ON THE DENSITY FOR SEVERI-BRAUER CONICS AND PERIOD-INDEX DISTRIBUTIONS FOR ELLIPTIC CURVES

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Abstract. Let K be a number field with class number 1. In this paper, we study the family of Severi-Brauer conics over K defined by equations $z^2 - ax^2 - by^2 = 0$, where $(a, b) \in K^*/(K^*)^2 \times K^*/(K^*)^2$. We prove that, within this family, the density of the conics admitting at least one nontrivial K-rational point is zero. As an application, let E/K be an elliptic curve satisfying $E(K) \supset E[4]$, we show that the subset of $H^1(K, E)[2]$ consisting of classes with equal period and index has density zero. This is closely related to the question proposed in [3, Problem 2].

1. Introduction

Let K be a number field with class number 1. In this paper, we prove that within the family of Severi–Brauer conics parameterized by $K^*/(K^*)^2 \times K^*/(K^*)^2$, the density (cf. Definition 1.1) of the conics admitting at least one nontrivial K-rational point is zero. Our motivation originates from a period-index distribution problem for elliptic curves (cf. [3, Problem 2]). Through O'Neil's obstruction map (cf. Equation (3.3)), which coincides with the Hilbert symbol in our setting, we establish a connection between the period-index distribution problem and the density problem for Severi–Brauer conics. Furthermore, we clarify part of [3, Problem 2] and give an answer for the case that $P = 2$ for number fields with class number 1.

We fix some notation and explain our main results in the following. Let K be a number field. For each pair $(a, b) \in K^* \times K^*$, the associated *Severi–Brauer conic* is defined by the homogeneous equation $z^2 - ax^2 - by^2 = 0$. We study the density of such conics that have at least one nontrivial K-rational point. In the case that K has class nubmer 1, for each prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, we can fix a generator $\pi \in \mathfrak{p}$ and call π a *prime* of $\mathcal{O}_K$. We say that π is an *odd* prime if $2 \notin (\pi)$. Every $x \in K^*$ admits a unique factorization of the form

$$x = u\pi_1^{\alpha_1}\pi_2^{\alpha_2}\cdots\pi_n^{\alpha_n}, \tag{1.1}$$

where $\alpha_i \in \mathbb{Z}$, $u \in \mathcal{O}_K^*$ and π_i are distinct primes of K. Let $U = \{1,\ u_2, \cdots,\ u_r\} \subseteq \mathcal{O}_K^*$ be a system of representatives of the quotient group $\mathcal{O}_K^*/{\mathcal{O}_K^*}^2$, where 1 is the representatives of the identity element. For any $[a] \in K^*/{K^*}^2$, we can choose a representative $a \in K^*$ of the form $a = u_a\pi_1\pi_2\cdots\pi_n$, where $u_a \in U$ and π_i are distinct primes of K. This construction gives a one-to-one correspondence ϕ between $K^*/{K^*}^2$ and the set of "square-free number":

$$\Sigma_K := \{a \in K^* \mid a = u\prod_{i=1}^{n}\pi_i,\ u \in U,\ \pi_i \text{ are distinct primes}\}.$$

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In the following, for any $[a] \in K^*/K^{*2}$, the image of $[a]$ under ϕ is denoted by a.

Definition 1.1. Let K be a number field with class number 1. For $X \in \mathbb{R}_{>0}$, define

$$\Sigma_K^X := \{a \in \Sigma_K \mid -X \leq N(a) \leq X\},$$

and

$$\Sigma_K^{H,X} := \left\{(a,b) \in \Sigma_K^X \times \Sigma_K^X \mid (a,b)_H = 1\right\},$$

where $(a,b)_H$ is the Hilbert symbol (cf. Definition 2.1) and N is the field norm $N_{K/\mathbb{Q}}$. The *rational density* of Severi–Brauer conics over K is defined by

$$\delta_{H,K} := \limsup_{X \to +\infty} \frac{|\Sigma_K^{H,X}|}{|\Sigma_K^X|^2}.$$

Our main result is the following theorem.

Theorem 1.2. *Let K be a number field with class number 1. The rational density of Severi–Brauer conics over K is zero.*

As an immediate application to the period-index problem discussed in Section 3, we obtain the following result. See Theorem 3.4 for the precise meaning of density.

Theorem 1.3. *Let K be a number field with class number 1, and let E/K be an elliptic curve with $E(K) \supset E[4]$. Then in $H^1(K,E)[2]$, the density of the subset of classes with coinciding period and index is zero.*

In Section 2, we review basic notions of the Hilbert symbol and prove Theorem 1.2 through analytic number theory techniques.

In Section 3, we review basic notions of period and index of homogeneous spaces for elliptic curves and prove Theorem 3.4, by relating the rational density (cf. Definition 1.1) and the period-index density (cf. Definition 3.3) via O'Neil's obstruction map (cf. Equation (3.3)).

Notation Let $f(x)$ and $g(x)$ be functions defined on $\mathbb{Z}_{>0}$. We employ the following asymptotic notation:

- $f(x) = O(g(x))$ if $\exists\, C > 0$ such that $|f(x)| \leq C|g(x)|$;
- $f(x) = o(g(x))$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$;
- $f(x) \sim g(x)$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Let K be a number field:

- $\mathcal{O}_K$ denotes the ring of integers of K;
- For $a \in \mathcal{O}_K$ (when K with class number 1), $\omega(a)$ counts distinct prime divisors of a;
- For ideals $\mathfrak{a} \subseteq \mathcal{O}_K$, $\omega(\mathfrak{a})$ counts distinct prime ideal divisors of $\mathfrak{a}$;

- $N(\alpha)$ denotes the field norm $N_{K/\mathbb{Q}}(\alpha)$ for $\alpha \in K^*$, and $N(\mathfrak{a})$ denotes the ideal norm $N_{K/\mathbb{Q}}(\mathfrak{a})$ for ideals $\mathfrak{a} \subseteq \mathcal{O}_K$;
- Let K be of degree $n = r_1 + 2r_2$, where r_1 denotes the number of real embeddings and r_2 the number of conjugate pairs of complex embeddings. For $\alpha \in K$, let $\alpha^{(i)}$ $(1 \leq i \leq r_1)$ denote its real embeddings, and $\alpha^{(j)}$ $(r_1 + 1 \leq j \leq r_1 + r_2)$ denote a fixed choice of non-conjugate complex embedings;
- For every $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, $\|v\| := \sqrt{(v_1)^2 + \cdots + (v_n)^2}$.

2. The Hilbert symbol and the rational density

In this section, we provide necessary background on the Hilbert symbol (cf. [14, Chapter 14]) connecting the period-index problem in Section 3 with Theorem 1.2. We then proceed to prove Theorem 1.2, utilizing the computability of Hilbert symbols alongside tools from analytic number theory.

Let K be a field of character 0 that contains an n-th primitive root of unity, where $n \in \mathbb{Z}_{>0}$. It is well known that $K^*/K^{*n} \cong H^1(K, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(K, \mathbb{Z}/n\mathbb{Z})$. Thus for any $a \in K^*$, we can associate a character $\chi_a \in H^1(K, \mathbb{Z}/n\mathbb{Z})$.

Definition 2.1. Let K be field of character 0 and a, b in K^*. Denote by $\mathrm{Br}(K)$ the Brauer group of K and by $\mathrm{Br}(K)[n]$ its n-torsion subgroup. We define the symbol $(a, b)_{H_n,K} \in \mathrm{Br}(K)[n]$ to be the cup-product $b \cup \ \delta(\chi_a)$ of $b \in H^0(K, \bar{K}^*) = K^*$ with $\chi_a \in H^1(K, \mathbb{Q}/\mathbb{Z}) \xrightarrow[\delta]{\cong} H^2(K, \mathbb{Z})$.

The symbol is a bilinear map from $K^* \times K^*$ to $\mathrm{Br}(K)[n]$. In fact, when K contains an n-th primitive root of unity, it induces a bilinear map $(\ ,\ )_{H_n,K}$ from $K^*/K^{*n} \times K^*/K^{*n}$ to $\mathrm{Br}(K)[n]$. We call it the *Hilbert symbol* over K. When $n = 2$, we denote $(\ ,\ )_{H_2,K}$ by $(\ ,\ )_{H,K}$ and a necessary and sufficient condition for the Hilbert symbol to be trivial is given by the following proposition (cf. [14, Ch. 14, Sec. 2, Prop. 4] or [15, Chapter 3]).

Proposition 2.2. *Let K be a field of character 0. Let $a, b \in K^*$. Then $(a, b)_{H,K} = 1$ if and only if b is a norm of the extension $K(a^{1/2})/K$. In other words, $(a, b)_{H,K} = 1$ if and only if the conic $z^2 - ax^2 - by^2 = 0$ has a nontrivial K-rational point.*

Let $\pi \in \mathcal{O}_K$ be a prime of $\mathcal{O}_K$. For any $a \in \mathcal{O}_K$, define $\left(\frac{a}{\pi}\right)_2$ to be the quadratic residue symbol of a mod (π), i.e.,

$$\left(\frac{a}{\pi}\right)_2 = \begin{cases} 1 & \text{if } a \mod (\pi) \text{ is a square in } \kappa_\pi^*, \\ -1 & \text{if } a \mod (\pi) \text{ is not a square in } \kappa_\pi^*, \\ 0 & \text{if } a \in (\pi). \end{cases}$$

where κ_π is the residue field $\mathcal{O}_K/(\pi)$. For any ideal $I = (\pi_1 \cdots \pi_n) \subseteq \mathcal{O}_K$, denote by $\left(\frac{a}{\pi_1 \cdots \pi_n}\right)_2$ the product $\left(\frac{a}{\pi_1}\right)_2 \cdots \left(\frac{a}{\pi_n}\right)_2$ for any $a \in \mathcal{O}_K$. It is a character of $(\mathcal{O}_K/I)^*$.

Remark 2.3. When $b=-a$, the conic $z^2-ax^2+ay^2=0$ admits the solution $(x,y,z)=(1,1,0)$. Hence there exist infinitely many pairs $(a,b)\in K^*\times K^*$ such that $(a,b)_{H,K}=1$. Thus Theorem 1.2 is not trivial.

Theorem 2.4. *Let $(a,b)\in\Sigma_K$ such that $a=u_1\pi_1\dots\pi_n\cdot\varpi_1\dots\varpi_l$ and $b=u_2\pi_1'\dots\pi_m'\cdot\varpi_1\dots\varpi_l$, with π_i,π_j',ϖ_k distinct primes. Assume that π is an odd prime among π_i for $1\le i\le n$, then $(a,b)_{H,K_\mathfrak{p}}$ is trivial if and only if $(\frac{b}{\pi})_2$ is trivial, where $K_\mathfrak{p}$ denote the completion of K at $\mathfrak{p}:=(\pi)$. The symbol $(a,b)_{H,K_\mathfrak{p}}$ is abbreviated as $(a,b)_\mathfrak{p}$ in the following.*

Proof. Let π be an odd prime with $(\pi)=\mathfrak{p}$, and suppose $\pi\mid a$ but $\pi\nmid b$. Let $a'=a\pi^{-1}$. We claim that $(a',b)_\mathfrak{p}=1$. Assuming this, it follows that $(a,b)_\mathfrak{p}=(a',b)_\mathfrak{p}(\pi,b)_\mathfrak{p}=(\pi,b)_\mathfrak{p}=\left(\frac{b}{\pi}\right)_2$, which completes the proof. To prove the claim, it suffice to show that $a'x^2+by^2=z^2$ has a nontrivial solution in $K_\mathfrak{p}$ by Proposition 2.2. Fix a unit $c\in\mathcal{O}_{K_\mathfrak{p}}^*$. As y^2 ranges over squares in $\kappa_\mathfrak{p}=\mathbb{F}_q$ ($q=|\kappa_\mathfrak{p}|$), the values $c^2-by^2 \mod \mathfrak{p}$ take $\frac{q+1}{2}$ distinct residues. Similarly for $a'x^2$. Thus there exist $x_0,y_0\in\mathcal{O}_{K_\mathfrak{p}}$ such that

$$a'x_0^2+by_0^2\equiv c^2 \pmod{\mathfrak{p}}.$$

The multi-variable version of Hensel's lemma (cf. [17, Ch. 4, Ex. 4.27]) lifts this to a solution $(x_1,y_1,z_1)\neq(0,0,0)$ of $a'x^2+by^2=z^2$ in $K_\mathfrak{p}$. Thus $(a',b)_\mathfrak{p}=1$, as claimed. □

By Theorem 2.4, we can provide a necessary condition for the vanishing of the Hilbert symbol.

Lemma 2.5. *Let $([a],[b])\in K^*/(K^*)^2\times K^*/(K^*)^2$. If $([a],[b])_H=1$, then the product $\prod_{\substack{\pi\mid a,\pi\nmid b\\ \pi\ odd\ prime}}\frac{(1+\left(\frac{b}{\pi}\right))}{2}$ is equal to 1.*

We now turn to the proof of our main theorem on the rational density of Severi-Brauer conics over K. The proof relies on the following preparatory lemmas, which provide the necessary analytic tools for Theorem 2.13. Let K be a number field. We say a function f defined on all ideals of K is a multiplicative arithmetic function if for any two coprime ideals $\mathfrak{a},\mathfrak{b}$, we have $f(\mathfrak{ab})=f(\mathfrak{a})f(\mathfrak{b})$. For any two ideals $\mathfrak{a},\mathfrak{b}$, $\gcd(\mathfrak{a},\mathfrak{b})$ denotes the greatest common divisor of $\mathfrak{a},\mathfrak{b}$, i.e. $\gcd(\mathfrak{a},\mathfrak{b})=\mathfrak{a}+\mathfrak{b}$. In particular, if the class number of K is 1, for any elements $a,b\in\mathcal{O}_K$, $\gcd(a,b)$ also denotes the greatest common divisor of the elements.

Lemma 2.6. *Let $\lambda_1,\ \lambda_2$ be constants, such that $\lambda_1>0$, $0\le\lambda_2<2$. For any multiplicative arithmetic function f satisfying*

$$0\le f(\mathfrak{p}^v)\le\lambda_1\lambda_2^{v-1}\quad(N(\mathfrak{p})\ge2, v=1,2,...), \tag{2.1}$$

we have

$$\sum_{\substack{N(\mathfrak{a})\le X\\ \mathfrak{a}\subseteq\mathcal{O}_K}}f(\mathfrak{a})=O(X\cdot\prod_{\substack{N(\mathfrak{p})\le X\\ \mathfrak{p}\ prime\ ideal}}(1-N(\mathfrak{p})^{-1})\sum_{v=0}^{\infty}f(\mathfrak{p}^v)(N(\mathfrak{p}))^{-v})\ (X\ge2). \tag{2.2}$$

Proof. By adapting the method of [18, Part III, Ch. 3, Sec. 3, Cor. 3.6] to number fields (with primes replaced by prime ideals and p^{-s} systematically substituted by $\mathrm{N}(\mathfrak{p})^{-s}$), we establish the corresponding number field analogue. This completes the proof. □

Lemma 2.7. *Let K be a number field.*

(1) *For any $y \in \mathbb{R}_{>0}$,* $\sum_{\substack{1\le N(\mathfrak{a})\le X\\ \mathfrak{a}\subseteq\mathcal{O}_K}} (y)^{\omega(\mathfrak{a})} = O(X(\log X)^{y-1})$,

(2) $\sum_{\substack{1\le N(\mathfrak{a})\le X\\ \mathfrak{a}\subseteq\mathcal{O}_K}} \sum_{\substack{1\le N(\mathfrak{b})\le X\\ \mathfrak{b}\subseteq\mathcal{O}_K}} (\frac{1}{2})^{\omega(\frac{\mathfrak{a}}{\gcd(\mathfrak{a},\mathfrak{b})})} = O(\frac{X^2}{(\log X)^{\frac{1}{6}}})$.

Proof. **(1).** We adapt the strategy of [18, Part III, Ch. 3, Sec. 3, Thm. 6]. Note that the function $\mathfrak{a} \to (y)^{\omega(\mathfrak{a})}$ is a multiplicative arithmetic function , and satisfies the condition (2.1) for $\lambda_1 = 1+y,\ \lambda_2 = 1$. Thus by Equation (2.2),

$$\sum_{1\le N(\mathfrak{a})\le X} (y)^{\omega(\mathfrak{a})} = O(X \prod_{N(\mathfrak{p})\le X} (1-N(\mathfrak{p})^{-1})\{1+yN(\mathfrak{p})^{-1}+O(N(\mathfrak{p})^{-2})\}).$$

By the Mertens' formula (cf. [7, Theorem 1, M(3)]), the proof is complete.

(2). Note that for any fixed ideal $\mathfrak{b}$, the function $f_{\mathfrak{b}}(\mathfrak{a}) := (\frac{1}{2})^{\omega(\frac{\mathfrak{a}}{\gcd(\mathfrak{a},\mathfrak{b})})}$ is a multiplicative arithmetic function such that $0 \le f_{\mathfrak{b}}(\mathfrak{p}^v) \le 1\cdot 1^{v-1}$. Therefore

(2.3)

$$\sum_{1\le N(\mathfrak{a})\le X} \sum_{1\le N(\mathfrak{b})\le X} (\frac{1}{2})^{\omega(\frac{\mathfrak{a}}{\gcd(\mathfrak{a},\mathfrak{b})})}$$

$$\le X \sum_{1\le N(\mathfrak{b})\le X} \left(\prod_{\substack{N(\mathfrak{p})\le X\\ \mathfrak{p}\mid\mathfrak{b}}} (1-N(\mathfrak{p})^{-1}) \sum_{v=0}^{\infty} f_{\mathfrak{b}}(\mathfrak{p}^v)N(\mathfrak{p})^{-v}\right) \cdot \left(\prod_{\substack{N(\mathfrak{p})\le X\\ \mathfrak{p}\nmid\mathfrak{b}}} (1-N(\mathfrak{p})^{-1}) \sum_{v=0}^{\infty} (f_{\mathfrak{b}}(\mathfrak{p}^v))N(\mathfrak{p})^{-v}\right)$$

$$\le X \sum_{N(\mathfrak{b})\le X} \prod_{N(\mathfrak{p})\le X} (1-N(\mathfrak{p})^{-1})(1+\sum_{v=1}^{\infty}(\frac{1}{2})N(\mathfrak{p})^{-v}) \prod_{\mathfrak{p}\mid\mathfrak{b}} (\frac{\sum_{v=0}^{\infty} N(\mathfrak{p})^{-v}}{1+\sum_{v=1}^{\infty}(\frac{1}{2})N(\mathfrak{p})^{-v}})$$

$$\le X \prod_{N(\mathfrak{p})\le X} (1-N(\mathfrak{p})^{-1})(1+\sum_{v=1}^{\infty}(\frac{1}{2})N(\mathfrak{p})^{-v}) \sum_{N(\mathfrak{b})\le X} \prod_{\mathfrak{p}\mid\mathfrak{b}} (\frac{\sum_{v=0}^{\infty} N(\mathfrak{p})^{-v}}{1+\sum_{v=1}^{\infty}(\frac{1}{2})N(\mathfrak{p})^{-v}}).$$

Moreover,

$$\sum_{N(\mathfrak{b})\le X} \prod_{\mathfrak{p}\mid\mathfrak{b}} (\frac{\sum_{v=0}^{\infty} N(\mathfrak{p})^{-v}}{1+\sum_{v=1}^{\infty}(\frac{1}{2})N(\mathfrak{p})^{-v}})$$

$$\begin{aligned}
&\leq \sum_{N(\mathfrak{b})\leq X} \prod_{\mathfrak{p}|\mathfrak{b}} \left(\frac{N(\mathfrak{p})}{N(\mathfrak{p})-\frac{1}{2}}\right) \\
&\leq \sum_{N(\mathfrak{b})\leq X} \prod_{\mathfrak{p}|\mathfrak{b}} \left(1+\frac{\frac{1}{2}}{N(\mathfrak{p})-\frac{1}{2}}\right) \\
&\leq 2\cdot \sum_{N(\mathfrak{b})\leq X} \left(1+\frac{1}{3}\right)^{\omega(\mathfrak{b})}.
\end{aligned}$$

By Lemma 2.6, $\sum_{N(\mathfrak{b})\leq X} (1+\frac{1}{3})^{\omega(\mathfrak{b})} = O(X(\log X)^{\frac{1}{3}})$. By the Mertens' formula (cf. [7, Theorem 1, M(3)]), $X\cdot \prod_{N(\mathfrak{p})\leq X} (1-N(\mathfrak{p})^{-1})(1+\sum_{v=1}^{\infty}(\frac{1}{2})N(\mathfrak{p})^{-v}) = O(\frac{X}{(\log X)^{\frac{1}{2}}})$. Thus

$$\sum_{N(\mathfrak{a})\leq X}\sum_{N(\mathfrak{b})\leq X} \left(\frac{1}{2}\right)^{\omega\left(\frac{\mathfrak{a}}{\gcd(\mathfrak{a},\mathfrak{b})}\right)} = O\left(\frac{X}{(\log X)^{\frac{1}{2}}}\right)\cdot O(X(\log X)^{\frac{1}{3}}) = O\left(\frac{X^2}{(\log x)^{\frac{1}{6}}}\right),$$

which completes the proof. □

Let K be a number field with degree $n = r_1 + 2r_2$. From now on, we fix a basis $\{\epsilon_i\}_{i=1}^{r_1+r_2-1}$ of $\mathscr{O}_K^*/\mu_K$, where μ_K is the group of roots of unity in K. For a constant $C > 0$, let $\mathscr{R}_{K,C}$ be the set $\{\alpha \in \mathscr{O}_K \mid 0 < |\alpha^{(i)}| < C|N(\alpha)|^{\frac{1}{n}},\ 0 < |\alpha^{(j)}|^2 < C|N(\alpha)|^{\frac{2}{n}} \text{ for } i = 1, ..., r_1,\ j = r_1+1, ..., r_1+r_2\}$. In the following we shall demonstrate that K contains sufficiently many elements in the set $\mathscr{R}_{K,C}$ for some suitble constant C. This construction is designed to facilitate the application of Lemma 2.10 for analytic estimation of the density.

Lemma 2.8. *Let Λ be a lattice of $\mathbb{R}^n$ with basis $\{v_i\}_{i=1}^n$. For any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, define*

$$S_a = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_i - a_i| \leq \sum_{i=1}^{n} ||v_i|| \text{ for all } 1 \leq i \leq n \right\}.$$

Then $\Lambda \cap S_a \neq \emptyset$.

Proof. Since $\{v_i\}$ is an $\mathbb{R}$-basis, write $a = \sum_{i=1}^n b_i v_i$ with $b_i \in \mathbb{R}$. Choose integers m_i satisfying $|b_i - m_i| \leq \frac{1}{2}$. Then

$$\|a - \sum_{i=1}^{n} m_i v_i\| \leq \sum_{i=1}^{n} \frac{1}{2}\|v_i\|,$$

which implies $\sum_{i=1}^n m_i v_i \in S_a$ and this completes the proof. □

Let K be a number field of degree $n = r_1 + 2r_2$, define the map

$$l : \mathscr{O}_K^* \to \mathbb{R}^{r_1+r_2}$$

by

$$l(\eta) = \left(\log\left|\eta^{(1)}\right|, \ldots, \log\left|\eta^{(r_1)}\right|, 2\log\left|\eta^{(r_1+1)}\right|, \ldots, 2\log\left|\eta^{(r_1+r_2)}\right|\right).$$

Lemma 2.9. *Let $\{\epsilon_i\}_{i=1}^{r_1+r_2-1}$ be the fixed basis of $\mathcal{O}_K^*/\mu_K$. For any $\alpha \in \mathcal{O}_K$, there exists a unit $u \in \mathcal{O}_K^*$ such that $\alpha u \in \mathscr{R}_K$, where $\mathscr{R}_K := \mathscr{R}_{K,\mathrm{e}^{(n-1)\sum_{k=1}^{r_1+r_2-1}\|v_k\|}}$ and $\{v_i\}_{i=1}^{r_1+r_2-1}$ is the image of $\{\epsilon_i\}_{i=1}^{r_1+r_2-1}$ under l.*

Proof. Let $\alpha \in \mathcal{O}_K$. By Dirichlet's unit theorem, the image $l(\mathcal{O}_K^*)$ forms a full lattice in the hyperplane

$$H := \left\{ (a_1, \ldots, a_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2} \;\middle|\; \sum_{i=1}^{r_1+r_2} a_i = 0 \right\}.$$

Since $\{v_i\}_{i=1}^{r_1+r_2-1}$ is the image of $\{\epsilon_i\}_{i=1}^{r_1+r_2-1}$ under l, $\{v_i\}_{i=1}^{r_1+r_2-1}$ is a basis of $l(\mathcal{O}_K^*)$. For each $m \in \mathbb{Z}_{>0}$, let S_m be the set

$$S_m := \left\{ (b_1, \ldots, b_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2} \;\middle|\; \begin{array}{l} \left| b_i - \dfrac{\log|N(\alpha)|}{n} + \log|\alpha^{(i)}| \right| \\ \quad < m \displaystyle\sum_{k=1}^{r_1+r_2-1} \|v_k\|, \quad 1 \le i \le r_1 \\ \left| b_j - \dfrac{2\log|N(\alpha)|}{n} + 2\log|\alpha^{(j)}| \right| \\ \quad < m \displaystyle\sum_{k=1}^{r_1+r_2-1} \|v_k\|, \quad r_1+1 \le j \le r_1+r_2 \end{array} \right\}.$$

This lemma is equivalent to showing that $l(\mathcal{O}_K^*) \cap (S_{n-1}) \neq \emptyset$. Let $p\colon \mathbb{R}^{r_1+r_2} \to \mathbb{R}^{r_1+r_2-1}$ be the linear projection

$$p(a_1, \ldots, a_{r_1+r_2}) = (a_1, \ldots, a_{r_1+r_2-1}).$$

Then $p(l(\mathcal{O}_K^*))$ becomes a full lattice in $\mathbb{R}^{r_1+r_2-1}$. By Lemma 2.8, there exists a

$$b = p(l(u)) \in p(l(\mathcal{O}_K^*)) \cap p(S_1)$$

for some unit $u \in \mathcal{O}_K^*$, which implies $l(u) \in S_{n-1}$. This completes the proof. □

Let K be a number field of degree $n = r_1 + 2r_2$, for each $\alpha \in \mathcal{O}_K$, we can fix one $u_\alpha \in \mathcal{O}_K^*$ by Lemma 2.9, such that $u_\alpha \alpha \in \mathscr{R}_K$. Then we can choose a suitble constant C', depending on K and the choice of basis $\{\epsilon_i\}_{i=1}^{r_1+r_2-1}$ of $\mathcal{O}_K^*/\mu_K$, such that $u_i^{-1}u_j u_\alpha \alpha \in \{\alpha \in \mathcal{O}_K \mid 0 < |\alpha^{(i)}| < C'N(\alpha)^{\frac{1}{n}},\ 0 < |\alpha^{(j)}|^2 < C'N(\alpha)^{\frac{2}{n}} \text{ for } i = 1, ..., r_1,\ j = r_1+1, ..., r_1+r_2\}$ for any $u_i,\ u_j \in U$ and any $\alpha \in \mathcal{O}_K$, where U is the fixed representatives of $\mathcal{O}_K^*/{\mathcal{O}_K^*}^2$ in Section 1. Define $\mathscr{R}_K^X$ to be the set $\{\alpha \in \mathcal{O}_K \mid 0 < |\alpha^{(i)}| < C'X^{\frac{1}{n}},\ 0 < |\alpha^{(j)}|^2 < C'X^{\frac{2}{n}} \text{ for } i = 1, ..., r_1,\ j = r_1+1, ..., r_1+r_2\}$ for any $X \in \mathbb{R}_{>0}$.

The following lemma is an immediate consequence of [12, Theorem 2].

Lemma 2.10. *Let K be a number field of degree $n = r_1 + 2r_2$, $\mathfrak{q}$ is an ideal of $\mathcal{O}_K$. Let χ be a nonprincipal character of $(\mathcal{O}_K/q)^*$. Then we have*

$$\sum_{\alpha \in \mathscr{R}_K^X} \chi(\alpha) = O((N\mathfrak{q})^{\frac{1}{(r_2+2)}} X^{\frac{r_2}{r_2+2}} \log^{\frac{2r_1}{r_2+2}} X),$$

where O depends only on K.

By Lemma 2.9, for each b of the form $u\pi_1\cdots\pi_t \in \Sigma_K^X$, there exists $u_{\pi_1\cdots\pi_t} \in \mathscr{O}_K^*$ (fixed for each $\pi_1\cdots\pi_t$) such that $u_{\pi_1\cdots\pi_t}\pi_1\cdots\pi_t \in \mathscr{R}_K$. Let $u'_{\pi_1\cdots\pi_t} \in U$ satisfy $u'_{\pi_1\cdots\pi_t} \equiv u_{\pi_1\cdots\pi_t} \pmod{\mathscr{O}_K^{*\,2}}$. Then $u'^{-1}_{\pi_1\cdots\pi_t} u u_{\pi_1\cdots\pi_t}\pi_1\cdots\pi_t \in \mathscr{R}_K^X$, which induces an injective map:

$$\begin{aligned}\{b = u\pi_1\cdots\pi_t \mid b \in \Sigma_K^X\} &\longrightarrow \mathscr{R}_K^X,\\ u\pi_1\cdots\pi_t &\longmapsto u'^{-1}_{\pi_1\cdots\pi_t} u u_{\pi_1\cdots\pi_t}\pi_1\cdots\pi_t.\end{aligned}$$

We use this injective map to apply Lemma 2.10 in subsequent computations. Fix $\alpha \in \mathscr{O}_K$ and $X > 0$, we estimate the cardinality $\left|\alpha\mathscr{O}_K^* \cap \mathscr{R}_K^X\right|$, which is important for the proof of Theorem 2.13.

Lemma 2.11. *Let Λ be a full lattice in $\mathbb{R}^n$. Then there exists a constant C depending only on Λ, such that the number $Num(X)$ of points of Λ in a sphere $\{x \in \mathbb{R}^n | ||x|| < X\}$ is at most $C(X^n + 1)$ for any $X \in \mathbb{R}_{>0}$.*

Proof. See [13, Chapter 3, Lemma 1]. □

Lemma 2.12. *Let K be a number field of degree n. For any $\alpha \in O_K$, there exists $u \in \mathscr{O}_K^*$, such that $\alpha u \in \mathscr{R}_K$ (denote α_u) by Lemma 2.9. Then there exist constants $\tilde{C}', \tilde{C}'' > 0$ such that for all $X \in \mathbb{R}_{>0}$,*

$$|\alpha_u\mathscr{O}_K^* \cap \mathscr{R}_K^X| < \tilde{C}''((\log X - \log\tilde{\alpha_u}) + \tilde{C}')^n,$$

where $\tilde{\alpha_u} = \min_i\{|\alpha_u^{(i)}|\}$.

Proof. Note that $|\alpha_u\mathscr{O}_K^* \cap \mathscr{R}_K^X| = |\mathscr{O}_K^* \cap \alpha_u^{-1}\mathscr{R}_K^X|$. Define the logarithmic map

$$\ell : \mathscr{O}_K \setminus \{0\} \to \mathbb{R}^{r_1+r_2}, \quad \eta \mapsto \left(\{\log|\eta^{(i)}|\}_{i=1}^{r_1},\ \{2\log|\eta^{(i)}|\}_{i=r_1+1}^{r_1+r_2}\right).$$

Let $H = \{(b_i) \in \mathbb{R}^{r_1+r_2} \mid \sum_{i=1}^{r_1+r_2} b_i = 0\}$ and $S = \ell(\alpha_u^{-1}\mathscr{R}_K^X) \cap H$. For any $s \in S$ with $s = \ell(\alpha_u^{-1}b)$, since $s \in H$, the coordinate bounds

$$\log|b^{(i)}| - \log|\alpha_u^{(i)}| \geq -\left(\sum_{\substack{j=1\\ j\neq i}}^{r_1} \log\frac{X^{1/n}}{|\alpha_u^{(j)}|} + \sum_{\substack{j=r_1+1\\ j\neq i}}^{r_1+r_2} \log\frac{X^{2/n}}{|\alpha_u^{(j)}|^2} + C\right)$$

hold for some $C > 0$. We can therefore choose constants $C', C'' > 0$ and define

$$S' := \left\{(c_i) \in \mathbb{R}^{r_1+r_2} \;\middle|\; |c_i| < C'(\log X - \log\tilde{\alpha}_u) + C'', 1 \leq i \leq r_1 + r_2\right\},$$

such that $S \subset S'$. Consider the projection $p : \mathbb{R}^{r_1+r_2} \to \mathbb{R}^{r_1+r_2-1}$ dropping the last coordinate. Let $S'' = \{x \in \mathbb{R}^{r_1+r_2-1} \mid \|x\| < \sqrt{n}C'(\log X - \log\tilde{\alpha}_u) + C''\}$. By Lemma 2.11,

$$|S'' \cap p\ell(\mathscr{O}_K^*)| < \tilde{C}((\log X - \log\tilde{\alpha}_u) + \tilde{C}')^n,$$

for some suitable constants $\tilde{C}, \tilde{C}' > 0$. Since $p|_{\ell(\mathscr{O}_K^*)\cap S}$ is injective and $p(S') \subseteq S''$, we conclude

$$|l(\mathscr{O}_K^* \cap \alpha_u^{-1}\mathscr{R}_K^X)| < \tilde{C}((\log X - \log\tilde{\alpha}_u) + \tilde{C}')^n.$$

Since $l|_{\mathscr{O}_K^*}$ has finite kernel, $|\mathscr{O}_K^* \cap \alpha_u^{-1}\mathscr{R}_K^X| < \tilde{C}''((\log X - \log\tilde{\alpha_u}) + \tilde{C}')^n$ for a suitable constant $\tilde{C}'' > 0$ and this completes the proof. □

With the above preparations, we establish our main theorem.

Theorem 2.13. *Let K be a number field of degree $n = r_1 + 2r_2$ with class number 1. Then the rational density $\delta_{H,K} = 0$.*

Proof. Let $\Sigma_K^{H',X} := \{(a,b) | (a,b) \in \Sigma_K^X \times \Sigma_K^X,\ (\frac{b}{\pi})_2 = 1 \text{ for any prime } \pi | a, \pi \nmid b\}$. Since $\Sigma_K^{H,X} \subseteq \Sigma_K^{H',X}$, it suffices to show that

$$\delta_{H',K} := \limsup_{X \to +\infty} \frac{|\Sigma_K^{H',X}|}{|\Sigma_K^X|^2}$$

is zero. To establish this, we first recall that applying the Wiener-Ikehara Tauberian theorem [18, Chapter 7] to the Dirichlet L-series associated with the square of the Möbius function yields the asymptotic formula for square-free positive integers:

$$\#\big\{1 \le n \le X : n \text{ square-free}\big\} \sim \frac{6}{\pi^2} X + o(X). \tag{2.4}$$

By adapting this strategy to the ideal-theoretic setting—replacing integers with ideals—we analyze the Dirichlet L-series for the squared Möbius function over ideals. This leads to the analogous asymptotic estimate:

$$\#|\Sigma_K^X| \sim C_K X + o(X), \tag{2.5}$$

where $C_K > 0$ is a constant depending on K. From above, $|\Sigma_K^{H',X}| = \sum\limits_{(a,b) \in \Sigma_K^X \times \Sigma_K^X} \prod\limits_{\substack{\pi | a, \pi \nmid b \\ \pi \text{ odd prime}}} \frac{(1 + (\frac{b}{\pi})_2)}{2}$, and it suffices to show that

$$|\Sigma_K^{H',X}| = o(X^2). \tag{2.6}$$

By direct computation,

$$\begin{aligned}
&\sum_{(a,b) \in \Sigma_K^X \times \Sigma_K^X} \prod_{\substack{\pi | a, \pi \nmid b \\ \pi \text{ odd prime}}} \frac{(1 + \left(\frac{b}{\pi}\right)_2)}{2} \\
\le & r \cdot \sum_{\substack{1 \le N((a)) \le X \\ (a) \subseteq \mathcal{O}_K}} \sum_{b \in \mathscr{R}_K^X} \prod_{\substack{\pi | a, \pi \nmid b \\ \pi \text{ odd prime}}} \frac{(1 + \left(\frac{b}{\pi}\right)_2)}{2} \\
\le & r \cdot \sum_{\substack{1 \le N((a)) \le X \\ (a) \subseteq \mathcal{O}_K}} \sum_{b \in \mathscr{R}_K^X} \sum_{\substack{1 \ne (d) | (a) \\ 2 \notin (d) \\ d \text{ square free}}} (\frac{1}{2})^{\omega(d)} \left(\frac{b}{d}\right)_2 + 2r \cdot \sum_{\substack{1 \le N((a)) \le X \\ (a) \subseteq \mathcal{O}_K}} \sum_{b \in \mathscr{R}_K^X} (\frac{1}{2})^{\omega(\frac{a}{\gcd(a,b)})},
\end{aligned}$$

where $r = |U|$ and U is the fixed representatives of $\mathcal{O}_K^* / {\mathcal{O}_K^*}^2$ in Section 1. We begin by estimating the summation

$$r \cdot \sum_{\substack{1 \le N((a)) \le X \\ (a) \subseteq \mathcal{O}_K}} \sum_{b \in \mathscr{R}_K^X} \left(\frac{1}{2}\right)^{\omega\left(\frac{a}{\gcd(a,b)}\right)}.$$

Notice that

$$\mathscr{R}_K^X = \coprod_{\mathfrak{b}\subseteq\mathscr{O}_K} \left(b_u\mathscr{O}_K^* \cap \mathscr{R}_K^X\right),$$

where b_u is a fixed generator of the ideal $\mathfrak{b}$ such that $b_u \in \mathscr{R}_K$. By Lemma 2.11, we have

$$\left|b_u\mathscr{O}_K^* \cap \mathscr{R}_K^X\right| < \tilde{C}'' \left(\log X - \log \tilde{b}_u + \tilde{C}'\right)^n,$$

where $\tilde{b}_u = \min_i\{|b_u^{(i)}|\}$. We divide the ideals (b_u) into two classes:

- **Class I**: Ideals satisfying

$$\tilde{C}'' \left(\log X - \log \tilde{b}_u + \tilde{C}'\right)^n > \log^{1/7} X.$$

- **Class II**: Ideals violating the above inequality.

Equivalently, Class I consists of ideals with

$$\log \tilde{b}_u < \log X - \frac{\log^{1/(7n)} X}{(\tilde{C}'')^{1/n}} + \tilde{C}',$$

while Class II contains the remaining ideals. Since $b_u \in \mathscr{R}_K$, elements in Class I satisfy

$$|N(b_u)| < \tilde{C}_1 \mathrm{e}^{n\left(\log X - \frac{\log^{1/(7n)} X}{(\tilde{C}'')^{1/n}}\right)}$$

for some constant $\tilde{C}_1 > 0$. Let $A_X := \log X - \frac{\log^{1/7n} X}{\tilde{C}''^{1/n}}$. Notice that $\tilde{C}_1 \mathrm{e}^{n(\log X - A_X)} = O(\frac{X}{\log^{n+1} X})$. Then

$$\begin{aligned}
&\sum_{\substack{1\le N(\mathfrak{a})\le X\\ \mathfrak{a}\subseteq\mathscr{O}_K}} \sum_{b\in\mathscr{R}_K^X} (\frac{1}{2})^{\omega(\frac{\mathfrak{a}}{\gcd(\mathfrak{a},(b))})} \\
=&\sum_{\substack{1\le N(\mathfrak{a})\le X\\ \mathfrak{a}\subseteq\mathscr{O}_K}} \sum_{b\in \coprod_{\mathfrak{b}\subseteq\mathscr{O}_K} (b_u\mathscr{O}_K^*\cap\mathscr{R}_K^X)} (\frac{1}{2})^{\omega(\frac{\mathfrak{a}}{\gcd(\mathfrak{a},\mathfrak{b})})} \\
\le&\sum_{\substack{1\le N(\mathfrak{a})\le X\\ \mathfrak{a}\subseteq\mathscr{O}_K}} \sum_{1\le N(\mathfrak{b})\le \tilde{C}_1 e^{nA_X}} \tilde{C}'' \left(\log X - \log \tilde{b}_u + \tilde{C}'\right)^n \cdot 1 \\
&+\sum_{\substack{1\le N(\mathfrak{a})\le X\\ \mathfrak{a}\subseteq\mathscr{O}_K}} \sum_{\tilde{C}_1 e^{nA_X} \le N(\mathfrak{b})\le C'^n X} \log^{1/7} X \cdot (\frac{1}{2})^{\omega(\frac{\mathfrak{a}}{\gcd(\mathfrak{a},\mathfrak{b})})} \\
=&O(\frac{X^2}{\log X}) + O(\frac{X^2}{\log^{\frac{1}{42}} X}) \quad \text{(By Lemma 2.7 (2))}.
\end{aligned}$$

Thus

$$\sum_{(a,b)\in\Sigma_K^X} \prod_{\substack{\pi|a,\pi\nmid b\\ \pi \text{ odd prime}}} \frac{(1+\left(\frac{b}{\pi}\right)_2)}{2}$$

$$\leq r \cdot \sum_{\substack{1\leq N((a))\leq X\\(a)\subseteq\mathscr{O}_K}} \sum_{\substack{-X\leq N(b)\leq X\\b\in\mathscr{R}_K^X}} \sum_{\substack{1\neq(d)|(a)\\2\notin(d)\\d \text{ square free}}} (\frac{1}{2})^{\omega(d)} \left(\frac{b}{d}\right)_2 + o(X^2)$$

$$= r \cdot \sum_{\substack{1\leq N((a))\leq X\\(a)\subseteq\mathscr{O}_K}} \sum_{\substack{1\neq(d)|(a)\\2\notin(d)\\d \text{ square free}}} \sum_{\substack{-X\leq N(b)\leq X\\b\in\mathscr{R}_K^X}} (\frac{1}{2})^{\omega(d)} \left(\frac{b}{d}\right)_2 + o(X^2)$$

$$\leq r \cdot \sum_{\substack{1\leq N((a))\leq X\\(a)\subseteq\mathscr{O}_K}} \sum_{(d)|(a)} (\frac{1}{2})^{\omega(d)} (N((d)))^{\frac{1}{r_2+2}} X^{\frac{r_2}{r_2+2}} \log^{\frac{2r_1}{r_2+2}} X + o(X^2) \quad \text{(By Lemma 2.10)}$$

$$\leq r \cdot X^{\frac{r_2+1}{r_2+2}} \log^{\frac{2r_1}{r_2+2}} X \cdot \sum_{\substack{1\leq N((a))\leq X\\(a)\subseteq\mathscr{O}_K}} \sum_{(d)|(a)} (\frac{1}{2})^{\omega(d)} + o(X^2).$$

Here

$$\sum_{\substack{1\leq N((a))\leq X\\(a)\subseteq\mathscr{O}_K}} \sum_{(d)|(a)} (\frac{1}{2})^{\omega(d)}$$

$$= \left(\sum_{\substack{1\leq N((d_1))\leq\sqrt{X}\\(d_1)\subseteq\mathscr{O}_K}} (\frac{1}{2})^{\omega(d_1)} \cdot \Big(\sum_{\substack{\sqrt{X}<N((d_2))\leq\frac{X}{N((d_1))}\\(d_2)\subseteq\mathscr{O}_K}} 1 \Big) \right) + \left(\sum_{\substack{1\leq N((d_2))\leq\sqrt{X}\\(d_2)\subseteq\mathscr{O}_K}} 1 \cdot \Big(\sum_{\substack{\sqrt{X}\leq N((d_1))\leq\frac{X}{d_2}\\(d_1)\subseteq\mathscr{O}_K}} (\frac{1}{2})^{\omega(d_1)} \Big) \right).$$

We estimate the first term

$$\sum_{\substack{1\leq N((d_1))\leq\sqrt{X}\\(d_1)\subseteq\mathscr{O}_K}} (\frac{1}{2})^{\omega(d_1)} \cdot \Big(\sum_{\substack{\sqrt{X}<N((d_2))\leq\frac{X}{N((d_1))}\\(d_2)\subseteq\mathscr{O}_K}} 1 \Big),$$

and the estimation for the second term is the same. By Lemma 2.7 (1)

$$\sum_{\substack{\sqrt{X}<N((d_2))\leq\frac{X}{N((d_1))}\\(d_2)\subseteq\mathscr{O}_K}} 1 \leq C_5 \left(\frac{X}{N((d_1))}\right)$$

for some constant C_5. Thus

$$\sum_{\substack{1\leq N((d_1))\leq\sqrt{X}\\(d_1)\subseteq\mathscr{O}_K}} (\frac{1}{2})^{\omega(d_1)} \cdot \Big(\sum_{\substack{\sqrt{X}<N((d_2))\leq\frac{X}{N((d_1))}\\(d_2)\subseteq\mathscr{O}_K}} 1 \Big)$$

$$\leq C_5 \cdot \sum_{\substack{1\leq N((d_1))\leq\sqrt{X}\\(d_1)\subseteq\mathscr{O}_K}} (\frac{1}{2})^{\omega(d_1)} \frac{X}{N((d_1))}$$

$$\leq C_5 X \cdot \sum_{1\leq N((d_1))\leq\sqrt{X}} (\frac{1}{2})^{\omega(d_1)} \cdot \frac{1}{N((d_1))}.$$

But

$$\begin{aligned}\sum_{1\le N((d_1))\le\sqrt{X}}(\frac{1}{2})^{\omega(d_1)}\cdot\frac{1}{N((d_1))} &= (\sum_{1\le N((d_1))\le\sqrt{X}}(\frac{1}{2})^{\omega(d_1)})\cdot\frac{1}{\sqrt{X}}+\int_1^{\sqrt{X}}\sum_{1\le N((d_1))\le t}(\frac{1}{2})^{\omega(d_1)}\frac{1}{t^2}dt\\ &\le \frac{C_6}{\log^{\frac{1}{2}}\sqrt{X}}+C_7\int_2^{\sqrt{X}}\frac{t}{\log^{\frac{1}{2}}t}\frac{1}{t^2}dt\\ &\le \frac{C_6}{\log^{\frac{1}{2}}\sqrt{X}}+C_7\cdot\log^{\frac{1}{2}}\sqrt{X}+C_8.\end{aligned}$$

Here C_6, C_7, C_8 are constants. Therefore

$$\sum_{1\le N((d_1))\le\sqrt{X}}(\frac{1}{2})^{\omega(d_1)}\cdot(\sum_{\sqrt{X}<N((d_2))\le\frac{X}{N((d_1))}}1)=O(X\log^{\frac{1}{2}}X).$$

This completes the proof. □

3. The period-index density of elliptic curves

The Hilbert symbol is closely related to the period-index problem of elliptic curves. As an application, we can use Theorem 2.13 to compute the period-index density of elliptic curves. In the following, we review the notions of period and index of homogeneous spaces for elliptic curves, and explain the relationship between the Hilbert symbol and the period-index obstruction map..

Let K be a field with absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$, and let E/K be an elliptic curve over K. The Weil-Châtelet group $\mathrm{WC}(E/K)$, which classifies homogeneous spaces of E, is canonically isomorphic to the Galois cohomology group $H^1(K, E(\bar{K}))$ (cf. [6, Proposition 4]). Under this isomorphism, each homogeneous space $[C]$ of E corresponds to a cohomology class in $H^1(K, E(\bar{K}))$, which we still denote by $[C]$. The *period* of C, denoted by $P(C)$, is defined as the order of $[C]$ in the group $H^1(K, E(\bar{K}))$. The *index* of C, denoted by $I(C)$, is the smallest positive integer d for which there exists a K-rational divisor of degree d on C. The period-index problem asks whether $P(C)$ equal $I(C)$ for all $C \in \mathrm{WC}(E/K)$. When K is a local field, Lichtenbaum [8] (see also [11, Section 5]) proved that $P(C) = I(C)$ holds for all homogeneous spaces C of E/K. In contrast, when K is a number field, the period and index do not necessarily coincide (cf. [1]), but they satisfy the divisibility relation

$$P(C) \mid I(C) \mid P(C)^2 \tag{3.1}$$

as shown in [9, Theorem 8] (see also [11, Proposition 2.4]). Building on this constraint, the necessary conditions for a positive integer pair (P, I) to arise as the period and index of a homogeneous space C are that $I = Pl$ for some positive integer l, and that l divides P. Furthermore, Sharif [16, Theorem 2] proved that for any such pair (P, Pl) with $l \mid P$, there exist infinitely homogeneous spaces C of E satisfying $P(C) = P$ and $I(C) = Pl$.

A natural question arises: If the period is fixed, what is the distribution of distinct indices (cf. [3, Problem 2])? In the following, suppose $C \in H^1(K, E)$ satisfies $P(C) = 2$. From the divisibility relation (3.1), $I(C)$ must equal 2 or 4. This reduces to determining the density of the subset

$$\{C \mid C \in H^1(K, E)[2], I(C) = 2\}$$

within the entire set

$$\{C \mid C \in H^1(K, E)[2]\}.$$

In the rest of this section, we will answer this question under some constraints.

In the rest of this paper, let K be a number field and E/K be an elliptic curve defined over K, and n be a positive integer.

Recall the following Kummer sequence for elliptic curve E:

$$0 \to E(K)/nE(K) \xrightarrow{\delta} H^1(K, E[n]) \xrightarrow{\rho} H^1(K, E)[n] \to 0. \tag{3.2}$$

The main tool of period-index problem is the following O'Neil's obstruction map :

$$Ob : H^1(K, E[n]) \to Br(K). \tag{3.3}$$

The details of obstruction map can be found in [11]. The map Ob can be used to determine whether or not $I(C) = P(C)$ for a given homogeneous space C. Concretely, we have the following proposition.

Proposition 3.1. *Let E/K be an elliptic curve defined over number field K. Let $C \in H^1(K, E)$ be of period n. Then C has index n if and only if there exists a lift $\zeta \in H^1(K, E[n])$ of C such that $Ob(\zeta) = 1$.*

Proof. This is [2, Theorem 5]. □

We explain the relationship between the Hilbert symbol and the obstruction map. Assume that the entire n-torsion group $E[n] \subset E(K)$. Via the theory of the Weil pairing, the n-th roots of unity μ_n are contained in K. Fix a basis (S, T) for $E[n]$ once and for all. Again by the Weil pairing, $\zeta = e_n(S, T)$ is a generator of μ_n. After making this choice, we get an isomorphism

$$\psi_n : H^1(K, \mu_n) \times H^1(K, \mu_n) \xrightarrow{\sim} H^1(K, E[n]). \tag{3.4}$$

Via the canonical Kummer isomorphism $H^1(K, \mu_n) = K^*/K^{*n}$, we may equally well view ψ_n as maps defined on $(K^*/K^{*n})^2$.

Theorem 3.2. *For $n \in \mathbb{Z}_{>0}$, let n^* be n if n is odd and $2n$ if n is even. If $E[n^*] \subset E(K)$, then $Ob \circ \psi_n = (\ ,\ )_{H_n,K}$, where $(\ ,\ )_{H_n,K}$ is the Hilbert symbol defined in Section 2.*

Proof. This is [4, Theorem 10]. □

Suppose that $C \in H^1(K, E[n])$ with $P(C) = 2$, then $I(C) = 2$ or 4. Moreover, we suppose that $E[4] \subset E(K)$. As the above discussion, we have an isomoprhism $H^1(K, E[2]) \xrightarrow{\sim} K^*/K^{*2} \times K^*/K^{*2}$. In this isomorphism, we view the map Ob as a map from $K^*/K^{*2} \times K^*/K^{*2}$ to $Br(K)$. Then by

Theorem 3.2 , the map Ob coincides with the Hilbert symbol. In summary, replacing $H^1(K, E[2])$ with $K^*/K^{*2} \times K^*/K^{*2}$, we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(K)/2E(K) & \xrightarrow{\delta} & K^*/K^{*2} \times K^*/K^{*2} & \xrightarrow{\rho} & H^1(K,E)[2] & \longrightarrow & 0. \\ & & & & \Big\downarrow{\scriptstyle Ob=(\ ,\ )_{H,K}} & & & & \\ & & & & Br(K) & & & & \end{array} \tag{3.5}$$

Definition 3.3. Let K be a number field with class number 1, and let E/K be an elliptic curve satisfying $E[4] \subset E(K)$. Retaining the notation introduced earlier, for homogeneous spaces of period 2, the period-index density Θ is defined as the following limit:

$$\Theta := \limsup_{X \to +\infty} \frac{\left|\{C \mid C \in \rho(\Sigma_K^X \times \Sigma_K^X), I(C) = 2\}\right|}{\left|\rho(\Sigma_K^X \times \Sigma_K^X)\right|}$$

Theorem 3.4. *Let K be a number field with class number 1, and let E/K be an elliptic curve satisfying $E[4] \subset E(K)$. Then $\Theta = 0$.*

Proof. By Proposition 3.1, the set $\{C \mid C \in \rho(\Sigma_K^X \times \Sigma_K^X), I(C) = 2\}$ is equal to the set

$$\{C \in \rho(\Sigma_K^X \times \Sigma_K^X) \mid \exists\ (a,b) \in K^*/K^{*2} \times K^*/K^{*2}, \text{ such that } \rho(a,b) = C \text{ and } (a,b)_{H,K} = 1\}.$$

Let $\{(a_i, b_i)\}_{i=1}^t$ denote the image of $E(K)/2E(K)$ in $K^*/K^{*2} \times K^*/K^{*2}$. Let A be defined as

$$A = \max\left\{\max_{1 \le i \le t} |N(a_i)|, \max_{1 \le j \le t} |N(b_j)|\right\}.$$

Then $\{C \mid C \in \rho(\Sigma_K^X \times \Sigma_K^X),\ I(C) = 2\} \subset \{C \mid C \in \rho(\Sigma_K^{H,A\cdot X})\}$. Thus by Equation (2.6) and $\{C \mid C \in \rho(\Sigma_K^X \times \Sigma_K^X),\ I(C) = 2\} \subseteq \{C \mid C \in \rho(\Sigma_K^{H,A\cdot X})\}$, we have $|\{C \mid C \in \rho(\Sigma_K^X \times \Sigma_K^X),\ I(C) = 2\}| = o(X^2)$. Hence

$$\begin{aligned} \Theta &= \limsup_{X \to +\infty} \frac{\left|\{C \mid C \in \rho(\Sigma_K^X \times \Sigma_K^X),\ I(C) = 2\}\right|}{\left|\rho(\Sigma_K^X \times \Sigma_K^X)\right|} \\ &= \limsup_{X \to +\infty} \frac{o(X^2)}{X^2} \\ &= 0. \end{aligned}$$

□

Remark 3.5. By the divisibility relation (3.1), every non-trivial homogeneous space $C \in H^1(K, E)[2]$ satisfies $P(C) = 2$ and $I(C) \in \{2, 4\}$. This theorem establishes that within the 2-torsion subgroup $H^1(K, E)[2]$, the subset of elements with $I(C) = 2$ is sparse (exhibiting period-index density 0), whereas those with $I(C) = 4$ dominate (attaining period-index density 1) under the conditions specified in Theorem 3.4. Notably, there exist elliptic curves satisfying the hypotheses of Theorem 3.4. A concrete example is the curve **256.b1** in the LMFDB database, whose Weierstrass equation is explicitly given by

$$y^2 = x^3 - 2x.$$

Further instances from the LMFDB database, such as $\mathbf{200.2-a3}$, $\mathbf{225.2-a6}$ and $\mathbf{5525.5-b9}$, also satisfy the conditions in Theorem 3.4.

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