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General Section

Distinguishing newforms by the prime divisors of their Fourier coefficients

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Given two non-CM newforms with integral Fourier coefficients, in this paper we study the number of distinct prime divisors of their Fourier coefficients in a probability way. Based on a multivariate version of the Erdős-Kac theorem, using the Galois representations attached to newforms and the effective Chebotarev density theorem, and assuming the generalized Riemann hypothesis, we show that the distribution of the number of distinct primes dividing the Fourier coefficients behaves like the standard multivariate normal distribution if these newforms are not twists of each other. As a consequence, we prove a multiplicity one result for modular forms under the generalized Riemann hypothesis.

1. Introduction

Denote by $S_k(N)$ the space of cusp forms of weight k for $\Gamma_0(N)$. Let $a_n(f)$ denote the n-th Fourier coefficient of $f \in S_k(N)$. A newform is an element in $S_k(N)$ which is an eigenvector of every Hecke operator and satisfies $a_1(f) = 1$ (cf. [1]). One natural

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question is whether, if we know certain information about the Fourier coefficients of a newform, we can reconstruct the newform itself. The classical multiplicity-one result for newforms says that if two such forms f, h satisfy that $a_p(f) = a_p(h)$ for primes p in a set of density 1, then $f = h$. Results like these have been studied by a number of authors (cf. [7,11,12]). In this paper, we take a probabilistic view of this problem. Recall that a newform f has complex multiplication (CM, for short) if there is a quadratic character χ such that $a_p(f) = a_p(f)\chi(p)$ for almost all primes p (cf. [13]). Denote by $\omega(n)$ the number of distinct prime divisors of nonzero integer n. Assuming the Riemann hypothesis for all Dedekind zeta functions of number fields (GRH), we prove the following result which can be viewed as a multiplicity-one theorem from a probabilistic point of view.

Theorem 1.1. *Let f and h be two newforms with integral Fourier coefficients of levels N_f and N_h, respectively. Suppose that f and h are not of CM type. If for some constant $C \geq 0$,*

$$\limsup_{x \to +\infty} \frac{1}{x/\log x} \{p \leq x : a_p(f) \neq 0, a_p(h) \neq 0,\ |\omega(a_p(f)) - \omega(a_p(h))| \leq C\} > 0,$$

then assuming GRH, there exists a Dirichlet character χ such that $a_p(f) = \chi(p)a_p(h)$ for p prime to $N_f N_h$.

In a word, one can distinguish newforms by the number of distinct primes dividing the Fourier coefficients. The proof is based on a multivariate version of the Erdős-Kac theorem in probabilistic number theory. The Erdős-Kac theorem provides a splendid connection between probability theory and number theory. It states that the random variables

$$\frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}$$

defined on the set of natural numbers less than x equipped with the uniform probability measure, as x goes to infinity converge in distribution to the standard normal distribution. More precisely, for any $\alpha \in \mathbb{R}$,

$$\lim_{x \to +\infty} \frac{1}{x} \# \left\{ n \leq x : \ \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} < \alpha \right\} = G(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

Erdős and Kac's proof is based on the central limit theorem and sieve methods [6]. They provide a method to study the properties of arithmetic functions by studying their statistical properties. Since then, various generalizations of the Erdős-Kac theorem have been studied by many mathematicians, for example see [5].

R. Murty and K. Murty proved a modular analogue of the Erdős-Kac theorem [10]. For the Ramanujan τ-function, assuming GRH, they proved that for all $\alpha \in \mathbb{R}$,

$$\lim_{x\to+\infty} \frac{1}{x/\log x} \# \left\{ p \le x : \ \tau(p) \ne 0 \text{ and } \frac{\omega\left(\tau(p)\right) - \log\log p}{\sqrt{\log\log p}} < \alpha \right\} = G(\alpha).$$

Liu proved another prime analogue of the Erdős-Kac theorem regarding elliptic curves [8]. Let E be a non-CM elliptic curve defined over $\mathbb{Q}$. For a prime p of good reduction, denote by $E(\mathbb{F}_p)$ the set of rational points of the elliptic curve E defined over the finite field $\mathbb{F}_p$. Under GRH Liu proved that for all $\alpha \in \mathbb{R}$,

$$\lim_{x\to+\infty} \frac{1}{x/\log x} \# \left\{ p \le x : \ p \text{ is of good reduction and } \frac{\omega(\#E\left(\mathbb{F}_p\right)) - \log\log p}{\sqrt{\log\log p}} < \alpha \right\}$$
$$= G(\alpha).$$

In a recent paper [4], El-Baz, Loughran and Sofos generalized the work of predecessors and established a multivariate version of the Erdős-Kac theorem. Roughly speaking, if a family of integer sequences satisfies certain hypotheses, the number of distinct prime divisors of these sequences has a probabilistic behavior which fits a multivariate normal distribution. El-Baz, Loughran and Sofos used their result to study the distributions of integral points on varieties.

Applying the result of El-Baz, Loughran, Sofos and generalizing the works of R. Murty, K. Murty and Liu, in this paper we establish a result regarding the joint distribution of the number of prime divisors of the Fourier coefficients of two distinct newforms.

Theorem 1.2. *Let f and h be two newforms with integral Fourier coefficients of weights k_1, k_2, respectively. Assume that f, h are not of CM type and f, h are not twists of each other. Let $r_1(x)$ and $r_2(x)$ be two polynomials with integral coefficients. For every $x > 0$, let*

$$T_x := \{p \le x : a_p(f) + r_1(p^{k_1-1}) \ne 0 \text{ and } a_p(h) + r_2(p^{k_2-1}) \ne 0\}.$$

For notational simplicity, write $\omega_1 := \omega\big(a_p(f) + r_1(p^{k_1-1})\big)$ and $\omega_2 := \omega\big(a_p(h) + r_2(p^{k_2-1})\big)$. Then under GRH, for any Borel set $A \subseteq \mathbb{R}^2$ with zero measure on the boundary,

$$\lim_{x\to+\infty} \frac{1}{x/\log x} \# \left\{ p \in T_x : \left(\frac{\omega_1 - \log\log x}{\sqrt{\log\log x}}, \frac{\omega_2 - \log\log x}{\sqrt{\log\log x}} \right) \in A \right\}$$
$$= \frac{1}{2\pi} \int_A e^{-\frac{1}{2}(x^2+y^2)} dx dy.$$

Taking $A = \{(x, y) \in \mathbb{R}^2 : x < y\}$ in Theorem 1.2, we obtain the following result.

Corollary 1.3. *With the notation and assumptions of Theorem 1.2,*

$$\lim_{x\to+\infty} \frac{1}{x/\log x} \# \{p \in T_x : \ \omega_1 < \omega_2\} = \frac{1}{2}.$$

In particular, given two non-isogenous non-CM elliptic curves E_1 and E_2 over $\mathbb{Q}$, by the modularity theorem and the above result with $r_1 = r_2 = -1 - x$, we have the following corollary.

Corollary 1.4. *If E_1 is not a quadratic twist of E_2, then under GRH,*

$$\lim_{x \to +\infty} \frac{1}{x/\log x} \# \left\{ p \leq x : p \text{ is of good reduction and } \omega\left(\#E_1(\mathbb{F}_p)\right) < \omega\left(\#E_2(\mathbb{F}_p)\right) \right\} = \frac{1}{2}.$$

Theorem 1.1 follows directly from Theorem 1.2. More precisely, taking $r_1 = r_2 = 0$ in Theorem 1.2, the following two random variables

$$\left(\frac{\omega\left(a_p(f)\right) - \log\log x}{\sqrt{\log\log x}}, \frac{\omega\left(a_p(h)\right) - \log\log x}{\sqrt{\log\log x}} \right)$$

behave like two independent normally distributed random variables when x goes to infinity, so the random variables

$$R_x(p) := \frac{\omega\left(a_p(f)\right) - \omega\left(a_p(h)\right)}{\sqrt{\log\log x}}$$

converge in distribution to a difference of two independent standard normal distributions, i.e. a normal distribution with mean 0 and variance 2. Hence for any $\epsilon > 0$,

$$\begin{aligned} &\lim_{x \to +\infty} \frac{1}{x/\log x} \# \left\{ p \leq x : a_p(f) \neq 0, a_p(h) \neq 0, \ \frac{\left|\omega\left(a_p(f)\right) - \omega\left(a_p(h)\right)\right|}{\sqrt{\log\log x}} > \epsilon \right\} \\ &= \frac{1}{2\sqrt{\pi}} \int_{\epsilon}^{\infty} e^{-x^2/4} dx. \end{aligned}$$

This implies that for any constant $C \geq 0$, the set

$$\left\{ p : a_p(f) \neq 0, a_p(h) \neq 0, \ \left|\omega\left(a_p(f)\right) - \omega\left(a_p(h)\right)\right| \geq C \right\}$$

has natural density 1. Thus for two newforms which are not twists of each other, the number of distinct prime divisors of their Fourier coefficients will always diverge.

This paper is organized as follows. El-Baz, Loughran and Sofos' theorem is briefly reviewed in Section 2. In Section 3, we use the Galois representations attached to newforms and the effective Chebotarev density theorem to prove Theorem 1.2 by applying El-Baz, Loughran and Sofos' result. Finally Section 4 contains some examples and generalizations of Theorem 1.2.

1.1. Notation

Let f, g be two complex-valued maps defined on some set D. If $g(x)$ is positive and there is a constant C such that $|f(x)| \leq Cg(x)$ for all $x \in D$, we write either $f(x) \ll g(x)$ or $f(x) = O(g(x))$. We write $f(x) = o(g(x))$ if $\lim_{x\to\infty, x\in D} f(x)/g(x) = 0$. Throughout this paper, $\pi(x)$ denotes the number of primes less than x; p, ℓ denote prime numbers; k_1, k_2 denote integers at least 2.

2. A multivariate version of the Erdős-Kac theorem

In this section, we reformulate El-Baz, Loughran and Sofos' result in a concise form which is sufficient for our application. Let T be an infinite subset of $\mathbb{N}$. For every $x > 1$, denote by T_x the subset of T consisting of elements less than x. Given a family of integer sequences $\{a_i(n)\}_{1\leq i\leq m, n\in T}$, we have the following conditions.

C1. The sequences have polynomial growth, in other words, there exists a constant $d > 0$ such that $a_i(n) = O\left(n^d\right)$ for all n. Note that this condition is stronger than the condition appeared in [4, (2.7)].

C2. For each m-tuple of square-free integers $(d_1, \dots, d_m)$, write

$$R(d_1, \dots, d_m; x) := \frac{1}{|T_x|} \# \{n \in T_x : d_1 \mid a_1(n), \dots, d_m \mid a_m(n)\}.$$

Then there exist two functions g and e such that

$$R(d_1, \dots, d_m; x) = g(d_1, \dots, d_m) + e(d_1, \dots, d_m; x)$$

for all m-tuples of square-free integers $(d_1, \dots, d_m)$ whose prime divisors are greater than a given constant P. The function g should possess a multiplicative property, that is to say

$$g(a_1b_1, \dots, a_mb_m) = g(a_1, \dots, a_m)g(b_1, \dots, b_m) \text{ if } \gcd(a_1a_2\cdots a_m, b_1b_2\cdots b_m) = 1.$$

C3. Let $y = x^{F(x)}$, $F(x) = \log\log\log x/\sqrt{\log\log x}$, then for all $\gamma > 0$,

$$\sum\nolimits' |e(d_1, \dots, d_m; x)| = O\left((\log\log x)^{-\gamma}\right), \tag{1}$$

where $\sum'$ runs through all m-tuples of square-free integers $(d_1, \dots, d_m)$ which satisfy that the prime divisors of d_i are greater than P and $d_i < y$ for every i.

C4. For each $1 \leq i, j \leq n$, let

$$g_i(d) := g(1, \dots, 1, \underset{\substack{\uparrow \\ i}}{d}, 1, \dots, 1) \text{ and } g_{i,j}(d) := g(1, \dots, 1, \underset{\substack{\uparrow \\ i}}{d}, 1, \dots, 1, \underset{\substack{\uparrow \\ j}}{d}, 1, \dots, 1).$$

Then for every $1 \le i \le m$,

$$\sum_{\ell > x} g_i^2(\ell) = O\left(\frac{1}{\log x}\right) \text{ and } \sum_{\ell \le x} g_i(\ell) = c_i \log\log x + c_i' + O\left(\frac{1}{\log x}\right) \tag{2}$$

for some $c_i > 0, c_i' \in \mathbb{R}$. Moreover for every $1 \le i, j \le m,\ i \ne j$,

$$\sum_{\ell} g_{i,j}(\ell) < +\infty. \tag{3}$$

Note that this condition implies that the covariance matrix in [4, (2.11)] is trivial.

For each integer $x > 0$, define a uniform measure P_x on T as follows. For any subset A of T, define the probability measure:

$$\mathrm{P}_x(A) := \frac{1}{|T_x|}\#\{n \le x : n \in A\},$$

then equipping with the discrete σ-algebra, T becomes a probability space. Define the random vector $K_x : T \to \mathbb{R}^n$ via

$$K_x(n) := \left(\frac{\omega\left(a_1(n)\right) - c_1 \log\log x}{\sqrt{c_1 \log\log x}}, \ldots, \frac{\omega\left(a_m(n)\right) - c_m \log\log x}{\sqrt{c_m \log\log x}}\right).$$

Recall that a sequence of $\mathbb{R}^m$-valued random vectors $(X_n)_{n\ge 1}$ converges in distribution to X if the distribution functions of $(X_n)_{n\ge 1}$ converge to the distribution function F of X for all continuous points of F, it is equivalent to saying that $\mathrm{P}_n[X_n \in A] \to \mathrm{P}[X \in A]$ for all Borel sets $A \subseteq \mathbb{R}^m$ with $\mathrm{P}[X \in \partial A] = 0$ (cf. [2, p. 26]).

The result of [4, Theorem 2.1] claims the convergence of the above random vectors.

Theorem 2.1. *If the family of sequences $\{a_i(n)\}_{1\le i\le m, n\in T}$ satisfies C1, C2, C3 and C4, then the random vectors*

$$(T, \mathrm{P}_x) \to \mathbb{R}^m :\ n \mapsto K_x(n),$$

converge in distribution as $x \to +\infty$ to the standard multivariate normal distribution.

Remark 1. Although in the statement of [4] g is defined on all $\mathbb{N}^m$, from El-Baz, Loughran and Sofos' proof it is enough to assume that the support of g is the set of vectors $(d_1, \ldots, d_m)$ with square-free entries whose prime divisors are greater than some prime P.

Remark 2. In order that the error function satisfies condition (1), it suffices to check the following stronger condition: there exist constants $k, \delta > 0$ such that

$$e(d_1, \ldots, d_m; x) = O\left((d_1 \cdots d_m)^k x^{-\delta}\right). \tag{4}$$

Indeed, if inequality (4) holds, then

$$\begin{aligned}\sum{}' |e(d_1, \ldots, d_m; x)| &\ll x^{-\delta} \sum{}' (d_1 \cdots d_m)^k \\ &\ll x^{-\delta} \sum{}' y^{mk} \ll x^{-\delta} y^{mk+m} \ll x^{-\delta},\end{aligned}$$

where $\sum'$ runs through all m-tuples of square-free integers $(d_1, \ldots, d_m)$ such that $d_i < y$ and $p \mid d_i \Rightarrow p > P$. The last inequality holds since $y = o(x^\epsilon)$ for any $\epsilon > 0$.

Remark 3. If the family of sequences $\{a_i(n)\}_{1 \le i \le m, n \in T}$ satisfies C1, C2, C3 and C4, by Theorem 2.1 we have the following Erdős-Kac type theorem: for any Borel set $A \subseteq \mathbb{R}^m$ with zero measure on the boundary,

$$\begin{aligned}\lim_{x \to +\infty} \frac{1}{|T_x|} \# \left\{ n \in T_x : \left(\frac{\omega(a_1(n)) - c_1 \log\log x}{\sqrt{c_1 \log\log x}}, \ldots, \frac{\omega(a_m(n)) - c_m \log\log x}{\sqrt{c_m \log\log x}} \right) \in A \right\} \\ = \frac{1}{(2\pi)^{m/2}} \int\limits_A e^{-\frac{1}{2}(x_1^2 + \cdots + x_m^2)} dx_1 \cdots dx_m.\end{aligned}$$

Moreover if $c_1 = \cdots = c_m$ and $A = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 < \cdots < x_m\}$, we have

$$\lim_{x \to +\infty} \frac{1}{|T_x|} \# \{ n \in T_x : \omega(a_1(n)) < \cdots < \omega(a_m(n)) \} = \frac{1}{m!}.$$

3. Proof of Theorem 1.2

In this section, we choose the elements of the sequences in Section 2 to be the Fourier coefficients of certain newforms. We then check that these sequences satisfy all the conditions in Section 2, then by Theorem 2.1 we get the desired result.

3.1. Images of Galois representations

Let $f = \sum_{n=1}^{\infty} a_n(f) q^n \in \mathbb{Z}[\![q]\!] \cap S_{k_1}(N_f)$ be a newform which does not have complex multiplication (non-CM, for short). Following the construction of Shimura, Deligne and Serre (cf. [3]), attached to f, there exists an ℓ-adic Galois representation $\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ which is unramified outside ℓN_f. Composing with the natural projection $\mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$, we obtain a mod ℓ Galois representation $\overline{\rho}_{f,\ell}$ such that for any $p \nmid \ell N_f$,

$$\mathrm{tr}\, \overline{\rho}_{f,\ell}(\mathrm{Frob}_p) \equiv a_p(f) \bmod \ell \quad \text{and} \quad \det \overline{\rho}_{f,\ell}(\mathrm{Frob}_p) \equiv p^{k_1 - 1} \bmod \ell.$$

By Ribet's work [14], the image of the mod ℓ representations can be well described. For any sufficiently large prime ℓ, the image of $\overline{\rho}_{f,\ell}$ is

$$G(\ell,1) := \left\{u \in \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \det u = v^{k_1-1} \text{ for some } v \in (\mathbb{Z}/\ell\mathbb{Z})^*\right\}.$$

Let $h = \sum_{n=1}^{\infty} a_n(h)q^n \in \mathbb{Z}[\![q]\!] \cap S_{k_2}(N_h)$ be another non-CM newform, and we assume that there is no Dirichlet character χ such that $f = h \otimes \chi$ or $h = f \otimes \chi$. Loeffler described the image of the adelic Galois representation $\widehat{\rho}_f \times \widehat{\rho}_h$, and he proved that the image of the adelic Galois representation is open in the sense of [9, Theorem 3.4.1].

For sufficiently large primes ℓ and ℓ', consider the direct sum

$$\overline{\rho}_{\ell,\ell'} := \overline{\rho}_{f,\ell} \oplus \overline{\rho}_{h,\ell'} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell'\mathbb{Z}).$$

If $\ell = \ell'$, Loeffler's result implies that the image of $\overline{\rho}_{\ell,\ell'}$ is

$$G(\ell,\ell) := \begin{Bmatrix} (u_1,u_2) \in \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \\ \det u_1 = v^{k_1-1},\ \det u_2 = v^{k_2-1} \text{ for some } v \in (\mathbb{Z}/\ell\mathbb{Z})^* \end{Bmatrix}.$$

If $\ell \neq \ell'$, by Loeffler's result again, the image of $\overline{\rho}_{\ell,\ell'}$ is

$$\begin{Bmatrix} (u_1,u_2) \in \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell'\mathbb{Z}); \\ \det u_1 = v_1^{k_1-1},\ \det u_2 = v_2^{k_2-1} \text{ for some } v_1 \in (\mathbb{Z}/\ell\mathbb{Z})^* \text{ and } v_2 \in (\mathbb{Z}/\ell'\mathbb{Z})^* \end{Bmatrix}.$$

For two square-free integers d_1, d_2, if their prime factorizations are $d_1 = p_1 \cdots p_r$ and $d_2 = q_1 \cdots q_s$, consider

$$\overline{\rho}_{d_1,d_2} := \overline{\rho}_{f,p_1} \oplus \cdots \oplus \overline{\rho}_{f,p_r} \oplus \overline{\rho}_{h,q_1} \oplus \cdots \oplus \overline{\rho}_{h,q_s}.$$

Without loss of generality, we write $d_1 = LP, d_2 = LQ$, $\gcd(P,Q) = 1$. By Loeffler's result and the Chinese remainder theorem, the image of $\overline{\rho}_{d_1,d_2}$ is $G(d_1,d_2) :=$

$$\begin{Bmatrix} (u_1,u_2,u_3,u_4) \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/P\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/Q\mathbb{Z}) : \\ \det u_1 = \alpha^{k_1-1},\ \det u_2 = \alpha^{k_2-1} \text{ for some } \alpha \in (\mathbb{Z}/L\mathbb{Z})^*, \\ \det u_3 = \beta^{k_1-1} \text{ for some } \beta \in (\mathbb{Z}/P\mathbb{Z})^*, \\ \det u_4 = \gamma^{k_2-1} \text{ for some } \gamma \in (\mathbb{Z}/Q\mathbb{Z})^* \end{Bmatrix}.$$

3.2. Chebotarev density theorem

To gain the arithmetic information from the Galois representations, we need the effective Chebotarev density theorem. The following version of Chebotarev density theorem is from Serre [15, Théorème 4].

Theorem 3.1. *Let $K/\mathbb{Q}$ be a finite Galois extension of number fields with Galois group G. Let C be a subset of G which is stable under conjugation, and let* Frob_p *be the Frobenius element at an unramified prime p. Denote by $\pi_C(x)$ the set of primes p unramified in*

K for which $\mathrm{Frob}_p \in C$ *and* $p \le x$*. Assuming that the Dedekind zeta function* $\zeta_K(s)$ *satisfies the Riemann Hypothesis, then*

$$\pi_C(x) = \frac{|C|}{|G|}\pi(x) + O\left(|C| x^{\frac{1}{2}} \left(\frac{\log d_K}{n_K} + \log x\right)\right),$$

where d_K *and* n_K *are the discriminant and the degree of the extension* $K/\mathbb{Q}$*, respectively.*

The following estimate is useful in our computation:

$$\log d_K \le (n_K - 1) \sum_{p \in P(K)} \log p + n_K |P(K)| \log n_K, \tag{5}$$

where $P(K)$ denotes the set of ramified primes [15, Proposition 6].

We follow the notation in Section 3.1. Given two bivariate polynomials F_1, F_2 with integral coefficients, and for two square-free integers d_1, d_2 whose prime divisors are large enough, define

$$C(d_1, d_2) := \left\{ (u_1, u_2, u_3, u_4) \in G(d_1, d_2) : \begin{matrix} F_1(\mathrm{tr}\ u_1, \det u_1) = 0,\ F_1(\mathrm{tr}\ u_3, \det u_3) = 0 \\ F_2(\mathrm{tr}\ u_2, \det u_2) = 0,\ F_2(\mathrm{tr}\ u_4, \det u_4) = 0 \end{matrix} \right\}.$$

It is a subset of $G(d_1, d_2)$ which is stable under conjugation.

Applying the effective Chebotarev density theorem for the fixed field of $\ker \overline{\rho}_{d_1,d_2}$, we get

$$\begin{aligned} &\frac{1}{\pi(x)} \left\{ p \le x : d_1 \mid F_1\left(a_p(f), p^{k_1-1}\right) \text{ and } d_2 \mid F_2\left(a_p(h), p^{k_2-1}\right) \right\} \\ &\quad = \frac{|C(d_1, d_2)|}{|G(d_1, d_2)|} + e(d_1, d_2; x). \end{aligned}$$

Let $g(d_1, d_2) := |C(d_1, d_2)|/|G(d_1, d_2)|$. The multiplicativity of g follows from the isomorphism $G(d_1 d_1', d_2 d_2') \cong G(d_1, d_2) \times G(d_1', d_2')$ for $\gcd(d_1 d_2, d_1' d_2') = 1$.

For the remainder term, the degree of the extension is $O\left((d_1 d_2)^4\right)$ (cf. Lemma 3.2), by inequality (5) we have

$$\pi(x) e(d_1, d_2; x) = O\left((d_1 d_2)^4 x^{\frac{1}{2}} \log\left((d_1 d_2)^5 N_f N_h x\right)\right).$$

So for some $\epsilon > 0$,

$$e(d_1, d_2; x) = O\left((d_1 d_2)^5 x^{\epsilon - \frac{1}{2}}\right).$$

Remark 4. Note that the above error estimation has a similar form to condition (4). Rather, according to Remark 2, a quasi-GRH, which assumes that the associated zeta functions have no zero in the region $\mathrm{Re}(s) > \delta$ for some $\delta \in (\frac{1}{2}, 1)$, is sufficient for our purpose, although it seems as difficult as the original GRH. The GRH assumption is only used in the proof of inequality (1), which in turn relies on a sufficiently good effective version of the Chebotarev density theorem.

3.3. Calculate conjugacy classes

In this section, we verify conditions (2) and (3) in some special cases. Throughout this section, we keep the notation in Section 3.2.

Lemma 3.2. *Let* $\delta = \gcd(\ell-1, k_1-1)$ *and* $d = \gcd(\ell-1, k_1-1, k_2-1)$*, then for sufficiently large prime* ℓ*,*

$$|G(\ell,1)| = \frac{(\ell-1)^2\ell(\ell+1)}{\delta} \text{ and } |G(\ell,\ell)| = \frac{(\ell-1)^3\ell^2(\ell+1)^2}{d}.$$

Proof. The first assertion follows from the exact sequence

$$1 \to \mathrm{SL}_2(\mathbb{F}_\ell) \longrightarrow G(\ell,1) \longrightarrow \mathbb{F}_\ell^{*\delta} \to 1.$$

Similarly, we have the exact sequence

$$1 \to \mathrm{SL}_2(\mathbb{F}_\ell) \times \mathrm{SL}_2(\mathbb{F}_\ell) \longrightarrow G(\ell,\ell) \longrightarrow D \to 1,$$

where $D = \{(v^{k_1-1}, v^{k_2-1}) : v \in \mathbb{F}_\ell^*\}$. The order of D can be calculated from

$$1 \to \langle g^{\frac{\ell-1}{d}} \rangle \longrightarrow \mathbb{F}_\ell^* \xrightarrow{\varphi} D \to 1,$$

where g is a generator of $\mathbb{F}_\ell^*$ and φ is given by $v \mapsto (v^{k_1-1}, v^{k_2-1})$, so

$$|D| = \frac{\ell-1}{d},$$

the lemma follows. □

Lemma 3.3. *Let* ℓ *be an odd prime. For given* $t \in \mathbb{F}_\ell$ *and* $d \in \mathbb{F}_\ell^*$*,*

$$\#\{u \in \mathrm{GL}_2(\mathbb{F}_\ell) : \mathrm{tr}\ u = t,\ \det u = d\} = \ell^2 + \left(\frac{t^2-4d}{\ell}\right)\ell,$$

where $\left(\frac{\cdot}{\ell}\right)$ *denotes the Legendre symbol modulo* ℓ*.*

Proof. This follows easily from Table 1. □

Table 1
Conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_\ell)$.

Representative	No. of elements in each class	No. of classes	$\mathrm{tr}^2 - 4\det$
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	1	$\ell - 1$	0
$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$\ell^2 - 1$	$\ell - 1$	0
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\ell^2 + \ell$	$(\ell-1)(\ell-2)/2$	$(x-y)^2$
$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$	$\ell^2 - \ell$	$\ell(\ell-1)/2$	$4\varepsilon y^2$

ε is a quadratic nonresidue (mod ℓ)

Lemma 3.4. *For every $\delta \mid k_1 - 1$, define*

$$L_\delta = \{\ell : \gcd(k_1 - 1, \ell - 1) = \delta\}.$$

Given a bivariate polynomial $F_1(x, y)$ with integral coefficients, for every sufficiently large $\ell \in L_\delta$, let

$$N_{F_1}(\ell) = \left\{(x, y) \in \mathbb{F}_\ell \times \mathbb{F}_\ell^{*\delta} : F_1(x, y) = 0\right\}.$$

Assuming that there exist constants $\epsilon \in (0, 1]$ and $c_\delta \in \mathbb{R}_{>0}$ such that

$$\#N_{F_1}(\ell) = c_\delta \ell + O(\ell^{1-\epsilon}),$$

then

$$g(\ell, 1) = \frac{c_\delta \delta}{\ell} + O\left(\ell^{-1-\epsilon}\right),$$

and there exist constants $c_1 \in \mathbb{R}_{>0}$ and $c' \in \mathbb{R}$ such that

$$\sum_{\ell > x} g(\ell, 1)^2 = O\left(\frac{1}{\log x}\right) \quad \text{and} \quad \sum_{\ell \le x} g(\ell, 1) = c_1 \log\log x + c' + O\left(\frac{1}{\log x}\right).$$

Proof. According to Lemma 3.3, we have

$$\begin{aligned}
|C(\ell, 1)| &= \#\left\{u \in \mathrm{GL}_2(\mathbb{F}_\ell) : F_1(\mathrm{tr}\ u, \det u) = 0,\ \ \det u \in \mathbb{F}_\ell^{*\delta}\right\} \\
&= \sum_{(x,y) \in N_{F_1}(\ell)} \#\left\{u \in \mathrm{GL}_2(\mathbb{F}_\ell) : \mathrm{tr}\ u = x,\ \ \det u = y\right\} \\
&= \ell^2 \# N_{F_1}(\ell) + \ell \sum_{(x,y) \in N_{F_1}(\ell)} \left(\frac{x^2 - 4y}{\ell}\right) \\
&= \ell^3 c_\delta + O(\ell^{3-\epsilon}).
\end{aligned}$$

By Lemma 3.2, $|G(\ell,1)| = \ell^4/\delta + O(\ell^3)$, hence

$$g(\ell,1) = \frac{c_\delta \delta}{\ell} + O\left(\ell^{-1-\epsilon}\right),\ \ell \in L_\delta.$$

The first assertion follows easily from the Euler summation formula.

To check that $g(\ell,1)$ has average order $c \log\log x$, we need the Mertens' theorem for arithmetic progressions [16]: for any integer $m \geq 1$ and integer a which is coprime with m, there exists a constant $c_{m,a}$ such that

$$\sum_{\ell \leq x, \ell \equiv a(m)} \frac{1}{\ell} = \frac{1}{\varphi(m)} \log\log x + c_{m,a} + O\left(\frac{1}{\log x}\right),$$

where $\varphi(m)$ denotes Euler's totient function. If a is not coprime with m, the above sum is bounded as x varies. Note that the set $\{n \in \mathbb{N} : \gcd(n-1, k_1-1) = \delta\}$ can be divided into disjoint arithmetic progressions modulo $k_1 - 1$, so there exist constants $\alpha_\delta, \beta_\delta$ such that

$$\sum_{\ell \leq x, \ell \in L_\delta} \frac{1}{\ell} = \alpha_\delta \log\log x + \beta_\delta + O\left(\frac{1}{\log x}\right).$$

Then we have

$$\begin{aligned}\sum_{\ell \leq x} g(\ell,1) &= \sum_{\delta | k_1 - 1} \sum_{\ell \leq x, \ell \in L_\delta} g(\ell,1) \\ &= \sum_{\delta | k_1-1} c_\delta \delta \sum_{\ell \leq x, \ell \in L_\delta} \frac{1}{\ell} + O(1) \\ &= \left(\sum_{\delta | k_1 - 1} c_\delta \alpha_\delta \delta\right) \log\log x + c' + O\left(\frac{1}{\log x}\right). \quad \square\end{aligned}$$

Lemma 3.5. *For every $d \mid \gcd(k_1 - 1, k_2 - 1)$, define*

$$P_d = \{\ell : \gcd(k_1 - 1, k_2 - 1, \ell - 1) = d\}.$$

Given two bivariate polynomials $F_1(x,y), F_2(x,y)$ with integral coefficients, for every sufficiently large $\ell \in P_d$, let

$$N_{F_1,F_2}(\ell) = \{(x_1, x_2, y_1, y_2) \in \mathbb{F}_\ell \times \mathbb{F}_\ell \times D : F_i(x_i, y_i) = 0,\ i = 1, 2\}.$$

Assuming that there exist constants $\epsilon \in (0,1]$ and $c_d \in \mathbb{R}_{\geqslant 0}$ such that

$$\#N_{F_1,F_2}(\ell) = c_d \ell + O(\ell^{1-\epsilon}), \tag{6}$$

then

$$\sum_{\ell} g(\ell,\ell) < +\infty.$$

Proof. For any $\ell \in P_\delta$, by Lemma 3.3 we have

$$\begin{aligned}
&|C(\ell,\ell)| \\
&= \#\left\{(u_1,u_2) \in \mathrm{GL}_2(\mathbb{F}_\ell)\times \mathrm{GL}_2(\mathbb{F}_\ell) : \begin{array}{l} F_1(\mathrm{tr}\ u_1, \det u_1)=0, \\ F_2(\mathrm{tr}\ u_2, \det u_2)=0, \end{array} (\det u_1, \det u_2)\in D\right\} \\
&= \sum_{(x_i,y_i)\in N_{F_1,F_2}(\ell)} \#\left\{u_1 \in \mathrm{GL}_2(\mathbb{F}_\ell) : \begin{array}{l} \mathrm{tr}\ \ u_1 = x_1 \\ \det u_1 = y_1 \end{array}\right\} \#\left\{u_2 \in \mathrm{GL}_2(\mathbb{F}_\ell) : \begin{array}{l} \mathrm{tr}\ \ u_2 = x_2 \\ \det u_2 = y_2 \end{array}\right\} \\
&= \ell^4 \# N_{F_1,F_2}(\ell) + O(\ell^4) \\
&= \ell^5 c_d + O(\ell^{5-\epsilon}).
\end{aligned}$$

By Lemma 3.2, $|G(\ell,\ell)| = \ell^7/d + O(\ell^6)$, we have

$$g(\ell,\ell) = \frac{c_d d}{\ell^2} + O\left(\ell^{-2-\epsilon}\right), \text{ for } \ell \in P_d.$$

Hence

$$\sum_{\ell\le x} g(\ell,\ell) = \sum_{d|(k-1,k'-1)} \sum_{\ell\le x, \ell\in P_d} g(\ell,\ell) \ll \sum_{\ell\le x}\frac{1}{\ell^2},$$

the last series converges, which completes the proof. □

Lemma 3.6. *If $F_1(x,y) = x + r_1(y)$ and $F_2(x,y) = x + r_2(y)$, $r_1(y), r_2(y) \in \mathbb{Z}[y]$, then conditions (2) (3) are satisfied and the constants c_1, c_2 in condition (2) are equal to* 1.

Proof. For any $\delta \mid k-1$, $\ell \in L_\delta$,

$$\begin{aligned}
\# N_{F_1}(\ell) &= \sum_{y\in \mathbb{F}_\ell^{*\delta}} \#\left\{x \in \mathbb{F}_\ell : x = -r_1(y)\right\} \\
&= \sum_{y\in \mathbb{F}_\ell^{*\delta}} 1 = \frac{\ell-1}{\delta}.
\end{aligned}$$

By Lemma 3.4, $g(\ell,1) = \ell^{-1} + O(\ell^{-2})$ for all sufficiently large ℓ and in the same manner $g(1,\ell) = \ell^{-1} + O(\ell^{-2})$. Therefore condition (2) follows from the Mertens' theorem. Similarly for any $d \mid (k-1,k'-1)$, $\ell \in P_d$,

$$\begin{aligned}\#N_{F_1,F_2}(\ell) &= \sum_{(y_1,y_2)\in D} \#\left\{(x_1,x_2)\in \mathbb{F}_\ell\times\mathbb{F}_\ell : x_1=-r_1(y_1), x_2=-r_2(y_2)\right\}\\ &= \sum_{(y_1,y_2)\in D} 1 = \frac{\ell-1}{d},\end{aligned}$$

this calculation combined with Lemma 3.5 completes the proof. □

3.4. Conclusion

The polynomial growth condition C1 follows from Hecke's bound, which states that for cusp form f of weight k, $a_p(f) = O(p^{k/2})$. We have checked the multiplicativity of g and the error condition in Section 3.2, then combined with Lemma 3.6, all the conditions C1-C4 have been verified. We claim that $|T_x| \sim x/\log x$ as x goes to infinity, this follows from [15, Théorème 15]: $\#\left\{p \le x :\ a_p(f) + r_1(p^{k_1}) = 0\right\} = o(x/\log x)$. Hence Theorem 1.2 follows from Theorem 2.1 and Remark 3.

4. Remarks and generalizations

4.1. Examples

Let f and h be two newforms as in Theorem 1.2 and assuming that the generalized Riemann hypothesis is true for all Dedekind zeta functions of number fields. We choose $r_1 = r_2 = 0, -y-1$, respectively. Then by Theorem 1.2, we have

$$\lim_{x\to+\infty} \frac{1}{x/\log x}\#\left\{p \le x : a_p(f)\neq 0, a_p(h)\neq 0,\ \omega\left(a_p(f)\right) < \omega\left(a_p(h)\right)\right\} = \frac{1}{2},$$

$$\begin{aligned}&\lim_{x\to+\infty} \frac{1}{x/\log x}\#\left\{p \le x : p\nmid N_f N_h,\ \omega\left(p^{k_1-1} - a_p(f) + 1\right) < \omega\left(p^{k_2-1} - a_p(h) + 1\right)\right\}\\ &\qquad = \frac{1}{2}.\end{aligned}$$

We remark here that Theorem 1.2 also holds if we replace $a_p(f) + r_1(p^{k_1})$ and $a_p(h) + r_2(p^{k_2})$ by $F_1(a_p(f), p^{k_1})$ and $F_2(a_p(h), p^{k_2})$ if the bivariate polynomials F_1, F_2 satisfy conditions in Lemma 3.4 and Lemma 3.5. For example, take $F_1 = F_2 = x^2 - y$ and write $\delta_i = \gcd(\ell-1, k_i-1)$. Since k_i is even, we have

$$|N_{F_i}(\ell)| = \#\left\{(x,y)\in\mathbb{F}_\ell\times\mathbb{F}_\ell^{*\delta_i} : x^2 = y\right\} = \frac{\ell-1}{\delta_i}.$$

Let $D = \{(v^{k_1-1}, v^{k_2-1}) : v \in \mathbb{F}_\ell^*\}$, then

$$\begin{aligned}|N_{F_1,F_2}(\ell)| &= \#\left\{(x_1,x_2,y_1,y_2)\in\mathbb{F}_\ell\times\mathbb{F}_\ell\times D : x_1^2 = y_1,\ x_2^2 = y_2\right\}\\ &= \sum_{(y_1,y_2)\in D}\left(1+\left(\frac{y_1}{\ell}\right)\right)\left(1+\left(\frac{y_2}{\ell}\right)\right) = O(\ell).\end{aligned}$$

By Theorem 2.1, Remark 3, Lemma 3.4 and Lemma 3.5, we conclude that

$$\lim_{x\to+\infty}\frac{1}{x/\log x}\#\left\{p\le x: p\nmid N_fN_h,\ \omega\left(a_{p^2}(f)\right)<\omega\left(a_{p^2}(h)\right)\right\}=\frac{1}{2}.$$

4.2. On m newforms

Loeffler pointed out that the open image theorem also holds for three or more newforms which are not twists of each other (see [9, Theorem 3.4.2]). Similar arguments can be applied to these newforms and we have the following generalization.

Theorem 4.1. *Let $f_1,\dots,f_m$ be a family of non-CM newforms with integral Fourier coefficients of weights $k_1,\dots,k_m$, respectively. Let $r_1(x),\dots,r_m(x)$ be polynomials with integral coefficients. For every $x>0$, let*

$$T_x:=\left\{p\le x:\ a_p(f_i)+r_i(p^{k_i-1})\ne 0,\ i=1,\dots,m\right\}.$$

For simplicity of notation, write ω_i instead of $\omega\left(a_p(f_i)+r_i(p^{k_i-1}))\right)$. Assuming GRH, then either

- *there is a Dirichlet character χ such that $f_i=f_j\otimes\chi$ for some $i\ne j$;*
- *or for any permutation $\sigma\in S_n$,*

$$\lim_{x\to+\infty}\frac{1}{x/\log x}\#\left\{p\in T_x:\ \omega_{\sigma(1)}<\cdots<\omega_{\sigma(m)}\right\}=\frac{1}{m!}.$$

4.3. On combinations of newforms

Let f be a cusp form of weight k_1 such that

(i) $f=\sum_{i=1}^m a_if_i$, where each f_i is a non-CM newform with integral Fourier coefficients and $a_i\in\mathbb{Z}\setminus\{0\}$.
(ii) For $i\ne j$, there exists no Dirichlet character χ such that $f_i=f_j\otimes\chi$.

Let h be another cusp form of weight k_2 satisfying the above conditions. Write h as a sum of distinct newforms: $h=\sum_{j=1}^n b_jh_j$. We further assume that there exists no Dirichlet character χ such that $f_i=h_j\otimes\chi$ or $h_i=f_j\otimes\chi$ for $1\le i\le m, 1\le j\le n$. For every $x>0$, let

$$T_x:=\{p\le x:\ a_p(f)\ne 0 \text{ and } a_p(h)\ne 0\}.$$

Proposition 4.2. *With the above notation, under GRH, for any constant $C\ge 0$,*

$$\lim_{x\to+\infty}\frac{1}{x/\log x}\#\{p\in T_x:\ |\omega\left(a_p(f)\right)-\omega\left(a_p(h)\right)|\ge C\}=1.$$

Proof. The strategy is the same as the proof of Theorem 1.2, we sketch the proof in the following. We need to check that the pair $(a_p(f), a_p(h))$ satisfies C1, C2, C3 and C4 in Section 2. The polynomial growth condition is obvious. For two square-free integers d_1, d_2 with sufficiently large prime divisors, we consider the Galois representation

$$\overline{\rho}_{d_1,d_2} = \bigoplus_{1\le i\le m} \overline{\rho}_{f_i,d_1} \times \bigoplus_{1\le j\le n} \overline{\rho}_{h_j,d_2}.$$

The image of $\overline{\rho}_{d_1,d_2}$ is well described by Loeffler's theorem. It has a similar form to $G(d_1, d_2)$ in Section 3.1, denote by $\widetilde{G}(d_1, d_2)$ the image of $\overline{\rho}_{d_1,d_2}$. Without loss of generality, we write $d_1 = LP, d_2 = LQ, \gcd(P, Q) = 1$, then $\widetilde{G}(d_1, d_2)$ is the direct product of the following two groups:

$$\widetilde{G}(L, L) = \left\{ \begin{aligned} &(u_1, \dots, u_m, v_1, \dots, v_n) \in \prod_{i=1}^{m} \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) \times \prod_{j=1}^{n} \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}) : \\ &\forall i, j,\ \ \det u_i = \alpha^{k_1-1}, \det v_j = \alpha^{k_2-1} \text{ for some } \alpha \in (\mathbb{Z}/L\mathbb{Z})^* \end{aligned} \right\},$$

$$\widetilde{G}(P, Q) = \left\{ \begin{aligned} &(u_1, \dots, u_m, v_1, \dots, v_n) \in \prod_{i=1}^{m} \mathrm{GL}_2(\mathbb{Z}/P\mathbb{Z}) \times \prod_{j=1}^{n} \mathrm{GL}_2(\mathbb{Z}/Q\mathbb{Z}) : \\ &\forall i,\ \ \det u_i = \alpha^{k_1-1} \text{ for some } \alpha \in (\mathbb{Z}/P\mathbb{Z})^*, \\ &\forall j,\ \ \det v_j = \beta^{k_2-1} \text{ for some } \beta \in (\mathbb{Z}/Q\mathbb{Z})^* \end{aligned} \right\}.$$

By the Chebotarev density theorem, we have

$$\frac{1}{\pi(x)} \{p \le x : d_1 \mid a_p(f),\ d_2 \mid a_p(h)\} = \frac{|\widetilde{C}(d_1, d_2)|}{|\widetilde{G}(d_1, d_2)|} + \widetilde{e}(d_1, d_2; x),$$

where $\widetilde{C}(d_1, d_2)$ is the union of the conjugacy classes of $\widetilde{G}(d_1, d_2)$ whose elements satisfy a trace zero condition. For example, if $\gcd(d_1, d_2) = 1$ or $d_1 = d_2$, then

$$\widetilde{C}(d_1, d_2) = \left\{ (u_1, \dots, u_m, v_1, \dots, v_n) \in G(d_1, d_2) : \sum_{i=1}^{m} a_i \operatorname{tr} u_i = 0,\ \ \sum_{j=1}^{n} b_j \operatorname{tr} v_j = 0 \right\}.$$

Let $\widetilde{g}(d_1, d_2) := |\widetilde{C}(d_1, d_2)|/|\widetilde{G}(d_1, d_2)|$, according to our construction, the multiplicativity of $\widetilde{g}$ is obvious. Under GRH, the error condition (4) is satisfied. We need to check conditions (2) and (3). We claim that for any sufficiently large ℓ,

$$\widetilde{g}(\ell, 1) = \frac{1}{\ell} + O\left(\frac{1}{\ell^2}\right) \text{ and } \widetilde{g}(1, \ell) = \frac{1}{\ell} + O\left(\frac{1}{\ell^2}\right).$$

The proof runs as in that of Lemma 3.4. Let $\delta = (\ell - 1, k_1 - 1)$, the order of

$$\widetilde{G}(\ell,1)=\left\{(u_1,\ldots,u_m)\in\prod_{i=1}^{m}\mathrm{GL}_2(\mathbb{F}_\ell):\forall i,\ \det u_i=\alpha^{k_1-1},\ \alpha\in\mathbb{F}_\ell^*\right\}$$

is $\ell^{3m+1}/\delta+O(\ell^{3m})$. It remains to calculate the order of $\widetilde{C}(\ell,1)$. Note that the order of

$$N(\ell):=\{(x_1,\ldots,x_m)\in\mathbb{F}_\ell\times\cdots\times\mathbb{F}_\ell:a_1x_1+\cdots+a_mx_m=0\}$$

is ℓ^{m-1}. Let $\Delta:=\{(v^{k_1-1},\ldots,v^{k_1-1}):\ v\in\mathbb{F}_\ell^*\}$, we have

$$\begin{aligned}|\widetilde{C}(\ell,1)|&=\#\left\{(u_1,\ldots,u_m)\in\widetilde{G}(\ell,1):a_1\operatorname{tr}u_1+\cdots+a_m\operatorname{tr}u_m=0\right\}\\&=\sum_{(x_1,\ldots,x_m)\in N(\ell)}\sum_{(y_1,\ldots,y_m)\in\Delta}\prod_{1\le i\le m}\#\left\{u_i\in\mathrm{GL}_2(\mathbb{F}_\ell):\mathrm{tr}\ u_i=x_i,\ \det u_i=y_i\right\}\\&=\sum_{(x_1,\ldots,x_m)\in N(\ell)}\sum_{(y_1,\ldots,y_m)\in\Delta}\left(\ell^{2m}+O\left(\ell^{2m-1}\right)\right)\\&=\ell^{3m}/\delta+O\left(\ell^{3m-1}\right).\end{aligned}$$

Similar calculation holds for $\widetilde{g}(1,\ell)$ and the claim follows. Using the same argument as before, we have

$$|\widetilde{G}(\ell,\ell)|=\ell^{3m+3n+1}/d+O\left(\ell^{3m+3n}\right)\ \text{and}\ |\widetilde{C}(\ell,\ell)|=\ell^{3m+3n-1}/d+O\left(\ell^{3m+3n-2}\right),$$

where $d=\gcd(k_1-1,k_2-1,\ell-1)$. Hence

$$\sum_{\ell}\widetilde{g}(\ell,\ell)<+\infty.$$

Finally, by a standard sieve method combining Galois representations (for example see [17]), we have $\#\{p\le x:a_p(f)=0\}=o(x/\log x)$, hence $|T_x|\sim x/\log x$ as x goes to infinity. The result then follows from Theorem 2.1. □

Data availability

No data was used for the research described in the article.

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