique decomposition

$$C_{\mathcal{M}} = C^f_{\mathcal{M}} \coprod C^0_{\mathcal{M}},$$

where $C^f_{\mathcal{M}}$ is finite and etale over S and $C^0_{\mathcal{M}}$ has empty special fibre and is affine as a closed subscheme of $C_{\mathcal{M}}$. To prove the proposition, it suffices to prove that $C^0_{\mathcal{M}}$ is empty.

Define a function on U by $\Delta : \eta \mapsto \sharp(C^0_{\mathcal{M},\eta})$. Note that $\sharp(C^0_{\mathcal{M},\eta}) = \sharp(C_{\mathcal{M},\eta}) - \sharp(C^f_{\mathcal{M},\eta})$ and the two terms on the right hand side are both constant on U, hence Δ is constant on U. Therefore all geometric fibres of the map $C^0_{\mathcal{M}} \to U$ have the same number of points. Suppose that $C^0_{\mathcal{M}}$ is not empty, then $C^0_{\mathcal{M}} \to U$ is surjective, which implies that U is affine. This is a contradiction to [12, Proposition 6.4]. The proposition follows. □

3.3.2. The Φ-decomposition. Let G be a locally free $\mathcal{O}$-group scheme over S endowed with a homomorphism

$$\Phi : G \to G^{(s)}.$$

Here $G^{(s)} = G \times_{S,\mathrm{Fr}_S^s} S$. First consider the simple case where $S = \mathrm{Spec}(K)$ with K a field. Assume that $G = \mathrm{Spec}\, A$ with $A \in \mathrm{Alg}_K$. Then Φ induces a semi-linear endomorphism

$$\phi : A \to A.$$

Let $A^\phi = \{a \in A \mid \phi(a) = a\}$ and $K' = \{x \in K \mid \mathrm{Fr}^s x = x\}$. Then A^ϕ is a bi-K'-algebra and $G^\Phi = \mathrm{Spec}(A^\phi)$ is a finite scheme. We call G^Φ the Φ-*etale part* of G. We call the short exact sequence

$$0 \to G^{\Phi-\mathrm{nil}} \to G \to G^\Phi \to 0$$

the Φ-*connected-etale sequence of* G. Note that Φ induces an isomorphism on $G^\Phi \to (G^\Phi)^{(s)}$ and acts nilpotently on $G^{\Phi-\mathrm{nil}}$. Here *nilpotent* means that the natural composition induced from Φ

$$G \to G^{(s)} \to G^{(2s)} \to \cdots \to G^{(ns)}$$

is trivial for n large.

For general scheme S over $\mathbb{F}$, we still obtain a Φ-connected-etale sequence if G satisfies a technical condition. More precisely, we have the following result, which follows easily from Proposition 3.30.

PROPOSITION 3.36. *Let S be an $\mathbb{F}$-scheme. Let G be a locally free $\mathcal{O}$-group scheme over S endowed with a homomorphism*

$$\Phi : G \to G^{(s)}.$$

Assume that the function

$$\begin{aligned} &S \to \mathbb{Z} \\ &x \mapsto \mathrm{Rank}((G_x)^{\Phi_x}) \end{aligned} \tag{3.3.6}$$

is constant. Then there exists an exact sequence

$$0 \to G^{\Phi-\mathrm{nil}} \to G \to G^\Phi \to 0$$

such that Φ induces an isomorphism $G^\Phi \to (G^\Phi)^{(s)}$ and acts nilpotently on $G^{\Phi-\mathrm{nil}}$.

Let X be a π-divisible $\mathcal{O}$-module over a field K with a homomorphism $\Phi : X \to X^{(s)}$. Define X^Φ, the Φ-etale part of X by

$$X^\Phi := \varinjlim X[\pi^n]^\Phi,$$

where $X[\pi^n]$ is the π^n-torsion of X. Certainly, X^Φ is a π-divisible $\mathcal{O}$-module. The following corollary is clear.

COROLLARY 3.37. *Let X be a π-divisible $\mathcal{O}$-module over an $\mathbb{F}$-scheme S. Assume that for each geometric point $\eta \to S$ the height of the Φ-etale part of X_η is the same. Then a Φ-connected-etale sequence of X*

$$0 \to X^{\Phi-\mathrm{nil}} \to X \to X^\Phi \to 0$$

exists and X^Φ commutes with arbitrary base change.

By Proposition 3.30 and Lemma 3.31, we have the following result (cf. [32, Corollary 1.10]).

Corollary 3.38. *Let S be a connected $\mathbb{F}$-scheme. Let $G \to S$ be a finite, locally free $\mathcal{O}$-group scheme. Let $\Phi : G \to G^{(s)}$ be an isomorphism. Then there exists a finite etale morphism $T \to S$, and a morphism $T \to \operatorname{Spec} \mathbb{F}_s$, such that G_T is obtained by base change from an $\mathcal{O}$-group scheme H over $\mathbb{F}_s$, i.e.,*

$$G_T \cong H \times_{\operatorname{Spec} \mathbb{F}_s} T.$$

Moreover, Φ is induced from the identity on H.

This result has a generalization to π-divisible $\mathcal{O}$-modules (cf. Corollary 3.52).

3.3.3. Completely slope divisible $\mathcal{O}$-modules. Let S be a scheme over $\mathbb{F}$. For a scheme G over S, recall that

$$G^{(n)} = G \times_{S, \mathrm{Fr}_S^n} S.$$

Denote by $\mathrm{Fr} = \mathrm{Fr}_G : G \to G^{(1)}$ the Frobenius morphism relative to S. If G is a finite locally free commutative group scheme, denote by $\mathrm{Ver} = \mathrm{Ver}_G : G^{(1)} \to G$ the Verschiebung morphism.

Definition 3.39. Let X be a π-divisible $\mathcal{O}$-module over k, where $k \in \mathrm{Alg}_{\mathcal{O}}$ is a field of characteristic p. Let λ be a rational number. We call X *isoclinic of slope* λ, if there exist integers $s \geq r \geq 0$, $s > 0$ such that $\lambda = r/s$, and a π-divisible $\mathcal{O}$-module Y over k, which is isogenous to X such that

$$\pi^{-r} \mathrm{Fr}_Y^s : Y \to Y^{(s)} \tag{3.3.7}$$

is an isomorphism. Note that the last condition is equivalent to saying that

$$\pi^{-(s-r)} \mathrm{Ver}_Y^s : Y^{(s)} \to Y \tag{3.3.8}$$

is an isomorphism.

In general, a π-divisible $\mathcal{O}$-module over an $\mathbb{F}$-scheme S is called *isoclinic of slope* λ, if for each point $s \in S$, the fiber X_s is isoclinic of slope λ.

Definition 3.40 (cf. [32, Definition 1.2]). Let S be an $\mathbb{F}$-scheme. Let $s > 0$ and $r_1, \ldots, r_m$ be integers such that $s \geq r_1 > r_2 > \cdots > r_m \geq 0$. Let X be a π-divisible $\mathcal{O}$-module over S. We say that X is *completely slope divisible* (short by *CSD*) with respect to these integers if X has a filtration of π-divisible $\mathcal{O}$-modules

$$0 = X_0 \subset X_1 \subset \cdots \subset X_m = X,$$

such that the following two properties hold:

- $\pi^{-r_i} \mathrm{Fr}_{X_i}^s : X_i \to X_i^{(s)}$ is an isogeny for $1 \leq i \leq m$;
- $\pi^{-r_i} \mathrm{Fr}_{X_i/X_{i-1}}^s : X_i/X_{i-1} \to (X_i/X_{i-1})^{(s)}$ is an isomorphism for $1 \leq i \leq m$.

Remark 3.41.

(1) We do not require that r_i and s are relatively prime. The key point is to give the set of rational numbers (r_i/s).

(2) In terms of the Verschiebung morphism, the two conditions in the definition are equivalent to the following
- $\pi^{r_i - s} \operatorname{Ver}^s_{X_i} : X_i^{(s)} \to X_i$ is an isogeny for $1 \leq i \leq m$;
- $\pi^{r_i - s} \operatorname{Ver}^s_{X_i/X_{i-1}} : (X_i/X_{i-1})^{(s)} \to X_i/X_{i-1}$ is an isomorphism for $1 \leq i \leq m$.

In this paper, we take the definition that is consistent with [32, Definition 1.2].

(3) The subobjects X_i are uniquely determined if they exist. Indeed, consider the isogeny

$$\Phi = \pi^{-r_m} \operatorname{Fr}^s_X : X \to X^{(s)}.$$

Then X/X_{m-1} is the Φ-etale part of X. Hence X_{m-1} is uniquely determined if it exists (cf. Section 3.3.1 and Corollary 3.37). The claim then follows by induction.

(4) Let $K \in \operatorname{Alg}_{\mathbb{F}}$ be a field. A π-divisible $\mathcal{O}$-module X over K is CSD if and only if the base change $X \otimes_K L$ is CSD for some field $L \supset K$. This follows from the remark before [32, Proposition 1.3].

Proposition 3.42. *Let $k \in \operatorname{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. An isoclinic π-divisible $\mathcal{O}$-module Y over k is CSD if and only if it can be defined over a finite field, i.e., if and only if there exists a π-divisible $\mathcal{O}$-module Y' over a finite extension $\mathbb{F}'$ of $\mathbb{F}$ and $Y \cong Y' \otimes_{\mathbb{F}'} k$.*

Proof. Assume that Y is slope divisible with respect to $s \geq r \geq 0$. Let (M, V) be the covariant Dieudonné $\mathcal{O}$-module of Y. Set $\Phi = \pi^{-r} V^s$. By assumption, $\Phi : M \to M$ is a semi-linear isomorphism of M. By Lemma 2.14, M has a basis consisting of Φ-invariant vectors. Let $\mathbb{F}'$ be the extension of $\mathbb{F}$ with degree s. Define $M_0 = M^{\Phi}$. Then M_0 is a Dieudonné $\mathcal{O}$-module over $\mathbb{F}'$. Let Y' be the π-divisible $\mathcal{O}$-module over $\mathbb{F}'$ attached to M_0. Then $Y \cong Y' \otimes_{\mathbb{F}'} k$ since $M = M_0 \otimes_{W_{\mathcal{O}}(\mathbb{F}')} W_{\mathcal{O}}(k)$.

Conversely, assume that $Y \cong Y' \otimes_{\mathbb{F}'} k$, where Y' is a π-divisible $\mathcal{O}$-module over $\mathbb{F}'$ and is isoclinic over $\mathbb{F}'$ of slope r/s. Let (M, V') be the Dieudonné $\mathcal{O}$-module over $\mathbb{F}'$ of Y'. Then there exists a finitely generated free $W_{\mathcal{O}}(\mathbb{F}')$-module $M' \subset M \otimes \mathbb{Q}$ such that $\pi^m M' \subset M \subset M'$ for some $m \in \mathbb{Z}_{\geq 0}$ and $\pi^{-r} V^s(M') = M'$. Then $\Phi = \pi^{-r} V^s$ is an automorphism of $M'/\pi^m M'$. Note that $M'/\pi^m M'$ is a finite set, hence Φ^t acts trivially on this set for some t. Therefore, $\Phi^t(M) = M$ and Φ^t induces an automorphism of Y, i.e., Y is slope divisible with respect to $st \geq rt \geq 0$. ∎

Proposition 3.43. *Let S be an integral $\mathbb{F}$-scheme with function field K. Let X be a π-divisible $\mathcal{O}$-module over S with constant Newton polygon. Assume that X_K is CSD with respect to integers $s \geq r_1 > r_2 > \cdots > r_m \geq 0$. Then X is CSD with respect to the same integers.*

Proof. Let $\Phi = \pi^{-r_m} \operatorname{Fr}^s : X \to X^{(s)}$. This quasi-isogeny is an isogeny since it is an isogeny over the generic point. By Corollary 3.37, X admits a Φ-connected-etale sequence

$$0 \to X^{\Phi-\mathrm{nil}} \to X \to X^{\Phi} \to 0.$$

The proposition then follows easily by induction. ∎

In the following, we prove the following result.

THEOREM 3.44. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a field of characteristic p. Any π-divisible $\mathcal{O}$-module X over k is isogenous to a CSD π-divisible $\mathcal{O}$-module over k. Moreover, the degree of the isogeny is bounded by a constant that depends only on the height of X.*

LEMMA 3.45. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a field of characteristic p. Let $X \to Y$ be a morphism of π-divisible $\mathcal{O}$-modules over k. Then there is a unique factorization in the category of π-divisible $\mathcal{O}$-modules*

$$X \to X' \to Y' \to Y$$

with the following properties:

1. *$X' \to Y'$ is an isogeny.*
2. *$Y' \to Y$ is a monomorphism of π-divisible $\mathcal{O}$-modules.*
3. *For each n, the morphism $X[\pi^n] \to X'[\pi^n]$ is an epimorphism of finite group schemes.*

Moreover, this factorization commutes with base change to another field.

Proof. This is entirely similar to [43, Proposition 8]. Roughly speaking, let A be the kernel of $X \to Y$ in the category of flat sheaves of abelian groups. There exists a unique $A' \subset A$ such that A' is a π-divisible $\mathcal{O}$-module and the quotient A/A' is a finite group scheme. Then we may define $X' = X/A'$ and $Y' = X'/(A/A')$. □

In the same setting as in the above lemma, the group Y' is the image of $X \to Y$ in the category of flat sheaves. We call Y' the *small image* of $X \to Y$.

LEMMA 3.46. *If k is perfect, then Theorem 3.44 holds.*

Proof. Let (M, V) be the (-1)-$\mathcal{O}$-crystal over A_k associated with X. Assume that $\mathrm{Newton}(M \otimes F_k, V) = r_1/s_1$. By Lemma 2.10, there exists a $\pi^{-r_1}V^{s_1}$-stable lattice in $M \otimes F_k$, say M'. Write $\Phi = \pi^{-r_1}V^{s_1}$. By Lemma 2.14, we have a short exact sequence

$$0 \to (M')_{\mathrm{nil}} \to M' \to (M')_{\mathrm{et}} \to 0$$

with respect to Φ. Note that $(M')_{\mathrm{et}} \neq \{0\}$ since $\mathrm{Newton}(M \otimes F_k, V) = r_1/s_1$. Hence $\mathrm{Rank}(M')_{\mathrm{nil}} < \mathrm{Rank}\, M'$. By induction, there exists a lattice $M_c \subset (M')_{\mathrm{nil}}$, such that M_c is CSD and is Φ-stable. Indeed, there exists a lattice $M_c' \subset (M')_{\mathrm{nil}}$, such that M_c' is CSD and is $\pi^{-r_2}\phi^{s_2}$-stable. Then we may take $M_c = M_c' + \Phi M_c' + \Phi^2 M_c' + \cdots$.

Pull back with $M_c \to (M')_{\mathrm{nil}}$, we obtain

$$0 \to M_c \to M'' \to (M')_{\mathrm{et}} \to 0.$$

Then M'' is CSD and the proposition follows.

To prove the last claim, note that we may take $M' = M + \Phi M + \cdots + \Phi^{h-1}M$ from the proof of Lemma 2.10. Hence $F^{s_1(h-1)}M' \subset M$ since $r_1 \le s_1$. Because we can always choose $s_1 \le h$. The lemma follows from the fact that the index of M in M' $(M' : M) \le q^{h^2(h-1)}$. □

Proof of Theorem 3.44. Assume now that k is not perfect. Base change X to k^{perf} and assume that the first Newton slope of the (-1)-$\mathcal{O}$-crystal of $X_{k^{\mathrm{perf}}}$ is r/s. Let Y be the small image of the following composition

$$X^{((h-1)s)} \times \cdots \times X^{(s)} \times X \xrightarrow{\alpha} X^{((h-1)s)} \xrightarrow{\beta} X,$$

where $\alpha \mid_{X^{((h-i)s)}} = \pi^{-(i-1)r}\,\mathrm{Fr}^{(i-1)s}$, β is induced from $\mathrm{Ver} : X^{(1)} \to X$. Base change to k^{perf}, we see that $Y \to X$ is an isogeny and Y is slope divisible with respect to $s \geq r$, i.e.

$$\Phi := \pi^{-r}\,\mathrm{Fr}_Y^s : Y \to Y^{(s)}$$

is an isogeny. Hence we have Φ-connected-etale sequence

$$0 \to Y^{\Phi-\mathrm{nil}} \to Y \to Y^{\Phi} \to 0.$$

From our construction, Y^{Φ} is CSD and isoclinic with respect to $s \geq r$ and the slopes of $Y^{\Phi-\mathrm{nil}}$ are strictly greater than r/s. Hence the theorem follows by induction on the height of X. □

3.3.4. The slope filtration.

Definition 3.47. Let X be a π-divisible $\mathcal{O}$-module over S. A *slope filtration of* X is a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_m = X$$

consisting of π-divisible sub-$\mathcal{O}$-modules of X, such that there exist rational numbers $1 \geq \lambda_1 > \cdots > \lambda_m \geq 0$ and the subquotient X_i/X_{i-1} is isoclinic of slope λ_i for $1 \leq i \leq m$.

If X is a CSD π-divisible $\mathcal{O}$-module, then X admits a slope filtration. Hence over a field of characteristic p, up to isogeny, every π-divisible $\mathcal{O}$-modules admits a slope filtration by Theorem 3.44. Over a connected base scheme S of positive dimension, it is easy to see that X admits a slope filtration only if X has constant Newton polygon. The converse is not true. Nevertheless, we have the following result.

Theorem 3.48. *Let h be a natural number. There exists a positive integer $N(h)$ with the following property. Let S be an integral, normal Noetherian $\mathbb{F}$-scheme. Let X be a π-divisible $\mathcal{O}$-module over S of height h with constant Newton polygon. Then there is a CSD π-divisible $\mathcal{O}$-module Y over S and an isogeny $\phi : X \to Y$ over S with* $\deg(\phi) \leq N(h)$.

In particular, a π-divisible $\mathcal{O}$-module over an integral, normal Noetherian $\mathbb{F}$-scheme S with constant Newton polygon is isogenous to a π-divisible $\mathcal{O}$-module over S that admits a slope filtration.

The proof is entirely similar to the proof of [32, Theorem 2.1]. The theorem follows from the following lemmas, whose proofs are also entirely similar to the proofs of the results in [32, Section 2]. We give the statements and point out the adjustments in the arguments.

Let X be a π-divisible $\mathcal{O}$-module over a scheme S, let d be a natural number. Define a functor $\mathcal{M}$ on S-schemes by

$$\mathcal{M}(T) = \{(Z,\alpha) \mid Z \in \pi\,\mathrm{Mod}_T,\ \alpha : X_T \to Z \text{ is an isogeny of degree } d\}. \tag{3.3.9}$$

Here $\pi\,\mathrm{Mod}_T$ denotes the category of π-divisible $\mathcal{O}$-modules over T.

Lemma 3.49. *The functor $\mathcal{M}$ is representable by a projective scheme over S.*

Proof. For any locally free sub-$\mathcal{O}$-group scheme $G \subset X_T$, there is a unique isogeny $\alpha : X_T \to X_T/G$ with kernel G. Let n be the natural number such that

$q^n \geq d$. Then G is a finite, locally free sub-$\mathcal{O}$-group scheme on $X[\pi^n]_T$. Assume that $X[\pi^n] = \mathrm{Spec}_S \mathcal{A}$. Then the affine algebra of G is a quotient of the locally free sheaf $\mathcal{A}_T$. Hence we obtain a point of the Grassmannian of $\mathcal{A}$. Therefore $\mathcal{M}$ is representable by a closed subscheme of this Grassmannian. □

LEMMA 3.50. *For every $h \in \mathbb{Z}_{>0}$, there exists a number $N(h) \in \mathbb{Z}$ with the following property. Let S be an integral Noetherian scheme over $\mathbb{F}$. Let X be a π-divisible $\mathcal{O}$-module of height h over S with constant Newton polygon. There is a non-empty open subset $U \subset S$, and a projective morphism $\mathfrak{p} : \tilde{S} \to S$ of integral schemes which induces an isomorphism $\mathfrak{p} : \mathfrak{p}^{-1}(U) \to U$ such that there exist a CSD π-divisible $\mathcal{O}$-module Y over $\tilde{S}$, and an isogeny $X_{\tilde{S}} \to Y$, whose degree is bounded by $N(h)$.*

Proof. This is entirely similar to [32, Lemma 2.4]. We use Theorem 3.44 instead of [43, Proposition 12]. □

LEMMA 3.51. *Let k be an algebraically closed field of characteristic p. Let $s \geq r_1 > r_2 > \cdots > r_m \geq 0$ and $d > 0$ be integers. Let X be a π-divisible $\mathcal{O}$-module over k. Then there are up to isomorphism only finitely many isogenies $X \to Z$ of degree d to a π-divisible $\mathcal{O}$-module Z, which is CSD with respect to $s \geq r_1 > r_2 > \cdots > r_m \geq 0$.*

Proof. This is entirely similar to [32, Lemma 2.5]. We use Lemma 2.22 instead of [47, Theorem 6.26]. □

Now Theorem 3.48 follows from the same argument of [32, Theorem 2.1]. It has interesting consequences. One may find more details in [32, Section 3]. As an example, we prove the following constancy result.

COROLLARY 3.52. *Let S be a Noetherian integral normal scheme over $\bar{\mathbb{F}}$. Let K be the function field of S and let $\bar{K}$ be an algebraic closure of K. Denote by $L \subset \bar{K}$ the maximal unramified extension of K with respect to S. Let T be the normalization of S in L. Let X be an isoclinic π-divisible $\mathcal{O}$-module over S. Then there exist a π-divisible $\mathcal{O}$-module X_0 over $\bar{\mathbb{F}}$ and an isogeny $X \times_S T \to X_0 \times_{\mathrm{Spec}(\bar{\mathbb{F}})} T$, such that the degree of this isogeny is smaller than an integer which depends only on the height of X.*

Proof. By Theorem 3.48, there exists an isogeny $\phi : X \to Y$, where Y is a CSD π-divisible $\mathcal{O}$-module over S. There are integers $s \geq r \geq 0$ such that

$$\Phi = \pi^{-r}\,\mathrm{Fr}^s : Y \to Y^{(s)}$$

is an isomorphism. By Corollary 3.38, for each $Y[n]$, there exists an $\mathcal{O}$-group scheme Z_n over $\mathbb{F}_s$ such that

$$Y[\pi^n]_T \cong Z_n \times_{\mathrm{Spec}(\mathbb{F}_s)} T.$$

Taking inductive limit of Z_n, we obtain a π-divisible $\mathcal{O}$-module Z over $\mathbb{F}_s$. Then we may take $X_0 = Z$ and the corollary follows. □

4. Minimal π-divisible $\mathcal{O}$-modules. In this section, following the idea of [31], we study minimal π-divisible $\mathcal{O}$-modules over an algebraic closed field of characteristic p. The main goal is to prove that these objects are determined (up to isomorphism) by their π-torsion parts (cf. Theorem 4.7).

4.1. Minimal π-divisible $\mathcal{O}$-modules. In this section, we fix an algebraically closed field $k \in \mathrm{Alg}_{\mathcal{O}}$ of characteristic p. Let m and n be a pair of non-negative integers such that $(m, n) = 1$. In Section 2.3, we have constructed a (-1)-$\mathcal{O}$-isocrystal $N_{m,m+n}$ over $\mathbb{F}$. Inside $N_{m,m+n}$, there exists a (-1)-$\mathcal{O}$-crystal $M_{m,m+n}$ with corresponding π-divisible $\mathcal{O}$-module $G_{m,m+n}$. Here following [6, Section 5.3], we construct another π-divisible $\mathcal{O}$-module $H_{m,m+n}$ that is isogenous to $G_{m,m+n}$ by writing down its covariant Dieudonné $\mathcal{O}$-module $M(H_{m,m+n})$ explicitly.

Specifically, $M(H_{m,m+n})$ is a free $W_{\mathcal{O}}(\mathbb{F})$-module of rank $m+n$ with basis $e_0, e_1, \dots, e_{m+n-1}$. For $j \in \mathbb{Z}_{\geq 0}$, we write $e_j = \pi^a e_i$ if $j = i + a(m+n)$. The actions of F and V on $M(H_{m,m+n})$ are given by $F(e_i) = e_{i+n}$ and $V(e_i) = e_{i+m}$. Note that there is a special object $\Pi \in \mathrm{End}(M(H_{m,m+n}))$ given by $\Pi(e_i) = e_{i+1}$.

Remark 4.1. It is easy to see that $M(H_{m,m+n})$ is isoclinic of slope $m/(m+n)$ and $M(H_{m,m+n}) \otimes \mathbb{Q} = N_{m,m+n}$. The isogeny between $H_{m,m+n}$ and $G_{m,m+n}$ is induced from an inclusion $M_{m,m+n} \hookrightarrow M(H_{m,m+n})$ by identifying $M_{m,m+n}$ with $W_{\mathcal{O}}(\mathbb{F})[F, V] \cdot e_0$.

Each Newton polygon β with slopes in $[0, 1]$ corresponds to a (-1)-$\mathcal{O}$-isocrystal (N, V) over $\mathbb{F}$, hence corresponds to an isogeny class of π-divisible $\mathcal{O}$-modules over $\mathbb{F}$. Let $\oplus_i G^{r_i}_{m_i, m_i+n_i}$ be a representative of this isogeny class. Define $H(\beta) := \oplus_i H^{r_i}_{m_i, m_i+n_i}$. It is a π-divisible $\mathcal{O}$-module determined by the Newton polygon β.

Definition 4.2 (Cf. [31, Section 1]). A π-divisible $\mathcal{O}$-module X is called *minimal* if there exist a Newton polygon β and an isomorphism $X_k \cong H(\beta)_k$.

Remark 4.3. By duality, one may define a *maximal* Dieudonné $\mathcal{O}$-modules $M'(H_{m,m+n})$ as follows. It is a free $W_{\mathcal{O}}(\mathbb{F})$-module with basis $f_0, f_1, \dots, f_{m+n-1}$. The actions of F and V are given by

$$F \cdot f_i = \begin{cases} \pi f_{i+m} & \text{if } 0 \leq i \leq n-1, \\ f_{i-n} & \text{if } n \leq i \leq m+n-1; \end{cases} \qquad V \cdot f_i = \begin{cases} \pi f_{i+n} & \text{if } 0 \leq i \leq m-1, \\ f_{i-m} & \text{if } m \leq i \leq m+n-1. \end{cases} \tag{4.1.1}$$

But it is easy to check that there is an injection of Dieudonné $\mathcal{O}$-modules $M'(H_{m,m+n}) \hookrightarrow M(H_{m,m+n})$ by $f_i \mapsto \pi e_{m+n-1-i}$, which identifies $M'(H_{m,m+n})$ with $\pi M(H_{m,m+n})$. Hence to understand maximal objects, it suffices to understand minimal ones.

Let $H = H_{m,m+n}$ be the minimal π-divisible $\mathcal{O}$-module of type $(m+n, m)$ over $\mathbb{F}$. Let $\mathbb{F}_{m+n}$ be the extension of $\mathbb{F}$ with degree $m+n$. Let σ be the Frobenius morphism of $W_{\mathcal{O}}(\mathbb{F}_{m+n})$. Choose a, $b \in \mathbb{Z}$ such that $am + bn = 1$. The following lemma is a generalization of [6, Lemma 5.4].

Lemma 4.4. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. With the notation as above,*

$$\mathrm{End}(H_k) = W_{\mathcal{O}}(\mathbb{F}_{m+n})[\Pi],$$

where $\lambda \cdot \Pi = \Pi \cdot \sigma^{b-a}(\lambda)$ for $\lambda \in W_{\mathcal{O}}(\mathbb{F}_{m+n})$.

Moreover, the ring $\mathrm{End}(H_k)$ is a discrete valuation ring with uniformizer Π. Consider the filtration $N^\bullet$ of $M(H_k) = M(H) \otimes W_{\mathcal{O}}(k)$ given by

$$M(H_k) = N^{(0)} \supset \cdots \supset N^{(j)} \supset \cdots$$

with

$$N^{(j)} = \langle e_j, e_{j+1}, \dots \rangle = \sum_{t \geq j} W_{\mathcal{O}}(k) e_t = \Pi^j(M(H_k)).$$

For every $\tau \in \mathrm{End}(H_k)$ and every j we have $\tau(N^{(j)}) = N^{(j+v)}$, where $v = v(\tau)$ is the valuation of τ.

Proof. Note that $F^bV^a(e_i) = e_{i+1} = \Pi(e_i)$. Let $f : M(H_k) \to M(H_k)$ be an endomorphism of Dieudonné $\mathcal{O}$-modules. Then f commutes with F and V, hence commutes with Π. From this, one deduces that $\mathrm{End}(M(H_k)) = W_{\mathcal{O}}(\mathbb{F}_{m+n})[\Pi]$. The relation $\lambda \cdot \Pi = \Pi \cdot \sigma^{b-a}(\lambda)$ for $\lambda \in W_{\mathcal{O}}(\mathbb{F}_{m+n})$ follows from $\Pi = F^bV^a$.

Note that $\Pi^{m+n} = \pi$. One sees that $W_{\mathcal{O}}(\mathbb{F}_{m+n})[\Pi] \otimes \mathbb{Q}$ is the central simple algebra over $W_{\mathcal{O}}(\mathbb{F})$ of rank $(m+n)^2$ and invariant $n/(m+n)$. Moreover, $W_{\mathcal{O}}(\mathbb{F}_{m+n})[\Pi]$ is a maximal order of this central simple algebra. The other claims follow easily. ◻

Lemma 4.5. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Let $\varphi : H_k \to X$ and $\psi : H_k \to X$ be isogenies of π-divisible $\mathcal{O}$-modules. Then either $\varphi = \psi \circ \tau$ or $\psi = \varphi \circ \tau$ for some $\tau \in \mathrm{End}(H_k)$. If $\deg(\varphi) = \deg(\psi)$, then τ is an automorphism of H_k. A similar result holds for isogenies $X \to H_k$.*

Proof. The proof is entirely similar to that of [6, Lemma 5.5]. Let $\beta : X \to H_k$ be any isogeny. Then both $\beta \circ \varphi$ and $\beta \circ \psi$ are in $\mathrm{End}(H_k)$, which is a discrete valuation ring. Hence either $(\beta \circ \psi)^{-1} \circ (\beta \circ \varphi) \in \mathrm{End}(H_k)$ or $(\beta \circ \varphi)^{-1} \circ (\beta \circ \psi) \in \mathrm{End}(H_k)$. The lemma follows. ◻

Lemma 4.6. *Let Q be a nonzero Dieudonné $\mathcal{O}$-module over $\mathbb{F}$. Suppose that there exists $C \in Q$, such that*

(1) *there exist coprime integers m and n with $F^{m+n}C = \pi^n C$,*
(2) *Q is generated by $\{\pi^{-[jn/(m+n)]} F^j C \mid 0 \leq j < m+n\}$ as a $W_{\mathcal{O}}(\mathbb{F})$-module.*

Then $Q \cong M(H_{m,m+n})$.

Proof. This is clear. The isomorphism is given by $C \mapsto e_0$. ◻

Our next goal is to prove a generalization of [31, Theorem (1.2)]. The strategy of the proof is similar to that in [31], i.e. we translate it into a question on Dieudonné $\mathcal{O}$-modules.

Theorem 4.7. *Let X be a π-divisible $\mathcal{O}$-module over an algebraically closed field $k \in \mathrm{Alg}_{\mathbb{F}}$. Let β be a Newton polygon with slopes in $[0,1]$. If $X[\pi] \cong H(\beta)[\pi]$, then $X \cong H(\beta)$.*

Remark 4.8. The following proof is adapted from, hence very similar to the proof in [31]. But Theorem 4.7 is not a consequence of [31, Theorem (1.2)].

Remark 4.9. Note that the Newton slope is defined differently in [31], e.g, the Newton slope of $H_{m,m+n}$ in [31] is $n/(m+n)$ and in this paper is $m/(m+n)$.

Remark 4.10. As in [31, Section 1, Convention], if the Newton slope of X is 1 with multiplicity 1, then the corresponding (-1)-$\mathcal{O}$-isocrystal N is given by $(\mathrm{Frac}(W_{\mathcal{O}}(k)), V = {}^V)$. It is easy to check that each (-1)-$\mathcal{O}$-crystal in N is isomorphic to $(W_{\mathcal{O}}(k), {}^V)$. Theorem 4.7 holds in this case. By duality, it also holds if the Newton slopes are 0. Hence to prove the theorem, we may assume that all group schemes are of local-local type.

In the following, we prove Theorem 4.7 with the strategy explained in Section 1.4.

4.2. Oort's slope filtration. Fix positive integers r_i, m_i, n_i $(1 \le i \le t)$, such that $(m_i, n_i) = 1$ for all i, and $m_i/n_i \neq m_j/n_j$ if $i \neq j$. Let $h_i = m_i + n_i$. Assume that the numbers are ordered in a way such that $\lambda_1 := n_1/h_1 < \cdots < \lambda_t := n_t/h_t$. Define

$$H := \prod_{1 \le i \le t} (H_{m_i, m_i+n_i})^{r_i}.$$

PROPOSITION 4.11. *Suppose that X is a π-divisible $\mathcal{O}$-module over k such that $X[\pi] \cong H[\pi]$. Suppose that $\lambda_1 = n_1/h_1 \le 1/2$. Then there exists a π-divisible sub-$\mathcal{O}$-module $X_1 \subset X$ such that*

$$X_1 \cong (H_{m_1, m_1+n_1})^{r_1}, \quad (X/X_1)[\pi] \cong \prod_{2 \le i \le t} (H_{m_i, m_i+n_i}[\pi])^{r_i}.$$

Let M be the covariant Dieudonné $\mathcal{O}$-module of X, Q_j the covariant Dieudonné $\mathcal{O}$-module of H_{m_j, m_j+n_j} for all $1 \le j \le t$. Since $X[\pi] \cong H[\pi]$, we have an isomorphism of $W_{\mathcal{O}}(k)[F, V]$-modules

$$M/\pi M \cong \oplus_{1 \le j \le t} (Q_j/\pi Q_j)^{r_j}.$$

As in [31, Section 2.5], we construct a map

$$v : M \to \mathbb{Q}_{\ge 0} \cup \{\infty\}.$$

Let Π_j be the uniformizer of $\mathrm{End}(Q_j)$ as defined in Section 4.1. For each $1 \le j \le t$, choose $A^{(j)}_{i,s} \in Q_j$ with $i \in \mathbb{Z}_{\ge 0}$ and $1 \le s \le r_j$ such that

(1) The elements $A^{(j)}_{i,s}$ generate Q_j;

(2) $\Pi_j A^{(j)}_{i,s} = A^{(j)}_{i+1,s}$, $F \cdot A^{(j)}_{i,s} = A^{(j)}_{i+n_j,s}$, $V \cdot A^{(j)}_{i,s} = A^{(j)}_{i+m_j,s}$.

As a vector space over k, $Q_j/\pi Q_j$ has a basis consisting of $A^{(j)}_{i,s} \pmod{\pi Q_j}$ $(0 \le i < h_j)$. We write

$$A^{(j)}_i = (A^{(j)}_{i,s} \mid 1 \le s \le r_j) \in (Q_j)^{r_j}$$

for the vector with coordinate $A^{(j)}_{i,s}$ in the summand on the s-th place.

Let $B \in M$. Then

$$B \pmod{\pi M} = \sum_{j,\ 0 \le i < h_j,\ 1 \le s \le r_j} b^{(j)}_{i,s} (A^{(j)}_{i,s} \pmod{\pi Q_j}),$$

for uniquely determined $b^{(j)}_{i,s} \in k$. Then the map v is defined as follows.

- $v(0) = \infty$;
- $v(B) = \min_{j,\ i,\ s,\ b^{(j)}_{i,s} \neq 0} \frac{i}{h_j}$ if $B \notin \pi M$;
- $v(B) = \beta + v(\pi^{-\beta} B)$ if $B \in \pi^\beta M - \pi^{(\beta-1)} M$.

For every $\rho \in \mathbb{Q}$, we define

$$M_\rho = \{B \in M \mid v(B) \ge \rho\} \subset M.$$

Then $\pi M_\rho \subset M_{\rho+1}$. Let T be the least common multiple of $h_1, \cdots, h_t$. Then $\mathrm{Im}(v) \subset \frac{1}{T}\mathbb{Z}_{\ge 0}$. Note that $v(B) \ge d \in \mathbb{Z}$ if and only if $B = \pi^d B'$ for some $B' \in M$.

Hence $\cap_{\rho\to\infty} M_\rho = \{0\}$.

For each j, i, s with $1 \le j \le t$, $0 \le i < h_j$, $1 \le s \le r_j$, choose $B_{i,s}^{(j)} \in M$ such that

$$B_{i,s}^{(j)} \pmod{\pi M} = A_{i,s}^{(j)} \pmod{\pi Q_j}.$$

Define $B_{i+\beta h_j,s}^{(j)} = \pi^\beta B_{i,s}^{(j)}$. Then

$$v(B_{i,s}^{(j)}) = i/h_j \text{ for all } i \ge 0, \text{ all } j, \text{ and all } s.$$

Hence $M_\rho = W_{\mathcal{O}}\langle B_{i,s}^{(j)} \mid v(B_{i,s}^{(j)}) \ge \rho\rangle$. Similarly, we write

$$B_i^{(j)} = (B_{i,s}^{(j)} \mid 1 \le s \le r_j) \in M^{r_j}.$$

Define

$$\begin{aligned} P &= W_{\mathcal{O}}(k)\langle B_{i,s}^{(j)} \mid j \ge 2,\ i < h_j\rangle \subset M, \\ N &= W_{\mathcal{O}}(k)\langle B_{i,s}^{(1)} \mid i < h_1\rangle \subset M. \end{aligned} \tag{4.2.1}$$

Then $M = N \oplus P$ is a direct sum decomposition of M as a $W_{\mathcal{O}}(k)$-module. Our plan is to modify N so that we obtain a direct summand of M as a Dieudonné $\mathcal{O}$-module.

Write $m_1 = m$, $n_1 = n$, $h = h_1 = m + n$, $r_1 = r$. By assumption, $m \ge n > 0$. For each $i \in \mathbb{Z}_{\ge 0}$, define integer δ_i by

$$ih \le \delta_i n < im + (i+1)n = ih + n$$

and non-negative integer γ_i by

$$\delta_0 = 0,\ \delta_1 = \gamma_1 + 1, \cdots,\ \delta_i = \sum_{j=1}^{i} (\gamma_i + 1), \cdots.$$

For $1 \le i \le n$, define

$$f(i) = \delta_{i-1} n - (i-1)h.$$

One checks easily that f defines a bijective map

$$f : \{1, 2, \cdots, n\} \to \{0, 1, \cdots, n-1\}.$$

The inverse map $f' : \{0, 1, \cdots, n-1\} \to \{1, 2, \cdots, n\}$ is given by $f'(x) \equiv 1 - \frac{x}{h} \pmod{n}$.

In $(Q_1)^r$ we have the vectors $A_i^{(1)}$. For $1 \le i \le n$, write $C'_i = A_{f(i)}^{(1)}$. Hence $C'_{f'(i)} = A_i^{(1)}$. Thus we have

- $F^{\gamma_i} C'_i = V C'_{i+1}$ for $1 \le i < n$;
- $F^{\gamma_n} C'_n = V C'_1$;
- $F^{\delta_i} C'_1 = \pi^i C'_{i+1}$ for $1 \le i < n$;
- $F^h C'_1 = \pi^n C'_1$. Note that $h = \delta_n$ and $m = \sum_{i=1}^n \gamma_i$.

One sees that

$$\{F^j C'_i \mid 1 \le i \le n,\ 0 \le j \le \gamma_i\} = \{A_i^{(1)} \mid 0 \le i < h\}.$$

Choose $C_{i,s} := B^{(1)}_{f(i),s}$ for $1 \le i \le n$. Then the set

$$\{F^j C_{i,s} \mid 1 \le i \le n,\ 0 \le j \le \gamma_i,\ 1 \le s \le r\}$$

is a $W_{\mathcal{O}}(k)$-basis for N, and

$$F^{\gamma_i} C_{i,s} - V C_{i+1,s} \in \pi M \text{ for } 1 \le i < n,\ F^{\gamma_n} C_{n,s} - V C_{1,s} \in \pi M.$$

Write $C_i = (C_{i,s} \mid 1 \le s \le r) \in M^r$.

Lemma 4.12. *With the notation as above, the objects we constructed satisfy the following properties.*

(1) *For every $\rho \in \mathbb{Q}_{\ge 0}$, the multiplication by π map $\pi : M_\rho \to M_{\rho+1}$ is surjective.*
(2) *$FM \subset M_{\frac{n}{h}}$.*
(3) *For every i and s, $FB^{(1)}_{i,s} \in M_{\frac{i+n}{h}}$; for every i, s, and every $j > 1$, $FB^{(j)}_{i,s} \in M_{\frac{i}{h_j}+\frac{n}{h}+\frac{1}{T}}$. Recall that T is the least common multiple of $h_1, \ldots, h_t$.*
(4) *For every $\rho \in \mathbb{Q}_{\ge 0}$, $FM_\rho \subset M_{\rho+\frac{n}{h}}$.*
(5) *For every $1 \le i \le n$, $F^{\delta_i} C_1 - \pi^i B^{(1)}_{f(i+1)} \in (M_{i+\frac{1}{T}})^r$; moreover, $F^{\delta_n} C_1 - \pi^n C_1 \in (M_{n+\frac{1}{T}})^r$.*
(6) *If $u \in \mathbb{Z}$ such that $u > Tn$, then for each $\xi_N \in (N \cap M_{\frac{u}{T}})^r$, there exists $\eta_N \in (N \cap M_{\frac{u}{T}-n})^r$ such that*

$$(F^h - \pi^n)\eta_N \equiv \xi_N \pmod{(M_{\frac{u+1}{T}})^r}.$$

Proof.

(1) The set $M_{\rho+1}$ is generated by elements $B^{(j)}_{i,s}$ with $i/h_j \ge \rho + 1$. Since $\rho \ge 0$, we have $i \ge h_j$. The claim follows since $B^{(j)}_{i,s} = \pi \cdot B^{(j)}_{i-h_j,s}$.
(2) By construction, we have
   - $FB^{(j)}_{i,s} = B^{(j)}_{i+n_j,s} + \pi\xi$ for some $\xi \in M$, if $0 \le i < m_j$;
   - $FB^{(j)}_{i,s} = \pi B^{(j)}_{i+n_j-h_j,s} + \pi\eta$ for some $\eta \in M$, if $m_j \le i < h_j$.

   The claim follows.
(3) The argument is similar as above. For all $1 \le j \le t$ and all $\beta \in \mathbb{Z}_{\ge 0}$, we have
   - $FB^{(j)}_{i,s} = B^{(j)}_{i+n_j,s} + \pi^{\beta+1}\xi$ for some $\xi \in M$, if $\beta h_j \le i < m_j + \beta h_j$;
   - $B^{(j)}_{i,s} = VB^{(j)}_{i-m_j,s} + \pi^{\beta+1}\eta$ for some $\eta \in M$, if $m_j + \beta h_j \le i < (\beta+1)h_j$.

   Note that $n_j/h_j > n/h$ if $j \ne 1$. In the first case, $v(B^{(j)}_{i+n_j,s}) = (i+n_j)/h_j$, $v(\pi^{\beta+1}\xi) \ge \beta + 1 > (i+n_j)/h_j$. The claim follows. In the second case,

$$v(FB^{(j)}_{i,s}) \ge \min\{v(\pi B^{(j)}_{i-m_j,s}),\ v(\pi^{\beta+1}F\eta)\}.$$

   Note that $v(\pi B^{(j)}_{i-m_j,s}) = (i+n_j)/h_j$, $v(\pi^{\beta+1}F\eta) \ge (\beta+1) + v(F\eta) \ge \beta + 1 + n/h$. The claim follows.
(4) This follows from (3).

(5) Write $C_{n+1} = C_1$. From construction, for $1 \le i \le n$, we have

$$F^{\gamma_i} C_i = VC_{i+1} + \pi\xi_i \text{ for some } \xi_i \in M^r.$$

Hence

$$F^{\delta_i} C_1 = \pi^i C_{i+1} + \sum_{1\le l\le i} \pi^l F^{\delta_i - \delta_l} F\xi_l,\ 1 \le i \le n.$$

By our construction, $ih \le \delta_i n < ih + n$. Then

$$lh + (\delta_i - \delta_l)n + n > ih.$$

The claim follows.

(6) If $(l-1)/h < u/T < l/h$ for some $l \in \mathbb{Z}$, then $u < u+1 \le lT/h$. In this case, $N \cap M_{\frac{u}{T}} = N \cap M_{\frac{u+1}{T}}$ and we may take $\eta_N = 0$.
Suppose that $u/T = l/h$ for some $l \in \mathbb{Z}$. Then $N \cap M_{\frac{u}{T}} = N_{\frac{l}{h}} \supset N_{\frac{l+1}{h}} = N \cap M_{\frac{u+1}{T}}$. In this case $(F^h - \pi^n)(N \cap M_{\frac{u}{T}-n}) \subset N_{\frac{l}{h}} + M_{\frac{u+1}{T}}$ and it induces a morphism

$$F^h - \pi^n : N \cap M_{\frac{u}{T}-n} \to N_{\frac{l}{h}} + M_{\frac{u+1}{T}}/M_{\frac{u+1}{T}} \cong N_{\frac{l}{h}}/N_{\frac{l+1}{h}}.$$

The last term $N_{\frac{l}{h}}/N_{\frac{l+1}{h}}$ is a vector space over k spanned by the residue classes $\bar{B}^{(1)}_{l,s}$ of $B^{(1)}_{l,s}$. Let $y_s\bar{B}^{(1)}_{l,s} \in N_{\frac{l}{h}}/N_{\frac{l+1}{h}}$ with $y_s \in k$. Since k is algebraically closed, we could find $x_s \in k$ such that $x_s^{q^n} - x_s = y_s$. Let $\tilde{x}_s \in W_{\mathcal{O}}(k)$ be a lifting of x_s. Define

$$\eta_N = \sum_s \tilde{x}_s B^{(1)}_{l-nh,s}.$$

Then η_N has the required property and the claim follows.

∎

Lemma 4.13. *Let $u \in \mathbb{Z}$ with $u \ge nT + 1$. Suppose that $D_1 \in M^r$ such that $D_1 \equiv C_1 \pmod{(M_{\frac{1}{T}})^r}$ and such that $\xi := F^h D_1 - \pi^n D_1 \in (M_{\frac{u}{T}})^r$. Then there exists $\eta \in (M_{\frac{u}{T}-n})^r$ such that for $E_1 := D_1 - \eta$ we have $E_1 \equiv C_1 \pmod{(M_{\frac{1}{T}})^r}$ and such that $F^h E_1 - \pi^n E_1 \in (M_{\frac{u+1}{T}})^r$.*

Proof. Since $M = N \times P$, there exists a unique pair $(\xi_N, \xi_P) \in N^r \times P^r$ with $\xi = \xi_N + \xi_P$. By assumption $\xi \in (M_{\frac{u}{T}})^r$, we have $\xi_* \in (* \cap M_{\frac{u}{T}})^r$ for $* = N,\ P$. By Lemma 4.12, there exists $\eta_N \in (N \cap M_{\frac{1}{T}})^r$ such that

$$(F^h - \pi^n)\eta_N \equiv \xi_N \pmod{(M_{\frac{u+1}{T}})^r}.$$

As $M_{\frac{u}{T}} \subset M_n$, choose η_P such that $\pi^n\eta_P = \xi_P$. Then $\eta_P \in (M_{\frac{u}{T}-n})^r \subset (M_{\frac{1}{T}})^r$. Define $\eta = \eta_N + \eta_P$. It is easy to check that this η satisfies the required properties and the lemma follows. ∎

Lemma 4.14. *There exists $E_1 \in M^r$ such that $(F^h - \pi^n)E_1 = 0$ and $E_1 \equiv C_1 \pmod{(M_{\frac{1}{T}})^r}$.*

Proof. For $u \in \mathbb{Z}_{\ge nT+1}$, by Lemma 4.13, there exists a $D_1(u) \in M^r$, such that

- $D_1(u) \equiv C_1 \pmod{(M_{\frac{1}{T}})^r}$;
- $(F^h - \pi^n)D_1(u) \in (M_{\frac{u}{T}})^r$;
- $D_1(u) - D_1(u+1) \in (M_{\frac{u}{T}-n})^r$.

The sequence $D_1(u)$ converges since $\cap_{\rho\to\infty} M_\rho = 0$. Let $E_1 = \lim_{u\to\infty} D_1(u)$. Then E_1 satisfies the expected properties and the lemma follows. $\square$

Choose E_1 as in Lemma 4.14. For each $j \in \mathbb{Z}_{\geq 0}$, $\pi^{-[\frac{jn}{h}]}F^j E_1 \in M^r$. Define

$$N' := \prod_{1\leq s\leq r}\prod_{1\leq j<h} W_{\mathcal{O}}(k)\langle \pi^{-[\frac{jn}{h}]}F^j E_{1s}\rangle \subset M.$$

Lemma 4.15. *With the notation as above, $N' \subset M$ is a sub Dieudonné $\mathcal{O}$-module of M. Moreover, there is an isomorphism*

$$M(H^r_{m,m+n}) \cong N'.$$

Proof. For the first claim, it suffices to show that N' is stable under F and V. This follows form Lemma 4.12. The last claim follows from Lemma 4.6. $\square$

Lemma 4.16. *With the notation as above, N' is a $W_{\mathcal{O}}(k)$-module direct summand of M and $M = N' \oplus P$.*

Proof. For a module Z over $W_{\mathcal{O}}(k)$, denote by $\bar{Z}$ the natural tensor product $Z \otimes_{W_{\mathcal{O}}(k)} k$. If $z \in Z$, denote by $\bar{z}$ the image of z in $\bar{Z}$. To prove the lemma, by Nakayama's lemma, it suffices to prove that $\bar{M} = \bar{N}' \oplus \bar{P}$.

Write $g(y) = yn - h[\frac{yn}{h}]$ for $y \in \mathbb{Z}_{\geq 0}$. Then by the construction,

$$\pi^{-[\frac{jn}{h}]}F^j C'_i = A^{(1)}_{g(j)}.$$

Hence $\bar{N}' + \bar{P} = \bar{M}$. Let $\tau := \sum_{0\leq j<h} \beta_{j,s}\pi^{-[\frac{jn}{h}]}F^j \bar{E}_{1,s} \in \bar{N}' \cap \bar{P}$. Here $\beta_{j,s} \in k$. Suppose that $\tau \neq 0$. Let (x, s) be a pair of indices such that $\beta_{x,s} \neq 0$ and for every y with $g(y) < g(x)$ we have $\beta_{y,s} = 0$. Then projecting the equation to the s-component of $\bar{N}$ in $\bar{M}$, we have

$$\tau \equiv \beta_{x,s}B^{(1)}_{g(x),s} \pmod{M_{\frac{g(x)}{h}+\frac{1}{T}} + P}.$$

On the other hand, we have $N \cap P = \{0\}$ and the residue class of $B^{(1)}_{g(x),s}$ generates the s-th component of $(M_{\frac{g(x)}{h}} + P)/(M_{\frac{g(x)}{h}+\frac{1}{T}} + P) \cong N_{\frac{g(x)}{h},s}/N_{\frac{g(x)}{h}+\frac{1}{h},s}$. We obtain a contradiction. Hence $\bar{N}' \cap \bar{P} = \{0\}$ and the lemma follows. $\square$

Proof of Proposition 4.11. The sub Dieudonné $\mathcal{O}$-module N' of M gives us a π-divisible $\mathcal{O}$-module $X_1 \subset X$. This X_1 satisfies the required properties. $\square$

4.3. Proof of Theorem 4.7. In order to prove Theorem 4.7, it suffices to prove the following proposition.

Proposition 4.17. *Let (m, n) and (d, e) be pairs of pairwise coprime positive integers. Suppose that $\frac{n}{m+n} < \frac{e}{d+e}$. Let*

$$0 \to Z := H_{m,m+n} \to T \to Y := H_{d,d+e} \to 0 \tag{4.3.1}$$

be an exact sequence of π-divisible $\mathcal{O}$-modules over k such that the induced sequence of π-torsions

$$0 \to Z[\pi] \to T[\pi] \to Y[\pi] \to 0 \tag{4.3.2}$$

splits. Then sequence (4.3.1) *splits and $T \cong Z \oplus Y$.*

REMARK 4.18. By duality, we may assume that $\frac{1}{2} \leq \frac{e}{d+e}$. Let M, N, Q be the covariant Dieudonné $\mathcal{O}$-modules of T, Z, Y respectively. Sequence (4.3.2) splits means that we have a splitting $\varphi_1 : Q/\pi Q \to M/\pi M$. To prove the proposition, for each $a \in \mathbb{Z}_{\geq 1}$, we construct homomorphisms

$$\varphi_a : Q/\pi^a Q \to M/\pi^a M$$

extending φ_1. Then we show that the limit of the φ_a provides the required splitting $Q \to M$.

Write $h = m + n$ and $g = d + e$. For the Dieudonné $\mathcal{O}$-module Q, there exists a standard $W_{\mathcal{O}}(k)$-basis as introduced before. More precisely, $Q = W_{\mathcal{O}}(k)\langle A_i \mid 0 \leq i < g\rangle$, $FA_i = A_{i+e}$, $VA_i = A_{i+d}$. Furthermore, $\Pi : A_i \mapsto A_{i+1}$ is a uniformizer of $\mathrm{End}(Q)$. We choose generators for Q in a new way. Define $\delta_i \in \mathbb{Z}$ by inequality

$$ig \leq \delta_i d < (i+1)d + ie = ig + d$$

and $\gamma_i \in \mathbb{Z}$ by equaitions

$$\delta_1 = \gamma_1 + 1, \cdots, \ \delta_i = \sum_{l=1}^{i} (\gamma_i + 1).$$

Note that $\delta_d = g = d+c$. Choose $C = A_0 = C_1$ and $\{C_1, \cdots, C_d\} = \{A_0, \cdots, A_{d-1}\}$ such that

$$V^{\gamma_i} C_i = FC_{i+1} \text{ for all } 1 \leq i \leq d.$$

Here $C_{d+1} = C_1$. Hence $V^{\delta_i} C = \pi^i C_{i+1}$ for $1 \leq i \leq d$. In particular, if $i = d$, then $\delta_i = g$ and we have $V^g C = \pi^d C$.

From the construction, we have

$$\{\pi^{[\frac{jd}{g}]} V^j C \mid 0 \leq j < g\} = \{V^j C \mid 1 \leq i \leq d, \ 0 \leq j \leq \gamma_i\} = \{A_l \mid 0 \leq l < g\}.$$

Choose $B = B_1 \in M$ such that B maps to C under the map $M \to Q$.

LEMMA 4.19. *With the notation as above, $V^{\delta_i} B$ is divisible by π^i for every $1 \leq i < d$ and $V^g B - \pi^d B \in N^{(dh+1)}$. Here $N = N^{(0)} \supset N^{(1)} \supset \cdots$ is the filtration of N defined in Lemma 4.4.*

Proof. Choose $B_i'' \in M$ which maps to C_i under the map $M \to Q$ for $1 \leq i < d$. Then $V^{\gamma_i} B_i'' = FB_{i+1}'' \pmod{\pi N}$. Let $\xi_i \in N$ such that $V^{\gamma_i} B_i'' - FB_{i+1}'' = \pi \xi_i$. Then $V^{\gamma_i+1} B_i'' - \pi B_{i+1}'' = \pi V \xi_i \in \pi VN$. Hence for $1 \leq i < d$, we have

$$V^{\delta_i} B - \pi^i B = \sum_{1 \leq j < i} V^{\delta_i - \delta_j} \pi^j V \xi_j.$$

By our assumption, we have $\frac{g}{d} > \frac{h}{m}$. If $i > j$, by definition of δ_i, we have

$$\delta_i - \delta_j + 1 > \frac{ig - (jg + d)}{d} + 1 = (i-j)\frac{g}{d} > (i-j)\frac{h}{m}.$$

Hence $(\delta_i - \delta_j)m + j(m+n) + m > ih$ and $V^{\delta_i - \delta_j}\pi^j V\xi \in \pi^i N^{(1)}$. In particular, $V^g B - \pi^d B \in \pi^d N^{(1)} = N^{(dh+1)}$. The lemma follows. □

LEMMA 4.20. *Suppose that for a choice $B \in M$ with $B \pmod N = C$, there exists an integer $s \geq dh+1$ such that $V^g B - \pi^d B \in N^{(s)}$. Then there exists a choice $B' \in M$ such that $B' - B \in N^{(s-dh)}$ and $V^g B' - \pi^d B' \in N^{(s+1)}$.*

Proof. By assumption, we may write $\pi^d B - V^g B = \pi^d \xi$, where $\xi \in N^{(s-dh)}$. Let $B' := B - \xi$. Then

$$V^g B' - \pi^d B' = V^g B - \pi^d B - V^g \xi + \pi^d \xi = -V^g \xi \in N^{(gm - dh + s)} \subset N^{(s+1)}.$$

Here the last inclusion follows from $gm - dh > 0$. The lemma follows. □

Proof of Proposition 4.17. For any integer $r \geq d+1$ and $w \geq rh$, by Lemmas 4.19 and 4.20, there exists $B = B_1$ such that $B \pmod N = C$ and $V^g - \pi^d B \in N^{(w)} = \pi^r N^{(w-rh)}$. By Lemma 4.19, we define $B_{i+1} := \pi^{-1} V^{\delta_i} B$ for every $1 \leq i < d$, which are well-defined elements in M. Then

$$\begin{cases} V^{\gamma_d} B_d - F B_1 = \pi \xi_d \text{ for some } \xi_d \in N, \\ V^g B - \pi^d B \in N^{(w)} \subset \pi^r N. \end{cases}$$

Hence $\pi \xi_d \in \pi^{r-d} N$. Therefore,

$$\begin{aligned} \varphi_{r-d} : Q/\pi^{r-d} Q &\to M/\pi^{r-d} M \\ C_i &\mapsto B_i \text{ for all } 1 \leq i \leq d \end{aligned}$$

defines a section of $M/\pi^{r-d} M \to Q/\pi^{r-d} Q$. The proposition follows by taking limits. □

5. On Traverso's isogeny conjecture. As remarked in [31, Remark 4.2], by Theorem 4.7, if G is the π-torsion of a minimal π-divisible $\mathcal{O}$-module X, then we can recover the Newton polygon β of X with the property $H(\beta)[\pi] \cong G$ from G. Such a Newton polygon determines an isogeny class of π-divisible $\mathcal{O}$-modules. In particular, *the isogeny class of a minimal π-divisible $\mathcal{O}$-module X is determined by its π-torsion* $X[\pi]$. This is a special case of Traverso's isogeny conjecture. In this section, we discuss the generalization of Traverso's isogeny conjecture (on π-divisible $\mathcal{O}$-modules over an algebraically closed field of characteristic p) and some related questions. Without further comments, all π-divisible $\mathcal{O}$-modules are local-local in the rest of this section (cf. Remark 4.10).

5.1. Traverso's isogeny conjecture for π-divisible $\mathcal{O}$-modules. In this section, if X is a π-divisible $\mathcal{O}$-module, we write $X[n] = X[\pi^n]$ for simplicity. To state Traverso's isogeny conjecture for π-divisible $\mathcal{O}$-modules, we first state the following result (cf. [30, Corollary 1.7]).

THEOREM 5.1. *Let $k \in \mathrm{Alg}_{\mathbb{F}}$ be algebraically closed. Let X be a π-divisible $\mathcal{O}$-module over k. There exists a minimal number $n_X \in \mathbb{Z}_{>0}$ such that X is uniquely* *determined up to isomorphism by $X[n_X]$, i.e. if X' is a π-divisible $\mathcal{O}$-module over k such that $X'[n_X] \cong X[n_X]$, then $X' \cong X$.*

CONJECTURE 5.2 (Traverso's isomorphism conjecture). *Let $k \in \mathrm{Alg}_{\mathbb{F}}$ be algebraically closed. Let X be a π-divisible $\mathcal{O}$-module over k. Assume that X is of type (h, d), then $n_X \leq \lfloor 2\frac{d(h-d)}{h} \rfloor$.*

REMARK 5.3. In the above conjecture, $\lfloor x \rfloor$ denotes the biggest integer that is less or equal to x for $x \in \mathbb{R}$. In [41], Traverso made the above conjecture for p-divisible groups, but with bound $\min(d, h-d)$. In [23], Lau, Nicole, and Vasiu disproved Traverso's bound and proved that $n_X \leq \lfloor 2\frac{d(h-d)}{h} \rfloor$ for p-divisible groups.

Our focus in this section is not Traverso's isomorphism conjecture, but a similar question. We hope to come back to Conjecture 5.2 in the future. By Theorem 5.1, there exists a minimal natural number $b_X \in \mathbb{Z}_{>0}$ such that the isogeny class of X is determined by $X[b_X]$. We call b_X the *isogeny cutoff* of X (cf. [39, 40, 41]).

REMARK 5.4.
(1) It is clear that $1 \leq b_X \leq n_X$.
(2) If X is minimal, then $n_X = b_X = 1$ by Theorem 4.7. Hence Traverso's isomorphism conjecture and the following conjecture hold in this case.

THEOREM 5.5 (Traverso's isogeny conjecture). *Let $k \in \mathrm{Alg}_{\mathbb{F}}$ be algebraically closed. Let X be a π-divisible $\mathcal{O}$-module over k. Assume that X is of type (h, d), then $b_X \leq \lceil \frac{d(h-d)}{h} \rceil$.*

In the following, we give a proof of Theorem 5.1 (cf. [30, Section 1]). Let X and Y be π-divisible $\mathcal{O}$-modules over k. Let $N \geq n \geq 0$ be integers. Denote by

$$\Phi_n : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X[n], Y[n])$$

and

$$\Phi_n^N : \mathrm{Hom}(X[N], Y[N]) \to \mathrm{Hom}(X[n], Y[n])$$

the natural restriction maps.

LEMMA 5.6. *With the notation as above, for every $n \in \mathbb{Z}_{\geq 0}$, there exists an integer $N(X, Y, n)$ such that for every $N \geq N(X, Y, n)$ we have $\mathrm{Im}(\Phi_n) = \mathrm{Im}(\Phi_n^N)$.*

Proof. For each $m \in \mathbb{Z}_{\geq 0}$, the functor $T \mapsto \mathrm{Hom}(X[m]_T, Y[m]_T)$ on k-schemes is representable by a group of finite type over k. Denote it by G_m. For $m \geq n$, the restriction map induces a homomorphism of algebraic groups $\rho_n^m : G_m \to G_n$. We then obtain a descending chain of algebraic groups

$$G'_n \supset \rho_n^{n+1}(G'_{n+1}) \supset \cdots \supset \rho_n^{n+i}(G'_{n+i}) \supset \cdots \supset \Phi_n(\mathrm{Hom}(X, Y)).$$

Here $G'_m = (G_m)_{\mathrm{red}}$. The claim follows as the sequence stabilizes to $\Phi_n(\mathrm{Hom}(X, Y))$. $\square$

LEMMA 5.7. *Given $h \in \mathbb{Z}_{\geq 1}$, there exists an integer d_h such that for every π-divisible $\mathcal{O}$-module X over k of height h and with Newton polygon $\mathbb{N}(X) = \beta$, there is an isogeny $\rho : H(\beta) \to X$ of degree d_h.*

Proof. This follows from the following two facts.

(1) In the isogeny class of X, there are only finitely many isomorphism classes. This follows from the first finiteness theorem [24, Page 44, Theorem 3.4] and the fact that X is a p-divisible group.
(2) $\mathrm{End}(H(\beta))$ has an element Π with degree q.

$\square$

REMARK 5.8. Let (M, V) be the (-1)-$\mathcal{O}$-crystal attached to X. The first finiteness theorem is equivalent to saying that inside the (-1)-$\mathcal{O}$-isocrystal $(N = M \otimes_{\mathbb{Z}} \mathbb{Q}, V)$, there are only finitely many (-1)-$\mathcal{O}$-crystals up to isomorphism. Without loss of generality, we may assume that M is isoclinic of type (h, d). By Lemma 2.22, to give an isomorphism class of (-1)-$\mathcal{O}$-crystals is equivalent to giving a conjugacy class of matrices $V \in M_{h\times h}(W_{\mathcal{O}}(k))$ such that $V^h = \pi^d \,\mathrm{id}_{h\times h}$. By a similar argument as in [24, Section 4], one may show that the latter is finite. Hence the finiteness follows.

LEMMA 5.9. *Let H be a π-divisible $\mathcal{O}$-module over k. Assume that there exists a function $b \mapsto N_H(b)$ such that for every $L \geq N_H(b)$ we have*

$$\mathrm{Im}(\Phi_b : \mathrm{End}(H) \to \mathrm{End}(H[b])) = \mathrm{Im}(\Phi_b^L : \mathrm{End}(H[L]) \to \mathrm{End}(H[b])).$$

Let Z be another π-divisible $\mathcal{O}$-module and $\rho : H \to Z$ be an isogeny of degree q^s. Then for every $N \geq N_H(n+s)+s$,

$$\mathrm{Im}(\Phi_n : \mathrm{End}(Z) \to \mathrm{End}(Z[n])) = \mathrm{Im}(\Phi_n^N : \mathrm{End}(Z[N]) \to \mathrm{End}(Z[n])).$$

Proof. Let Q and M be the covariant Dieudonné $\mathcal{O}$-modules attached to H and Z respectively. The isogeny $\rho : H \to Z$ induces an injection $Q \hookrightarrow M$. Fix $n \geq 0$ and $N > N_H(n+s)+s$. Suppose that $\varphi_N \in \mathrm{End}(Z[N])$ and $\varphi_n = \Phi_n^N(\varphi_N)$. We claim that φ_n can be lifted to $\varphi \in \mathrm{End}(Z)$. Indeed, we have inclusions

$$\pi^N M \subset \pi^{N_H(n+s)} Q \subset \pi^{n+s} Q \subset \pi^{n+s} M \subset \pi^n Q \subset Q \subset M \subset \pi^{-s} Q.$$

Consider $\psi_{n+s} \in \mathrm{End}(Z[n+s])$ defined by $\psi_{n+s} = \pi^s \varphi_n$. It induces an endomorphism on Dieudonné $\mathcal{O}$-modules $\psi_{n+s} : M/\pi^{n+s}M \to M/\pi^{n+s}M$. Consider the composition

$$\psi'_{n+s} : Q/\pi^{n+s}M \hookrightarrow M/\pi^{n+s}M \xrightarrow{\pi^s\varphi_n} \pi^s M/\pi^{n+s}M \hookrightarrow Q/\pi^{n+s}M.$$

By our assumption on N, φ_N restricts to

$$\psi'_{N_H(n+s)} \in \mathrm{End}(H[N_H(n+s)]) = \mathrm{End}(Q/\pi^{N_H(n+s)}Q),$$

hence the restriction to $\mathrm{End}(H[n+s])$ can be lifted to $\mathrm{End}(H)$. Therefore $\psi'_{n+s} \in \mathrm{End}(Q/\pi^{n+s}M)$ can be lifted to $\mathrm{End}(Q)$, hence $\varphi_n \in \mathrm{End}(M/\pi^n M)$ can be lifted to $\mathrm{End}(M)$. The lemma follows. $\square$

LEMMA 5.10. *For every $h \in \mathbb{Z}_{\geq 0}$, there exists $N(h) \in \mathbb{Z}_{>0}$ such that for every π-divisible $\mathcal{O}$-module Z over k of height h and for every $N \geq N(h)$ we have*

$$\mathrm{Im}(\Phi_n : \mathrm{End}(Z) \to \mathrm{End}(Z[n])) = \mathrm{Im}(\Phi_n^N : \mathrm{End}(Z[N]) \to \mathrm{End}(Z[n])).$$

Proof. By Lemma 5.6 and the fact that there are only finitely many Newton polygons with height h, for every $h \in \mathbb{Z}_{\geq 0}$, there exists $N(h) \in \mathbb{Z}_{>0}$ such that for every minimal π-divisible $\mathcal{O}$-module Z of height h and for every $N \geq N(h)$ we have

$$\mathrm{Im}(\Phi_n : \mathrm{End}(Z) \to \mathrm{End}(Z[n])) = \mathrm{Im}(\Phi_n^N : \mathrm{End}(Z[N]) \to \mathrm{End}(Z[n])).$$

The general case then follows from Lemmas 5.7 and 5.9. □

Applying Lemma 5.10 for $Z = X \oplus Y$, we have the following result (cf. [30, Proposition 1.6]).

LEMMA 5.11. *For every $n \in \mathbb{Z}_{>0}$, there exists an integer $N(h', h'', n)$ such that for any π-divisible $\mathcal{O}$-modules X and Y over k of height h' and h'' respectively, and for every $N \geq N(h', h'', n)$ we have*

$$\begin{aligned} &\mathrm{Im}(\Phi_n : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X[n], Y[n])) \\ = &\mathrm{Im}(\Phi_n^N : \mathrm{Hom}(X[N], Y[N]) \to \mathrm{Hom}(X[n], Y[n])). \end{aligned}$$

Applying Lemma 5.11 to the case $n = 1$, we obtain Theorem 5.1.

5.2. A special case of Theorem 5.5.

DEFINITION 5.12. Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a field of characteristic p. Let X be a π-divisible formal $\mathcal{O}$-module over k. Let $\mathcal{P} = (P, Q, F, F_1)$ be the $\mathcal{O}$-display over k attached to X. The *a-number* of X is defined to be $a(X) = \mathrm{Rank}_k(P/(Q + F(P)))$.

REMARK 5.13.
(1) In Definition 5.12, if k is perfect and X is a π-divisible $\mathcal{O}$-module over k. Let $\mathcal{P} = (P, Q, F, F_1)$ be the Dieudonné $\mathcal{O}$-display over k attached to X. One may define the *a-number* of X to be $a(X) = \mathrm{Rank}_k(P/(Q + F(P)))$. If X is etale, then $\mathcal{P}$ is etale, i.e., $P = Q$ (cf. [2, Section 5]). In this case $a(X) = 0$. This invariant is determined by the formal part of X.
(2) Assume that k is perfect. Let X be a π-divisible $\mathcal{O}$-module over k and let (M, F, V) be the covariant Dieudonné $\mathcal{O}$-module attached to X. Then $a(X) = \dim_k M/(F(M) + V(M))$.
(3) Let s and r be integers with $s \geq r > 0$ and $(r, s) = 1$. Then $a(G_{r,s}) = 1$.
(4) Let m and n be positive integers with $(m, n) = 1$. Then $a(H_{m,m+n}) = |m-n|$.

As the first step, we prove Theorem 5.5 under the assumption that $a(X) \leq 1$.

THEOREM 5.14. *Let $k \in \mathrm{Alg}_{\mathbb{F}}$ be algebraically closed. Let X be a π-divisible $\mathcal{O}$-module over k with type (h, d) and $a(X) \leq 1$. Then $b_X \leq \lceil \frac{h(h-d)}{h} \rceil$.*

Let (M, F, V) be the Dieudonné $\mathcal{O}$-module attached to X. Let $x \in M$ such that its reduction modulo π generates the k-vector space $M/(F(M) + V(M))$.

LEMMA 5.15. *Let $c = h - d$. With the notation as above, the following claims hold.*
(1) $W_{\mathcal{O}}(k)[F, V] \cdot x = M$.
(2) $\{x,\ Fx,\ \ldots,\ F^d x,\ Vx,\ \cdots, V^{c-1}x\}$ *is a basis of M as a $W_{\mathcal{O}}(k)$-module.*
(3) $\{x,\ Fx,\ \ldots,\ F^{d-1} x,\ Vx,\ \cdots, V^{c}x\}$ *is a basis of M as a $W_{\mathcal{O}}(k)$-module.*

Proof. The first claim is obvious. We prove the second claim and the third claim follows by an entirely similar argument.

By assumption, $\{x,\ Vx,\ \cdots, V^{c-1}x\}$ is a k-basis of M/FM and $\{x,\ Fx,\ \ldots,\ F^{d-1}x\}$ is a k-basis of M/VM. The later tells us that $\{Fx,\ F^2x,\ \ldots,\ F^d x\}$ is a k-basis of $FM/\pi M$. Hence $\{x,\ Fx,\ \ldots,\ F^d x,\ Vx,\ \cdots, V^{c-1}x\}$ is a k-basis of $M/\pi M$. The claim then follows by Nakayama's lemma. □

LEMMA 5.16. *Let r be a positive integer. Let $a_i \in W_{\mathcal{O}}(k)[\pi^{1/r}]$ $(1 \le i \le n)$ with at least one a_i in $W_{\mathcal{O}}(k)^\times$. Then the equation*

$$ {}^{F^n}x + a_1{}^{F^{n-1}}x + \cdots + a_n x = 0 \tag{5.2.1} $$

has a solution in $(W_{\mathcal{O}}(k)[\pi^{1/r}])^\times$.

Proof. We solve the equation by successive approximation. Modulo $\pi^{1/r}$, equation (5.2.1) gives

$$ \bar{x}^{q^n} + \bar{a}_1\bar{x}^{q^{n-1}} + \cdots + \bar{a}_n\bar{x} = 0. $$

This has a non-zero solution since at least one of the $\bar{a}_i$ is non-zero and k is algebraically closed. Assume that we have found $x_j \in W_{\mathcal{O}}(k)^\times$ with

$$ {}^{F^n}x_j + a_1{}^{F^{n-1}}x_j + \cdots + a_n x_j \equiv 0 \pmod{\pi^{j/r}}. $$

Writing $x_{j+1} = x_j + \pi^{j/r}x$ and solving

$$ {}^{F^n}x_{j+1} + a_1{}^{F^{n-1}}x_{j+1} + \cdots + a_n x_{j+1} \equiv 0 \pmod{\pi^{(j+1)/r}}, $$

we obtain

$$ {}^{F^n}x_j + a_1{}^{F^{n-1}}x_j + \cdots + a_n x_j + \pi^{j/r}({}^{F^n}x + a_1{}^{F^{n-1}}x + \cdots + a_n x) \equiv 0 \pmod{\pi^{(j+1)/r}}. $$

To solve this, it suffices to solve

$$ \bar{x}^{q^n} + \bar{a}_1\bar{x}^{q^{n-1}} + \cdots + \bar{a}_n\bar{x} + \kappa = 0 $$

for some $\kappa \in k$, which has a solution in k since k is algebraically closed. The lemma then follows. □

LEMMA 5.17. *Assume that $V^h + a_1V^{h-1} + \cdots + a_h = 0$, where $a_i \in W_{\mathcal{O}}(k)$ $(1 \le i \le h)$ and $a_h \ne 0$. Then there exist positive integers r and s, such that*

$$ V^h + a_1V^{h-1} + \cdots + a_h = (b_0V^{h-1} + b_1V^{h-2} + \cdots + b_{h-1})(V - \pi^{s/r})u, $$

where $u,\ b_i \in W_{\mathcal{O}}(k)[\pi^{1/r}]$ $(0 \le i \le h-1)$ and u is invertible.

Proof. Let $s/r = \inf\{\frac{\mathrm{ord}_\pi(a_i)}{i} \mid 1 \le i \le h\}$. Let $a_i = \pi^{is/r}\alpha_i$. Then $\alpha_i \in W_{\mathcal{O}}(k)[\pi^{1/r}]$ $(1 \le i \le h)$ and at least one α_i is a unit in $W_{\mathcal{O}}(k)$. We need to find $v,\ b_i \in W_{\mathcal{O}}(k)[\pi^{1/r}]$ $(0 \le i \le h-1)$ with v invertible such that

$$ (V^h + a_1V^{h-1} + \cdots + a_h)v = (b_0V^{h-1} + b_1V^{h-2} + \cdots + b_{h-1})(V - \pi^{s/r}). \tag{5.2.2} $$

Comparing the coefficients, equation (6.1.2) is equivalent to

$$ \begin{aligned} \sigma^h(v) &= b_0 \\ a_1\sigma^{h-1}(v) &= b_1 - b_0\pi^{s/r} \\ &\cdots \\ a_{h-1}\sigma(v) &= b_{h-1} - b_{h-2}\pi^{s/r} \\ a_h v &= -b_{h-1}\pi^{s/r}. \end{aligned} \tag{5.2.3} $$

Here $\sigma = {}^{F^{-1}}$, hence $\sigma(x) \equiv x^{-q} \pmod{\pi}$. Write $b_i = \pi^{is/r}\beta_i$, we have

$$\begin{aligned} \sigma^h(v) &= \beta_0 \\ \alpha_1\sigma^{h-1}(v) &= \beta_1 - \beta_0 \\ &\cdots \\ \alpha_{h-1}\sigma(v) &= \beta_{h-1} - \beta_{h-2} \\ \alpha_h v &= -\beta_{h-1}. \end{aligned} \tag{5.2.4}$$

Summing up the equations in (5.2.4), we have

$$\sigma^h(v) + \alpha_1\sigma^{h-1}(v) + \cdots + \alpha_{h-1}\sigma(v) + \alpha_h v = 0.$$

This has a solution in $(W_{\mathcal{O}}(k)[\pi^{1/r}])^\times$ by Lemma 5.16. Tracing back the above steps, the lemma follows. □

Proof of Theorem 5.14. Let M be the covariant Dieudonné $\mathcal{O}$-module of X. Let $x \in M$ be such that its reduction modulo π generates the one-dimensional k-vector space $M/(FM + VM)$. Let $a_i \in W_{\mathcal{O}}(k)$ $(0 \leq i \leq h)$ be such that

$$\Psi := \sum_{i=0}^{c} a_{i+d}V^i + \sum_{j=1}^{d} a_{d-j}F^j : M \to M$$

maps x to 0. By Lemma 5.15, we may assume that $a_h = 1$ and $a_0 \in W_{\mathcal{O}}(k)^\times$. Then these elements are uniquely determined. Consider the composition $V^d \circ \Psi : M \to M$. It is easy to check that $V^d \circ \Psi = 0$ and we may write it as

$$V^d \circ \Psi = \sum_{i=0}^{c} a'_{i+d}V^{i+d} + \sum_{j=1}^{d} a'_{d-j}\pi^j V^{d-j},$$

where $\operatorname{ord}_\pi a_i = \operatorname{ord}_\pi a'_i$ $(0 \leq i \leq h)$ and $a'_h = 1$. Define a polynomial $Q(T) \in W_{\mathcal{O}}(k)$ by

$$Q(T) = \sum_{i=0}^{h} \alpha_i T^i,$$

where $\alpha_i = \begin{cases} a'_i & \text{if } d \leq i \leq h, \\ \pi^{d-i}a'_i & \text{if } 0 \leq i \leq d-1. \end{cases}$ By Lemma 5.17, the Newton polygon of X is the same as the Newton polygon of the polynomial $Q(T)$.

Let $J = \lceil \frac{d(h-d)}{h} \rceil$. Let M_g be another Dieudonné $\mathcal{O}$-module over $W_{\mathcal{O}}(k)$ such that $M_g/\pi^J M_g \cong M/\pi^J M$ as $W_{\mathcal{O}}(k)[F,V]$-modules. Let X_g be the associated π-divisible $\mathcal{O}$-module. We need to show that X_g and X are isogenous.

As $J \geq 1$, we have $X[\pi] \cong X_g[\pi]$. Hence $a(X_g) = a(X) \leq 1$. By the same argument as above, we obtain another polynomial $Q_g(T) \in W_{\mathcal{O}}(k)[T]$

$$Q_g(T) = \sum_{i=0}^{h} \beta_i T^i,$$

such that the Newton polygon of X_g is the same as the Newton polygon of $Q_g(T)$. From our assumption and construction, we have $\beta_i \equiv \alpha_i \pmod{\pi^J}$ for $d \leq i \leq h$, $\beta_i \equiv \alpha_i \pmod{\pi^{J+d-i}}$ for $0 \leq i \leq d-1$. Note that the Newton polygons of $Q(T)$ and $Q_g(T)$ are below the line connecting the points $(0,0)$ and (h,d), the above two congruences ensure that the Newton polygons of $Q(T)$ and $Q_g(T)$ are the same. Hence X and X_g are isogenous. The theorem follows. □

5.3. Further remarks on Theorem 5.5. In this part, $k \in \mathrm{Alg}_{\mathcal{O}}$ is an algebraically closed field of characteristic p.

5.3.1. A complete proof of Theorem 5.5. In the case of p-divisible groups, Nicole and Vasiu [27] proved that Theorem 5.5 holds. Their idea is to introduce another invariant $\tilde{b}_X$, which is called the *weak isogeny cutoff* of X, and prove inequalities $b_X \leq \tilde{b}_X \leq \lceil \frac{d(h-d)}{h} \rceil$.

DEFINITION 5.18. Let X be a π-divisible $\mathcal{O}$-module over k. We say that X satisfies the *Oort condition* (OC) if there exists a π-divisible $\mathcal{O}$-module $\mathcal{X}$ over $k[[x]]$ such that

(1) its fibre over k is X;
(2) if $\overline{k((x))}$ is an algebraic closure of $k((x))$, then $\mathcal{X}_{\overline{k((x))}}$ has the same Newton polygon as X and its a-number is at most one.

REMARK 5.19. For p-divisible groups over k, every X satisfies the above condition by [29, Proposition 2.8]. This is crucial in the argument of [27], as we shall see in the following.

THEOREM 5.20. *Every π-divisible $\mathcal{O}$-module over k satisfies the Oort condition.*

The proof of this theorem will be given in Section 6.2. We explain a proof of Theorem 5.5 assuming Theorem 5.20.

DEFINITION 5.21. Let X be a π-divisible $\mathcal{O}$-module over k which satisfies OC. By the *weak isogeny cutoff* of X, we mean the smallest number $\tilde{b}_X \in \mathbb{Z}_{>0}$ such that the following two properties hold:

(1) if X' is a π-divisible $\mathcal{O}$-module over k such that $X'[\pi^{\tilde{b}_X}]$ is isomorphic to $X[\pi^{\tilde{b}_X}]$, then its Newton polygon $N(X')$ is not strictly above $N(X)$;
(2) there exists a π-divisible $\mathcal{O}$-module $\mathcal{X}$ over $k[[x]]$ that has the following properties:
   (a) its fibre over k is X;
   (b) the fibre $\mathcal{X}_{k((x))}$ has the same Newton polygon as X;
   (c) the isogeny cutoff of $\mathcal{X}_{\overline{k((x))}}$ is at most $\tilde{b}_X$ and the a-number of $\mathcal{X}_{\overline{k((x))}}$ is at most one.

We have the following relation between isogeny cutoff and weak isogeny cutoff.

PROPOSITION 5.22. *Let X be a π-divisible $\mathcal{O}$-module over k. Then $b_X \leq \tilde{b}_X$.*

Proof. By Theorem 5.20, there exists a π-divisible $\mathcal{O}$-module $\mathcal{X}$ over $k[[x]]$ with constant Newton polygon such that $\mathcal{X}_k = X$ and the isogeny cutoff b of $\mathcal{X}_{\overline{k((x))}}$ is at most $\tilde{b}_X$. Let X' be a π-divisible $\mathcal{O}$-module over k such that $X'[\pi^{\tilde{b}_X}] = X[\pi^{\tilde{b}_X}]$. By a similar argument as in [21, Proposition 3.15] or [2, Lemma 4.4] (cf. [10, Section 8]), there exists a π-divisible $\mathcal{O}$-module $\mathcal{X}'$ over $k[[x]]$ such that $\mathcal{X}'_k = X'$ and

$$\mathcal{X}'[\pi^{\tilde{b}_X}] = \mathcal{X}[\pi^{\tilde{b}_X}].$$

Since the isogeny cutoff of $\mathcal{X}_{\overline{k((x))}}$ is less or equal to $\tilde{b}_X$, $\mathcal{X}'_{\overline{k((x))}}$ and $\mathcal{X}_{\overline{k((x))}}$ are isogenous and have the same Newton polygon. A similar argument as in [7, Chap. 4, Section 7] shows that the Newton polygons go up under specialization. Hence the Newton polygon of X' is above the Newton polygon of X. By assumption, X and X' have the same Newton polygon. Therefore $b_X \leq \tilde{b}_X$. □

Proof of Theorem 5.5. Let $J = \lceil \frac{d(h-d)}{h} \rceil$. By Proposition 5.22, it suffices to prove that $\tilde{b}_X \leq J$. Let $\mathcal{N}_{h,d}$ be the set of Newton polygons of π-divisible $\mathcal{O}$-modules over k of type (h, d). Let N_X be the Newton polygon of X. Let $\mathcal{D}_X$ be the set of π-divisible $\mathcal{O}$-modules over k which are of type (h, d), and whose Newton polygons are strictly above N_X. We prove the inequality $\tilde{b}_X \leq J$ by decreasing induction on $N_X \in \mathcal{N}_{h,d}$.

Assume that for every $Y \in \mathcal{D}_X$ we have $\tilde{b}_Y \leq J$. We show that $\tilde{b}_X \leq J$. By Proposition 5.22, we have $b_Y \leq \tilde{b}_Y \leq J$. Let $\mathcal{X}$ be a π-divisible $\mathcal{O}$-module over $k[[x]]$ such that the second condition in Definition 5.21 holds. Let b be the isogeny cutoff of $\mathcal{X}_{\overline{k((x))}}$. By the definition of weak isogeny cutoff, we have

$$\tilde{b}_X \leq \max\{b,\ b_Y \mid Y \in \mathcal{D}_X\}.$$

As $b \leq J$ by Theorem 5.14 and $b_Y \leq J$ by induction assumption, we have $\tilde{b}_X \leq J$. The proposition follows. □

5.3.2. Isogeny cutoff and minimal height.

DEFINITION 5.23. By the *minimal height* h_X of a π-divisible $\mathcal{O}$-module X over k we mean the smallest non-negative number h_X such that there exists an isogeny $\iota : X_0 \to X$ from a minimal π-divisible $\mathcal{O}$-module, whose kernel $\mathrm{Ker}(\iota)$ is annihilated by π^{h_X}.

We study the relation between isogeny cutoff and minimal height (cf. [42]). First we have the following lemma, which follows from the equivalence between π-divisible $\mathcal{O}$-modules over k and Dieudonné $\mathcal{O}$-modules over k.

LEMMA 5.24. *Let X be a π-divisible $\mathcal{O}$-module over k and (M, F, V) be the covariant Dieudonné $\mathcal{O}$-module of X. The isogeny cutoff b_X is the smallest positive integer such that for every element $g \in \mathrm{GL}_{W_{\mathcal{O}}(k)}(M)$ congruent to id_M modulo π^{b_X}, the Dieudonné $\mathcal{O}$-module $(M, g \circ F, V \circ g^{-1})$ is isogenous to (M, F, V).*

PROPOSITION 5.25. *Let X be a π-divisible $\mathcal{O}$-modules over k. Then $b_X \leq h_X + 1$.*

Proof. Let $\iota : X_0 \to X$ be an isogeny whose kernel is annihilated by π^{h_X}. Let $(M_0, F_0, V_0) \hookrightarrow (M, F, V)$ be the monomorphism of the corresponding Dieudonné $\mathcal{O}$-modules. We identify M_0 with its image under this monomorphism. The existence of ι shows that $\pi^{h_X} M \subset M_0 \subset M$. Let $g \in \mathrm{GL}_{W_{\mathcal{O}}(k)}(M)$ be such that $g \equiv \mathrm{id}_M \pmod{\pi^{h_X+1}}$. By Lemma 5.24, it suffices to show that (M, F, V) is isogenous to $(M, g \circ F, V \circ g^{-1})$.

Note that we have relation $\pi^{h_X} M \subset M_0 \subset M$, the endomorphism g induces an endomorphism $g \in \mathrm{GL}_{W_{\mathcal{O}}(k)}(M_0)$. Hence the triple $(M_0, g \circ F, V \circ g^{-1})$ is a Dieudonné $\mathcal{O}$-module. Since X_0 is minimal, we have $(M_0, F, V) \cong (M_0, g \circ F, V \circ g^{-1})$ by Theorem 4.7. Moreover, (M, F, V) and (M_0, F, V) are isogenous, so $(M, g \circ F, V \circ g^{-1})$ and $(M_0, g \circ F, V \circ g^{-1})$ are isogenous. The claim then follows. □

5.3.3. The bound is sharp. We construct two Dieudonné $\mathcal{O}$-modules M and M' of type (h, d) over k, hence two π-divisible $\mathcal{O}$-modules X and X' of type (h, d) over k, such that $X[J-1] \cong X'[J-1]$ with $J = \lceil \frac{d(h-d)}{h} \rceil$, but X and X' are not isogenous. Therefore the bound in Theorem 5.5 is sharp (cf. [27, Example 3.2]).

The case $J = 1$ is obvious. Assume now $J \geq 2$. Let $M = \langle e_1, \ldots, e_h \rangle$ and $M' = \langle f_1, \ldots, f_h \rangle$ be two Diedonné $\mathcal{O}$-modules. The corresponding F and V are given by

$$\begin{array}{c} e_h,\ \pi e_1,\ \ldots,\ \pi e_{d-1},\ \pi e_d,\ e_{d+1},\ \ldots,\ e_{h-1} \\ \Big\uparrow V \\ e_1,\ e_2,\ \ldots,\ e_d,\ e_{d+1},\ e_{d+2},\ \ldots,\ e_h \\ \Big\downarrow F \\ e_2,\ e_3,\ \ldots,\ e_{d+1},\ \pi e_{d+2},\ \pi e_{d+3},\ \ldots,\ \pi e_1 \end{array} \tag{5.3.1}$$

and

$$\begin{array}{c} f_h,\ \pi f_1,\ \ldots,\ \pi f_{d-1},\ \pi f_d - \pi^{J-1} f_h,\ f_{d+1},\ \ldots,\ f_{h-1} \\ \Big\uparrow V \\ f_1,\ f_2,\ \ldots,\ f_d,\ f_{d+1},\ f_{d+2},\ \ldots,\ f_h \\ \Big\downarrow F \\ f_2,\ f_3,\ \ldots,\ f_{d+1} + \pi^{J-1} f_1,\ \pi f_{d+2},\ \pi f_{d+3},\ \ldots,\ \pi f_1 \end{array} \tag{5.3.2}$$

respectively. Note that $M/\pi^{J-1}M \cong M'/\pi^{J-1}M'$ as $W_{\mathcal{O}}(k)[F,V]$-modules. We show that they have different Newton polygons. It is clear that $M/(FM+VM)$ is generated by e_1 and $M'/(FM'+VM')$ is generated by f_1. Note that $V^h e_1 = \pi^d e_1$, the Newton polygon of M is the same as the Newton polygon of $Q(T) = T^h - \pi^d$. On the other hand,

$$\begin{aligned} V^h(f_1) &= \pi^d f_1 - V^{h-d-1}(\pi^{J-1} f_h) \\ &= \pi^d f_1 - V^{h-d-1}(\pi^{J-1} V f_1) = \pi^d f_1 - \pi^{J-1} V^{h-d} f_1. \end{aligned} \tag{5.3.3}$$

The Newton polygon of M' is the same as the Newton polygon of

$$Q'(T) = T^h + \pi^{J-1} T^{h-d} - \pi^d.$$

The claim then follows since $Q(T)$ and $Q'(T)$ have different Newton polygons.

6. Deformations of π-divisible $\mathcal{O}$-modules. In this section, we study deformations of π-divisible $\mathcal{O}$-modules. The main goal is to give a proof of Theorem 5.20, which completes the proof of Traverso's isogeny conjecture as explained in Section 5.3.1. Because we have set up the basics for π-divisible $\mathcal{O}$-modules in a similar framework as for p-divisible groups, we could adapt the proofs from the references to prove most of the results in this section (cf. [6, 29, 46]).

6.1. The deformation functor and its representability. In the following, we study deformations of π-divisible $\mathcal{O}$-modules via displays. We focus on the case where X is formal. In this case, under the appropriate setting, deforming a formal π-divisible $\mathcal{O}$-module is equivalent to deforming the associated nilpotent $\mathcal{O}$-display by Theorem 3.11. Based on the discussion in [45, Section 2] and [2, Section 3.2], deformations of an $\mathcal{O}$-display are explicit as they could be describe by the structure equation. We give details in Sections 6.1.1 and 6.1.2. The π-divisible $\mathcal{O}$-module case follows by similar argument using the relation between π-divisible $\mathcal{O}$-modules and Dieudonné $\mathcal{O}$-displays in Theorem 3.11. Some ideas in the following are similar to those in [15].

6.1.1. Liftings of an $\mathcal{O}$-display. Let R be an $\mathcal{O}$-algebra. Let $\mathcal{P}$ be an $\mathcal{O}$-display over R. Let $S \to R$ be a surjection of $\mathcal{O}$-algebras. A *lifting* of $\mathcal{P}$ to S is an $\mathcal{O}$-display $\mathcal{P}'$ over S such that the base change of $\mathcal{P}'$ with respect to $S \to R$ is isomorphic to $\mathcal{P}$. See [45, Section 2] and [2, Section 3.2] for more details on liftings. In particular, if $\mathrm{Ker}(S \to R)$ has a divided power structure, it is known that to lift $\mathcal{P}$ to S is equivalent to lifting the Hodge filtration (cf. [45, Proposition 45, Equations (71) and (72)] and [4, Lemma 2.18])

$$\mathrm{Fil}^1_{\mathcal{P}}(R)(:= Q/I_{\mathcal{O}}(R)P) \subset \mathrm{Fil}_{\mathcal{P}}(R)(:= P/I_{\mathcal{O}}(R)P).$$

I.e. a lifting of $\mathcal{P}$ to S correspond to a direct summand of $\mathrm{Fil}_{\mathcal{P}}(S)$ that lifts $\mathrm{Fil}^1_{\mathcal{P}}(R)$. Note that this is denoted by $\mathcal{D}^1_{\mathcal{P}}(R) \subset \mathcal{D}_{\mathcal{P}}(R)$ in [45].

Let us consider the special case, where $S \to R$ is a surjection with kernel $\mathfrak{a}$, such that $\mathfrak{a}^2 = 0$. Define an abelian group $\mathcal{G}$ by

$$\mathcal{G} := \mathrm{Hom}(\mathrm{Fil}^1_{\mathcal{P}}(R), \mathfrak{a} \otimes_R (\mathrm{Fil}_{\mathcal{P}}(R)/\mathrm{Fil}^1_{\mathcal{P}}(R))). \tag{6.1.1}$$

We define an action of $\mathcal{G}$ on the set of liftings of $\mathcal{P}$ to S as follows. Two liftings of $\mathcal{P}$ to S correspond to two liftings E_1 and E_2 of the Hodge filtration, i.e., E_1 and E_2 are both direct summand of $\mathrm{Fil}_{\mathcal{P}}(S)$ that lifts $\mathrm{Fil}^1_{\mathcal{P}}(R)$. Consider the natural homomorphism

$$E_1 \subset \mathrm{Fil}_{\mathcal{P}}(S) \to \mathrm{Fil}_{\mathcal{P}}(S)/E_2. \tag{6.1.2}$$

Since $E_1 \equiv E_2 \pmod{\mathfrak{a}}$, the homomorphism (6.1.2) factors as

$$E_1 \to \mathfrak{a}(\mathrm{Fil}_{\mathcal{P}}(S)/E_2) \subset \mathrm{Fil}_{\mathcal{P}}(S)/E_2. \tag{6.1.3}$$

Moreover, since $\mathfrak{a}^2 = 0$, we have an isomorphism $\mathfrak{a}(\mathrm{Fil}_{\mathcal{P}}(S)/E_2) \cong \mathfrak{a} \otimes_R (\mathrm{Fil}_{\mathcal{P}}(R)/\mathrm{Fil}^1_{\mathcal{P}}(R))$. Hence we obtain a homomorphism

$$u : \mathrm{Fil}^1_{\mathcal{P}}(R) \to \mathfrak{a} \otimes_R (\mathrm{Fil}_{\mathcal{P}}(R)/\mathrm{Fil}^1_{\mathcal{P}}(R)).$$

Define $E_1 - E_2 = u$. It is easy to check from the construction that

$$E_2 = \{e - \widetilde{u(e)} \mid e \in E_1\}, \tag{6.1.4}$$

where $\widetilde{u(e)} \in \mathfrak{a}\,\mathrm{Fil}_{\mathcal{P}}(S)$ denotes any lifting of $u(e)$. We have the following result (cf. [45, Corollary 49]).

Proposition 6.1. *Let $\mathcal{P}$ be an $\mathcal{O}$-display over R. Let $S \to R$ be a surjection with kernel $\mathfrak{a}$ such that $\mathfrak{a}^2 = 0$. The action of $\mathcal{G}$ on the set of liftings of $\mathcal{P}$ to S constructed as above is simply transitive. If $\mathcal{P}_0$ is a lifting of $\mathcal{P}$ and $u \in \mathcal{G}$, we denote the action by $\mathcal{P}_0 + u$.*

Proof. The transitivity follows from the construction. Moreover, if $E_1 = E_2$, then the object u constructed above is trivial. Hence the action is simple. The proposition follows. □

Remark 6.2. The above action could be described more explicitly. Consider $\mathfrak{a}$ as an ideal of $W_{\mathcal{O}}(\mathfrak{a})$ and we equip $\mathfrak{a}$ with the trivial divided $\mathcal{O}$-pd-structure (cf. [4, Section 2.8]). Let $\mathcal{P}_0 = (P_0, Q_0, F, F_1)$ be a lifting of $\mathcal{P}$ to S. Let $\alpha : P_0 \to \mathfrak{a}P_0 \subset W_{\mathcal{O}}(\mathfrak{a})P_0$ be a homomorphism. For the pair (P_0, Q_0), we define a new $\mathcal{O}$-display structure by setting

$$\begin{aligned} F_\alpha x &= Fx - \alpha(Fx) \text{ for } x \in P_0, \\ F_{1\alpha} y &= F_1 y - \alpha(F_1 y) \text{ for } y \in Q_0. \end{aligned} \tag{6.1.5}$$

By Proposition 6.1, there is an element $u \in \mathcal{G}$ such that $\mathcal{P}_\alpha = \mathcal{P}_0 + u$. This u could be described as follows. We have a natural isomorphism $\mathfrak{a}P_0 \cong \mathfrak{a} \otimes_R P/I_{\mathcal{O}}(R)P$. Hence the homomorphism α factors uniquely through a homomorphism

$$\tilde{\alpha} : P/I_{\mathcal{O}}(R)P \to \mathfrak{a} \otimes_R P/I_{\mathcal{O}}(R)P.$$

Conversely, any such R-module homomorphism $\tilde{\alpha}$ determines a unique α. Let $u \in \mathcal{G}$ be the composition of

$$Q/I_{\mathcal{O}}(R)P \subset P/I_{\mathcal{O}}(R)P \xrightarrow{\tilde{\alpha}} \mathfrak{a} \otimes_R P/I_{\mathcal{O}}(R)P \to \mathfrak{a} \otimes_R P/Q.$$

Then it is easy to check that $\mathcal{P}_\alpha = \mathcal{P}_0 + u$.

6.1.2. Deformations of an $\mathcal{O}$-display. Let Λ be a topological $\mathcal{O}$-algebra of the following type. The topology on Λ is given by a filtration of $\mathcal{O}$-ideals

$$\Lambda = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n \supset \dots, \tag{6.1.6}$$

such that $\mathfrak{a}_i\mathfrak{a}_j \subset \mathfrak{a}_{i+j}$. We assume that π is nilpotent in $\Lambda/\mathfrak{a}_1$ and hence in any quotient $\Lambda/\mathfrak{a}_i$. Let $R \in \mathrm{Alg}_{\mathcal{O}}$ with the discrete topology. Suppose we are given a continuous surjective homomorphism $\varphi : \Lambda \to R$.

Let $\mathrm{Aug}_{\Lambda\to R}$ be the category of morphisms of discrete Λ-algebras $\psi_S : S \to R$, such that ψ_S is surjective and has a nilpotent kernel. If $\Lambda = R$, we denote this category simply by Aug_R.

Let Nil_R be the category of nilpotent R-algebras. Let $\mathcal{N} \in \mathrm{Nil}_R$. We associate with $\mathcal{N}$ an augmented R-algebra $R|\mathcal{N}|$ as follows. As an R-module, $R|\mathcal{N}| = R \oplus \mathcal{N}$. The multiplication is given by

$$(r_1 \oplus n_1)(r_2 \oplus n_2) = (r_1r_2) \oplus (r_1n_2 + r_2n_1 + n_1n_2) \text{ for all } r_1,\ r_2 \in R \text{ and } n_1,\ n_2 \in \mathcal{N}.$$

Let M be an R-module. We regard M as an object in Nil_R by setting $M^2 = 0$. Hence we obtain fully faithful functors $\mathrm{Mod}_R \subset \mathrm{Nil}_R \subset \mathrm{Aug}_{\Lambda\to R}$.

Definition 6.3. Let F be a set-valued functor on $\mathrm{Aug}_{\Lambda\to R}$. The restriction of this functor to the category of R-modules is denoted by t_F and is called the *tangent functor* of F.

Definition 6.4. Let $\mathcal{P}$ be an $\mathcal{O}$-display over R. Let $S \to R$ be a surjection of $\mathcal{O}$-algebras such that the kernel is nilpotent. A *deformation* of $\mathcal{P}$ to S is an isomorphism class of pairs $(\mathcal{P}', \iota)$, where $\mathcal{P}'$ is an $\mathcal{O}$-display over S and $\iota : \mathcal{P} \to \mathcal{P}'_R$ is an isomorphism. Here $\mathcal{P}'_R$ is the base change of $\mathcal{P}'$ with respect to $S \to R$ (cf. [2, Section 2.2]).

The *deformation functor* of $\mathcal{P}$ is defined by

$$\begin{aligned} \mathbb{D}_{\mathcal{P}} : \mathrm{Aug}_{\Lambda\to R} &\to \mathrm{Sets} \\ S &\mapsto \{\text{deformations of } \mathcal{P} \text{ to } S\}. \end{aligned} \tag{6.1.7}$$

We show that the functor $\mathbb{D}_{\mathcal{P}}$ is pro-representable and construct the universal object. First we compute the tangent functor of $\mathbb{D}_{\mathcal{P}}$. Let M be an R-module. We study the liftings of $\mathcal{P}$ to $R|M|$ with respect to the canonical map $R|M| \to R$. In this case, the kernel of $R|M| \to R$ is square-zero, we may apply Proposition 6.1 to this situation. In particular, we have an isomorphism:

$$\mathrm{Hom}_R(Q/I_{\mathcal{O}}(R)P, M \otimes_R P/Q) \to \mathbb{D}_{\mathcal{P}}(R|M|).$$

Note that in this case, we have a canonical choice for $\mathcal{P}_0 = \mathcal{P}_{R|M|}$ (cf. Remark 6.2). The tangent space of the functor $\mathbb{D}_{\mathcal{P}}$ is isomorphic to the finitely generated projective R-module $\mathrm{Hom}_R(Q/I_{\mathcal{O}}(R)P, P/Q)$. Define $\omega = \mathrm{Hom}_R(P/Q, Q/I_{\mathcal{O}}(R)P)$. Then we have an isomorphism

$$\mathrm{Hom}_R(\omega, M) \to \mathbb{D}_{\mathcal{P}}(R|M|).$$

The identical endomorphism of ω defines a morphism of functors

$$\mathrm{Spf}\, R|\omega| \to \mathbb{D}_{\mathcal{P}}. \tag{6.1.8}$$

Let $\tilde{\omega}$ be a finitely generated projective Λ-module with $\tilde{\omega} \otimes_\Lambda R \cong \omega$. Let $S_\Lambda(\tilde{\omega})$ be the symmetric algebra. Let A be the completion of the augmented algebra $S_\Lambda(\tilde{\omega})$ with respect to the augmentation ideal. The morphism (6.1.8) may be lifted to a morphism

$$\mathrm{Spf}\, A \to \mathbb{D}_{\mathcal{P}}. \tag{6.1.9}$$

By our construction, the morphism (6.1.9) induces an isomorphism on the tangent spaces. Hence it is an isomorphism. Now we could describe the universal $\mathcal{O}$-display $\mathcal{P}^{\mathrm{univ}}$ as follows. Let $u : Q/I_{\mathcal{O}}(R)P \to \omega \otimes_R P/Q$ be the map induced by the identical endomorphism of ω. Let $\alpha : P \to \omega \otimes_R P/Q$ be any map that induces u (cf. Remark 6.2). Then we obtain an $\mathcal{O}$-display $\mathcal{P}_\alpha$ over $R|\omega|$. Lifting $\mathcal{P}_\alpha$ to A, we obtain $\mathcal{P}^{\mathrm{univ}}$.

Remark 6.5. We may write down the universal object explicitly in terms of structure equation as follows (cf. [28, Section (1.12)] and [45, Equation (87)]). Assume that $\mathcal{P} = (P, Q, F, F_1)$ and $P = L \oplus T$ is a normal decomposition of $\mathcal{P}$. Then $\mathcal{P}$ is determined by its structure equation

$$\Phi := F \oplus F_1 : T \oplus L \to P.$$

Here $F \oplus F_1$ is an ^{F}-linear isomorphism. Assume further that L and T are finitely generated free $W_{\mathcal{O}}(R)$-modules, which is automatic if $W_{\mathcal{O}}(R)$ is local. Assume that the rank of L is c and the rank of T is d. Fix a basis of L and T, hence a basis of P, $F \oplus F_1$ is given by a matrix $M_{\mathcal{P}} \in \mathrm{GL}_h(W_{\mathcal{O}}(R))$. Here $h = c + d$. We choose indeterminates $\{t_{ij} \mid 1 \le i \le d,\ 1 \le j \le c\}$ and set $A = \Lambda[[t_{ij}]]$. Define an invertible matrix in $\mathrm{GL}_h(W_{\mathcal{O}}(A))$ by

$$\begin{pmatrix} \mathrm{id}_d & [t_{ij}] \\ 0 & \mathrm{id}_c \end{pmatrix} \tilde{M}_{\mathcal{P}}.$$

Here $\tilde{M}_{\mathcal{P}}$ is a lifting of $M_{\mathcal{P}}$ in $\mathrm{GL}_h(W_{\mathcal{O}}(A))$ and $[t_{ij}]$ is the Teichmüller representative of t_{ij}. This matrix defines an $\mathcal{O}$-display $\mathcal{P}^{\mathrm{univ}}$ over the topological ring A. Then the pair $(A, \mathcal{P}^{univ})$ pro-represents the functor $\mathbb{D}_{\mathcal{P}}$ on the category $\mathrm{Aug}_{\Lambda \to R}$.

6.1.3. Deformations of π-divisible formal $\mathcal{O}$-modules. Let $R \in \mathrm{Alg}_{\mathcal{O}}$ with π nilpotent in it. Let Λ be as above. Let X be a formal π-divisible $\mathcal{O}$-module over R. Let $S \to R$ be a surjection with nilpotent kernel. A *deformation* of X to S is an isomorphism class of pairs (X', ι), where X' is a formal π-divisible $\mathcal{O}$-module over

S and $\iota : X' \times_S R \cong X$ is an isomorphism of formal π-divisible $\mathcal{O}$-modules. The *deformation functor* of X is defined by

$$\begin{aligned} \mathbb{D}_X : \mathrm{Aug}_{\Lambda \to R} &\to \mathrm{Sets} \\ S &\mapsto \{\text{deformations of } X \text{ to } S\}. \end{aligned} \tag{6.1.10}$$

Combining Theorem 3.11 and the discussion in Section 6.1.2, we obtain the following result.

THEOREM 6.6. *With the notation as above, if $X = \mathrm{BT}_{\mathcal{O}}(\mathcal{P})$, then the two functors $\mathbb{D}_X$ and $\mathbb{D}_{\mathcal{P}}$ are equivalent. Therefore, if $W_{\mathcal{O}}(R)$ is local, there exists a formal π-divisible $\mathcal{O}$-module $\mathcal{X} \to \mathrm{Spf}(\Lambda[[t_1, \cdots, t_{dc}]])$ which is universal for the functor $\mathbb{D}_X$, i.e.,*

$$\mathbb{D}_X(S) = \mathrm{Hom}(\Lambda[[t_1, \cdots, t_{dc}]], S) \tag{6.1.11}$$

and every deformation of X over S is a base change induced by a morphism in equation (6.1.11). *Here $c = h - d$ and X is of type (h, d).*

6.1.4. Deformations of π-divisible $\mathcal{O}$-modules. Applying the relation between π-divisible $\mathcal{O}$-modules and Dieudonné $\mathcal{O}$-displays, one could obtain the universal deformation of a π-divisible $\mathcal{O}$-module over a perfect field of characteristic p. Note that there is a difference on the base ring R in Theorem 3.11 between the display case and the Dieudonné display case, we make some necessary adjustments.

Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a perfect field of characteristic p. Let $\mathcal{C}_{\mathcal{O},k}$ be the category of complete Noetherian local $\mathcal{O}$-algebra with residue field k. Let X be a π-divisible $\mathcal{O}$-module over k. Let $S \in \mathcal{C}_{\mathcal{O},k}$. A *deformation* of X to S is an isomorphism class of pairs (X', ι), where X' is a π-divisible $\mathcal{O}$-module over S and $\iota : X' \times_S k \cong X$ is an isomorphism of π-divisible $\mathcal{O}$-modules. The *deformation functor* of X is defined by

$$\begin{aligned} \mathbb{D}_X : \mathcal{C}_{\mathcal{O},k} &\to \mathrm{Sets} \\ S &\mapsto \{\text{deformations of } X \text{ to } S\}. \end{aligned} \tag{6.1.12}$$

By Theorem 3.11 and Remark 3.13, deforming X is equivalent to deforming the Dieudonné $\mathcal{O}$-display $\mathcal{P}$ with $X = \mathrm{BT}_{\mathcal{O}}(\mathcal{P})$, in which case one could work as in Section 6.1.2 and write down the universal object explicitly by giving the structure equation as in Remark 6.5. Note that if $p = 2$, we use Dieudonné $\mathcal{O}$-displays over the Zink ring with modified Verschiebung. See [20, Section 4] and [4, Section 2.8] for the study of deformations of windows, which unifies some arguments in the display case and in the Dieudonné display case. We have the following result.

THEOREM 6.7. *With the notation as above, the functor $\mathbb{D}_X$ is representable. More precisely, there exists a π-divisible $\mathcal{O}$-module over $\mathrm{Spf}(W_{\mathcal{O}}(k)[[t_1, \cdots, t_{dc}]])$ which is universal for the functor $\mathbb{D}_X$, i.e.,*

$$\mathbb{D}_X(S) = \mathrm{Hom}(W_{\mathcal{O}}(k)[[t_1, \cdots, t_{dc}]], S) \tag{6.1.13}$$

and every deformation of X over S is a base change of the universal object induced by a morphism in equation (6.1.13). *Here $c = h - d$ and X is of type (h, d).*

6.1.5. The Newton polygon of the universal object. We use the notation in Section 6.1.4. We are interested in deformations of X to $\mathbb{F}$-algebras. Denote by $\mathbb{D}_{X,\mathbb{F}}$ the restriction of $\mathbb{D}_X$ on the subcategory of $\mathcal{C}_{\mathcal{O},k}$, whose objects are $\mathbb{F}$-algebras. By Theorem 6.7, $\mathbb{D}_{X,\mathbb{F}}$ is representable and there exists a π-divisible $\mathcal{O}$-module $\mathcal{X} \to \mathrm{Spf}(k[[t_1, \cdots, t_{dc}]])$ which is universal for the functor $\mathbb{D}_{X,\mathbb{F}}$. According to [5, Section 2.2.4], we may replace Spf by Spec. We have the following result regarding the Newton polygon of the generic fiber of the universal π-divisible $\mathcal{O}$-module $\mathcal{X}$. This corresponds to [6, Lemma 5.15] and our proof here is adapted from [6].

Proposition 6.8. *With the notation as in Theorem 6.7, the generic fiber of $\mathcal{X}$ has Newton polygon $\mathbb{N}(\mathcal{X}_\eta)$ equal to the Newton polygon of $G_{0,1}^d \times G_{1,1}^c$, i.e., the corresponding Dieudonné $\mathcal{O}$-module has d times slope 1 and c times slope 0.*

Proof. First, by Grothendieck's specialization theorem, the Newton polygon goes down after specialization. Hence it suffices to find some point of $\mathrm{Spec}\, k[[t_1, \ldots, t_{dc}]]$ where $\mathcal{X}$ has the desired Newton polygon. Secondly, write $X = X^m \oplus X' \oplus X^{\mathrm{et}}$, where the three direct summands are multiplicative, local-local, etale respectively. Deforming X while keeping X^m and X^{et} constant and keeping the direct summand decomposition, we obtain a closed subscheme of $\mathrm{Spec}\, k[[t_1, \ldots, t_{dc}]]$. Hence it suffices to prove the proposition for X'. In other words, we may assume that X is local-local.

Let $\mathcal{P}$ be the nilpotent $\mathcal{O}$-display with $\mathrm{BT}_{\mathcal{O}}(\mathcal{P}) = X$. Assume that the structure equation of $\mathcal{P}$ $\Phi := F \oplus F_1 : T \oplus L \to P$ is given by the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. By Remark 6.5, the structure equation of $\mathcal{X}$ is given by

$$\begin{pmatrix} \mathrm{id} & T \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Consider the map $A_0 + \bar{T}C_0 := A + TC \pmod{\pi}$. Suppose it has determinant zero in $k[[t_{ij}]]$, then the intersection $\mathrm{Ker}\, A_0 \cap \mathrm{Ker}\, C_0$ is nontrivial. This contradicts to the fact that $\Phi \pmod{\pi}$ is invertible. Hence $A_0 + \bar{T}C_0$ is invertible. Therefore $\mathcal{X}$ has at least d times slope 0. By duality, $\mathcal{X}$ has exactly d times slope 0 and c times slope 1. The proposition follows. $\square$

6.1.6. Catalogues of minimal π-divisible $\mathcal{O}$-modules. As explained in [6, Section 5], catalogues have many advantages in the study of p-divisible groups. In this section, we study catalogues of simple minimal π-divisible $\mathcal{O}$-modules and explain some of their important properties.

In the following, we fix the integers m and n with $(m, n) = 1$ and write $H = H_{m,m+n}$, where $H_{m,m+n}$ is defined in Section 4.1. Let $r = \frac{1}{2}(m-1)(n-1)$. Consider the contravariant functor

$$\begin{aligned} &\mathrm{Sch}_{\mathbb{F}} \to \mathrm{Sets} \\ &\quad S \mapsto \{(\varphi, X) \mid \varphi : H_S \to X,\ \deg \varphi = q^r\}/\sim . \end{aligned} \tag{6.1.14}$$

In other words, this functor associates to S the set of isomorphisms classes of isogenies $\varphi : H_S \to X$ of degree q^r, where $H_S = H \times_{\mathbb{F}} S$ is the base change of H and X is a π-divisible $\mathcal{O}$-module over S. Each $\varphi : H_S \to X$ corresponds to a finite locally free closed sub $\mathcal{O}$-group scheme of H_S of rank q^r. This functor is representable by a scheme $\mathbb{T} = \mathbb{T}_{m,n} \to \mathrm{Spec}\, \mathbb{F}$ projective over $\mathrm{Spec}\, \mathbb{F}$ (cf. Lemma 3.49). Let $(\mathcal{G}, \Phi : H_{\mathbb{T}} \to \mathcal{G})$ be the universal object.

Lemma 6.9. *The family $\mathcal{G} \to \mathbb{T}$ of π-divisible $\mathcal{O}$-modules over $\mathbb{T}$ is a catalogue for π-divisible $\mathcal{O}$-modules isogenous to $G_{m,m+n}$: if X is a π-divisible $\mathcal{O}$-module over an* *algebraically closed field* $k \in \mathrm{Alg}_{\mathcal{O}}$ *of characteristic* p *and* X *is isogenous to* $G_{m,m+n}$*, then there exists a point* $t : \operatorname{Spec} k \to \mathbb{T}$ *with* $X \cong \mathcal{G}_t$.

Proof. This is similar to [6, Proposition 5.10] and it follows from the theory of semi-modules (cf. [6, Sections 5.6-5.8, Section 6]). Let X be isogenous to $G_{m,m+n}$. Then there exists an isogeny $\psi : X \to H_{m,m+n}$, which induces an inclusion $M(X) \subset M(H_{m,m+n})$. The standard filtration of $M(H_{m,m+n})$ (cf. Lemma 4.4) induces a filtration on $M(X)$. Define $A_\psi \subset \mathbb{Z}$ to be the set of jumps of this induced filtration, i.e.

$$A_\psi = \{a \in \mathbb{Z} \mid M(X) \cap N^a \neq M(X) \cap N^{a+1}\}.$$

Then

- $A_\psi \subset \mathbb{Z}$ is determined up to translation. This follows from the existence of the special $\Pi \in \mathrm{End}(M(H_{m,m+n}))$.
- A_ψ is a semi-module (cf. [6, Section 6.1]), i.e., $a \in A_\psi \Rightarrow a+n,\ a+m \in A_\psi$. This follows from the action of F and V on $M(H_{m,m+n})$.

By [6, Lemma 6.6], there is a unique admissible semi-module A such that A is equivalent to A_ψ. This tells us (cf. [6, Section 6.4]) that there exists a relation

$$\Pi^{2r} M(H_{m,m+n}) \subset M(X) \subset M(H_{m,m+n}),$$

where $r = \frac{1}{2}(m-1)(n-1)$. From the construction of the filtration (cf. Lemma 4.4), each term $\Pi^j M(H_{m,m+n})$ is isomorphic to $M(H_{m,m+n})$ as Dieudonné $\mathcal{O}$-modules. Note that Π is of degree q, it follows that there exist isogenies $H_{m,m+n} \to X$ and $X \to H_{m,m+n}$ of degree q^r. The lemma then follows. □

Theorem 6.10. *The scheme* $\mathbb{T}$ *is geometrically irreducible of dimension* r *over* $\mathbb{F}$.

Proof. In the case of p-divisible groups, this is proved in [6, Section 5]. With the preparations above, the proof there which uses semi-modules as an important tool works for π-divisible $\mathcal{O}$-modules as well (cf. [6, Remark 5.27]). We omit the details here. □

Another ingredient in the proof of Theorem 6.10 is a purity result for Newton polygons. As it also has applications that are needed in Section 6.2, we recall it here.

Theorem 6.11. *Let* R *be a Noetherian local* $\mathcal{O}$*-algebra of Krull dimension* ≥ 2 *with* $\pi \cdot R = 0$. *Let* $\mathfrak{m}$ *be the maximal ideal of* R *and* $U = \operatorname{Spec} R \backslash \{\mathfrak{m}\}$. *Let* X *be a* π*-divisible* $\mathcal{O}$*-module over* $\operatorname{Spec} R$. *If* X *has constant Newton polygon over* U*, then* X *has constant Newton polygon over* $\operatorname{Spec} R$.

Note that the Newton polygon of a π-divisible $\mathcal{O}$-module is determined by the Newton polygon of the underlying p-divisible group (i.e. forgetting the $\mathcal{O}$-action), hence Theorem 6.11 is an immediate consequence of the purity result of p-divisible groups [6, Theorem 4.1]. The author thank one of the referees for this argument.

Combining Theorem 6.10 and Theorem 6.11, we have the following result, which will be needed and improved in next section (cf. [6, Corollary 5.11]).

Corollary 6.12. *Let* $k \in \mathrm{Alg}_{\mathcal{O}}$ *be an algebraically closed field of characteristic* p. *Let* X *be a* π*-divisible* $\mathcal{O}$*-module over* k *that is isogenous to* $G_{m,m+n}$. *Then there exists a deformation* $\mathcal{X}/k[[t]]$ *of* X *over* $k[[t]]$ *with constant Newton polygon such that* $a(\mathcal{X}_\eta) = 1$.

Proof. Let $t \in \mathbb{T}(k)$ be a point such that $\mathcal{G}_t \cong X$. The irreducibility of $\mathbb{T}_k$ tells us that the $(a = 1)$-locus of the family $\mathcal{G}$ is an open dense sub-scheme U of $\mathbb{T}_k$. Therefore, we could construct a morphism $\phi : \operatorname{Spec} k[[t]] \to \mathbb{T}_k$ over k, such that $\phi(\eta) \in U$ and $\phi(\operatorname{Spec} k) = t$. Then $\mathcal{X} = \phi^*\mathcal{G}$ is a π-divisible $\mathcal{O}$-module over $k[[t]]$ with the desired property. $\square$

6.2. Oort's condition. In this section, following the idea in [29], we prove Theorem 5.20. As a consequence, Traverso's isogeny conjecture for π-divisible $\mathcal{O}$-modules holds.

Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a field of characteristic p. Let G_0 be a π-divisible $\mathcal{O}$-module over k. Without loss of generality, we assume that G_0 is of local-local type. We give a slightly different version of deformations. We say that $\mathcal{G}_\eta$ is a *deformation* of G_0, if there exists a complete local domain B of characteristic p with residue field k and quotient field $Q(B)$ and a π-divisible $\mathcal{O}$-module $\mathcal{G}$ over B, such that

$$\mathcal{G} \otimes_B k \cong G_0, \quad \mathcal{G} \otimes_B Q(B) = \mathcal{G}_\eta.$$

In the following, we study deformations of a π-divisible $\mathcal{O}$-modules, which preserve a filtration constructed from the Newton polygon.

Definition 6.13. Let G_0 be a π-divisible $\mathcal{O}$-module over an algebraically closed field $k \in \mathrm{Alg}_{\mathcal{O}}$. Suppose that the Newton polygon $\beta = \mathbb{N}(G_0)$ has $m + 1$ points with integral coordinates. Then there exists simple groups Z_i $(1 \leq i \leq m)$ corresponding with the slopes between integral points in β and an isogeny $\sum_{i=1}^m Z_i \to G_0$. Define

$$G_0^j := \operatorname{Im}(\sum_{i \leq j} Z_i \to G_0) \subset G_0.$$

Then we obtain a filtration

$$0 = G_0^0 \subset \cdots \subset G_0^i \subset \cdots \subset G_0^m = G_0,$$

which is called a *maximal filtration* of G_0. Note that the successive quotients G_0^i/G_0^{i-1} $(0 < i \leq m)$ are simple π-divisible $\mathcal{O}$-modules.

In the following, we study deformations of G_0 with respect to a fixed maximal filtration. Let $\{G_0^i \mid 0 \leq i \leq m\}$ be a filtered π-divisible $\mathcal{O}$-module. We denote by $(d_i + c_i, d_i)$ the type of G_0^i and a_i the a-number $a(G_0^i/G_0^{i-1})$ $(1 \leq i \leq m)$. Write $d = \sum_{i=1}^m d_i$ and $c = \sum_{i=1}^m c_i$.

In terms of Dieudonné $\mathcal{O}$-modules, we introduce the notion *bases adapted to a filtration*. Suppose given a maximal filtration of G_0 over k. Let $M_0 = \mathbb{D}(G_0)$ be the covariant Dieudonné $\mathcal{O}$-module. The maximal filtration of G_0 induces a filtration

$$0 = M_0^0 \subset \cdots \subset M_0^i \subset \cdots \subset M_0^m = M_0,$$

where $M_0^i = \mathbb{D}(G_0^i)$ $(0 \leq i \leq m)$. We say that $\{x_1, \ldots, x_d; y_1, \ldots, y_c\}$ is a *basis adapted to the filtration* if it is a $W_{\mathcal{O}}(k)$-basis for M_0 with $y_j \in VM_0$ and

- $x_j \in M_0^i$ if and only if $j \leq d_i$,
- $y_j \in M_0^i$ if and only if $j \leq c_i$.

If $H_0 \subset G_0$ is a sub π-divisible $\mathcal{O}$-module, then the sub Dieudonné $\mathcal{O}$-module $\mathbb{D}(H_0) \subset \mathbb{D}(G_0)$ is a direct summand, as the quotient is torsion free. Hence a basis adapted to the filtration always exists by induction.

In terms of $\mathcal{O}$-displays, a basis adapted to the filtration corresponds to a special form of the structure equation. Let $\mathcal{P}_i = (P_i, Q_i, F, F_1)$ be the $\mathcal{O}$-display associated with G_0^i/G_0^{i-1}. Let $P_i = L_i \oplus T_i$ be a normal decomposition of P_i. Hence $Q_i = L_i \oplus IT_i$ where $I = {}^V W_{\mathcal{O}}(k)$. Let

$$\Phi_i := F \oplus F_1 : T_i \oplus L_i \to T_i \oplus L_i$$

be the structure equation of $\mathcal{P}_i$. It is given by a matrix $\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ with $A_i : T_i \xrightarrow{F} T_i$, $B_i : L_i \xrightarrow{F_1} T_i$, $C_i : T_i \xrightarrow{F} L_i$, $D_i : L_i \xrightarrow{F_1} L_i$. Let $\mathcal{P} = (P, Q, F, F_1)$ be the $\mathcal{O}$-display associated with M_0 with the basis adapted to the filtration $\{x_1, \dots, x_d; y_1, \dots, y_c\}$. Let $L = \langle y_1, \dots, y_c \rangle$ and $T = \langle x_1, \dots, x_d \rangle$. Then $P = L \oplus T$ is a normal decomposition. The structure equation of $\mathcal{P}$ is given by

$$\Phi := F \oplus F_1 : T \oplus L \to T \oplus L.$$

In terms of matrix form, $\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Here A is a block upper triangle matrix with diagonal blocks $A_1, \dots A_m$ and similar for B, C and D. We prove a stronger version of Corollary 6.12.

Proposition 6.14. *There exists a deformation $\{\mathcal{G}^i \mid 0 \le i \le m\}$ of filtered π-divisible $\mathcal{O}$-module $\{G_0^i \mid 0 \le i \le m\}$ such that every sub-quotient $\mathcal{Y}^i := \mathcal{G}^i/\mathcal{G}^{i-1}$ is absolutely simple of constant slope and with $a(\mathcal{Y}^i_\eta) \le 1$ $(1 \le i \le m)$.*

Proof. Since deforming the filtered π-divisible $\mathcal{O}$-module is equivalent to deforming the corresponding $\mathcal{O}$-display, we may apply Remark 6.5. By Corollary 6.12, for each G_0^i/G_0^{i-1}, there exists a deformation with constant slope and a-number ≤ 1. This deformation corresponds to a matrix $\begin{pmatrix} \mathrm{id} & \tau_i \\ O & \mathrm{id} \end{pmatrix}$. Consider the deformation $\tilde{\mathcal{P}}$ of $\mathcal{P}$ with structure equation

$$\begin{pmatrix}
A_1 + \tau_1 C_1 & * & \cdots & * & B_1 + \tau_1 D_1 & * & \cdots & * \\
O & A_2 + \tau_2 C_2 & \cdots & * & O & B_2 + \tau_2 D_2 & \cdots & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
O & O & O & A_m + \tau_m C_m & O & O & O & B_m + \tau_m D_m \\
C_1 & * & \cdots & * & D_1 & * & \cdots & * \\
O & C_2 & \cdots & * & O & D_2 & \cdots & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
O & O & O & C_m & O & O & O & D_m
\end{pmatrix}.$$

It is easy to check that $\mathrm{BT}_{\mathcal{O}}(\tilde{\mathcal{P}})$ is a deformation of the filtered π-divisible $\mathcal{O}$-module with the desired property. $\square$

Let G be a π-divisible $\mathcal{O}$-module of type $(d+c, d)$ and with $a(G) = 1$. Let $x \in M$ be such that $W_{\mathcal{O}}(k)[F, V] \cdot x = M$. By Lemma 5.15, $\{x, Fx, \cdots, F^d x, Vx, \cdots, V^{c-1} x\}$ is a basis of M. In particular, M has a basis $\{x_1, \cdots, x_d; y_1, \cdots y_c\}$, such that $F(x_i) = x_{i+1}$ for $i < d$ and $F(x_d) = y_1$. Such a basis is called *a basis in normal form* (cf. [28, Section 2]). It is easy to check that M admits a basis in normal form if and only if $a(M) = 1$. With respect to this basis, the $\begin{pmatrix} A \\ C \end{pmatrix}$-part of the structure equation of the corresponding $\mathcal{O}$-display has the following simple form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We say a basis adapted to the filtration is in *normal form* if for each i, the basis $\{x_{d_i+1}, \cdots, x_{d_{i+1}}; y_{c_i+1}, \cdots y_{c_{i+1}}\}$ is a basis in normal form for G_0^i/G_0^{i-1}. By induction on the length of the filtration, it is easy to see that G_0 admits a basis in normal form adapted to the filtration for any filtered π-divisible $\mathcal{O}$-module G_0 with $a_i = 1$

PROPOSITION 6.15. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be a field of characteristic p. Let $\{H_0^i \mid 0 \leq i \leq m\}$ be a filtered local-local π-divisible $\mathcal{O}$-module over k with $a(H_0^i/H_0^{i-1}) = 1$. Then there exists a deformation $\{\mathcal{H}^i \mid 0 \leq i \leq m\}$ of $\{H_0^i \mid 0 \leq i \leq m\}$, such that $\mathcal{H}^i/\mathcal{H}^{i-1}$ and H_0^i/H_0^{i-1} have the same slope, and such that $a(\mathcal{H}_\eta^m) = 1$.*

Proof. Without loss of generality, we may assume that k is algebraically closed. The proof is similar to the proof of Proposition 6.14. In order to find the desired deformation of the filtered π-divisible $\mathcal{O}$-module, we only need to find an appropriate deformation of the corresponding $\mathcal{O}$-display. The filtered π-divisible $\mathcal{O}$-module H_0 admits a basis of normal form adapted to the filtration and the corresponding $\mathcal{O}$-display has a structure equation $\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Here A is a block upper triangle matrix with diagonal blocks $A_{11}, \ldots, A_{mm}$ and similar for B, C and D. Moreover, $\begin{pmatrix} A_{ii} \\ C_{ii} \end{pmatrix}$ has a special form as above. Now, we deform $\mathcal{P}$ by deforming the structure equation with respect to

$$T = (T_{ij}) := \begin{pmatrix} 0 & T_{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & T_{23} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & T_{m-1,m} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Each $T_{i,i+1}$ is a matrix with all entries equal to 0 except the $(1,1)$-entry, which is the Teichmüller lift $[s_i]$ of indeterminant s_i. One then checks easily that $\mathrm{Rank}(A+TC) = d-1$ as the determinant of the sub-matrix of $(A+TC)$ without the d_1-column and (c_m+1)-row (with all entries 0) is $\prod_{i=1}^{s-1}(*+[s_i])$ with $* \in W_{\mathcal{O}}(K)$, which is non-zero. Note that $Q + F(P) = Q + F(L) + F(T) = Q + F(T)$ as $F(L) \subset Q$, the deformed $\mathcal{O}$-display has a-number 1. Furthermore, this deformation does not change the successive quotients as the diagonal blocks remain the same. Thus the corresponding π-divisible $\mathcal{O}$-module satisfies the desired property. □

Now we prove that every π-divisible $\mathcal{O}$-module over algebraically closed k satisfies OC. We have the following result.

THEOREM 6.16. *Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p and G_0 be a π-divisible $\mathcal{O}$-module over k. There is a deformation $\mathcal{G}$ of G_0 with $a(\mathcal{G}_\eta) \leq 1$ such that $\mathcal{G}_\eta$ and G_0 have the same Newton polygon.*

Proof. The proof is a combination of Proposition 6.14 and Proposition 6.15. Starting with G_0, by Proposition 6.14, there is a deformation to a filtered group H with successive quotients each having a-number 1. Without loss of generality, by taking an irreducible component of the base if necessary, we may assume that H is over a complete local domain B_1 of characteristic p with residue field k. Applying Proposition 6.15 to the filtered $\mathcal{O}$-module $H_\eta = H \otimes Q(B_1)$, there exists a complete local domain B_2 and a deformation $\mathcal{G}$ over B_2 of H_η such that $\mathcal{G}$ has the same Newton polygon of H_η and $a(\mathcal{G}_\eta) = 1$. From the construction in the proof of Proposition 6.15, the $\mathcal{O}$-display defining $\mathcal{G}$ over B_2 is defined over $B_3 = B_1 \times_{Q(B_1)} B_2$. Hence $\mathcal{G}$ is defined over B_3 and it satisfies the desired properties. □

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