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General Section

On the existence of maximal spanning vectors in $L^2(\mathbb{Q}_2)$ and $L^2(\mathbb{F}_2((T)))$ ☆

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We study the phase retrieval property for the Weyl-Heisenberg representation $\pi : \widehat{G} \times G \to \mathbf{U}(L^2(G))$, where G is a non-archimedean local field with residue characteristic 2. We prove that π does phase retrieval by explicitly constructing maximal spanning vectors in $L^2(G)$.

1. Introduction

The phase retrieval problem for frames was introduced in [1], with motivation coming from applications such as speech recognition and signal analysis. Yet in general, it is

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difficult to verify whether a frame has the phase retrieval property. We refer to [5,8] and their references for more details on this topic. On the other hand, the study of phase retrievable group frames has rich and interesting connections with abstract harmonic analysis, algebraic geometry, number theory, representation theory, etc. This paper is part of such an example.

Let H be a locally compact group and $r : H \to \mathbf{U}(V)$ be an irreducible (projective) group representation on a Hilbert space V. Recall that $v \in V$ is a *frame vector* for (r, H) if the map $H \to V$ $(h \mapsto r(h)v)$ is a frame (cf. [11, Definition 2.1]); a frame vector $v \in V$ is *phase retrievable* if the associated frame is phase retrievable, i.e., the map $H \to \mathbb{R}$ $(h \mapsto |\langle u, r(h)v\rangle|)$ determines $u \in V$ up to a unit constant; if there exists a phase retrievable frame vector for (r, H), we say that the representation r *does phase retrieval.* The main result of [9] proves that all irreducible projective representations of finite abelian groups do phase retrieval by showing the existence of the so called *maximal spanning vectors* (cf. [2,3,9]). This result has been generalized to nilpotent Lie groups and certain p-groups in a recent paper by Führ and Oussa [6].

In [4], Cheng-Lo-Xu studied the phase retrieval problem for continuous frames constructed from the Weyl-Heisenberg representation. Let G be a Hausdorff and second countable locally compact abelian group and $\widehat{G}$ be the Pontryagin dual of G. Let $\pi : \widehat{G} \times G \to \mathbf{U}(L^2(G))$ be the Weyl-Heisenberg representation, i.e. $\pi(u^*, u)f(v) = u^*(v)f(uv)$ for $(u^*, u) \in \widehat{G} \times G$ and $f \in L^2(G)$. For $f \neq 0$, the map $\widehat{G} \times G \to L^2(G)$ $((u^*, u) \mapsto \pi(u^*, u)f)$ is a continuous frame for $L^2(G)$ (cf. [7,11] and [4, Theorem 2.1]).

An element $f \in L^2(G)$ is called *maximal spanning* if

$$\overline{\mathrm{Span}}\{c_{\pi(u^*,u)f,\pi(u^*,u)f} :\ (u^*, u) \in \widehat{G} \times G\} = L^2(\widehat{G} \times G),$$

where $c_{x,y} : \widehat{G} \times G \to \mathbb{C}$ $(x,\ y \in L^2(G))$ is the matrix coefficient defined by

$$c_{x,y}(v^*, v) = \langle \pi(v^*, v)x, y\rangle.$$

Maximal spanning vectors are special as they are phase retrievable. This property provides a method to explicitly construct phase retrievable group frames. We refer to [2,9] and [4, Section 3.2.2] for more information on the relation between phase retrievable vectors and maximal spanning vectors.

In [4, Conjecture 1.4], Cheng-Lo-Xu conjectured that maximal spanning vectors exist for all such groups G. The main result of [4] verified the conjecture and constructed phase retrievable group frames for various groups G, in particular, for G a local field with residue characteristic $p \neq 2$. The condition $p \neq 2$ is essential for the computation in [4]. In this paper, we remove this condition, and hence provide further evidence for [4, Conjecture 1.4]. Our main result is the following theorem.

Theorem 1.1. *If $G = \mathbb{Q}_2$, $\mathbb{Z}_2$, $\mathbb{F}_2((T))$, $\mathbb{F}_2[[T]]$, then there exist maximal spanning vectors in $L^2(G)$. Therefore the Weyl-Heisenberg representation $\pi : \widehat{G} \times G \to \mathbf{U}(L^2(G))$ does phase retrieval.*

By [4, Proposition 3.3], f is maximal spanning if and only if $c_{f,f}(v^*,v) \neq 0$ for almost all $(v^*,v) \in \widehat{G} \times G$. Moreover, if f is a maximal spanning vector, then $\pi(u^*,u)f$ is a maximal spanning vector. We then prove Theorem 1.1 by explicitly constructing an $f \in L^2(G)$ with $c_{f,f}(v^*,v) \neq 0$ for all $(v^*,v) \in \widehat{G} \times G$. The proof of Theorem 1.1 is divided into two parts. In Section 3, we construct maximal spanning vectors in $L^2(\mathbb{Q}_2)$ and $L^2(\mathbb{Z}_2)$. In Section 4, we construct maximal spanning vectors in $L^2(\mathbb{F}_2((T)))$ and $L^2(\mathbb{F}_2[[T]])$.

For a general local field K with residue characteristic 2, one could explicitly construct maximal spanning vectors in $L^2(K)$ and prove a result analogous to Theorem 1.1. To save notation, we reduce the general case to the basic ones and obtain the following result.

Corollary 1.2. *Let K be a local field with residue characteristic 2, and let $\mathcal{O}_K$, U_K be the ring of integers of K and the group of units of K respectively. If $G = K$, $\mathcal{O}_K$, or $G = K^\times$, U_K in the case that K has characteristic 0, then there exist maximal spanning vectors in $L^2(G)$, hence the associated Weyl-Heisenberg representation does phase retrieval.*

Proof. As explained in [4, Remark 1.6], if I is a finite set and there exist maximal spanning vectors in $L^2(G_i)$ for every $i \in I$, then there exist maximal spanning vectors in $L^2(\oplus_{i\in I} G_i)$. Note that K is a finite direct sum of copies of $\mathbb{Q}_2$ or $\mathbb{F}_2((T))$ and $\mathcal{O}_K$ is a finite direct sum of copies of $\mathbb{Z}_2$ or $\mathbb{F}_2[[T]]$, so the case $G = K$ and case $G = \mathcal{O}_K$ follow easily from Theorem 1.1. The case $G = K^\times$ and case $G = U_K$ follow from Theorem 1.1, [4, Theorem 1.5], and the structure property of local fields [10, Chap 2, Section 5]. □

2. Notation and preliminaries

2.1. Notation

In the rest of this paper, $K = \mathbb{Q}_2$ or $\mathbb{F}_2((T))$. Let $\mathcal{O}$ be the ring of integers of K. Fix a uniformizer π of K. For example $\pi = 2$ if $K = \mathbb{Q}_2$ and $\pi = T$ if $K = \mathbb{F}_2((T))$. Denote by ord the valuation on K with $\operatorname{ord}(\pi) = 1$. For all integers n, let $\mathcal{O}_n$ be the fractional ideal $\pi^n\mathcal{O}$ and let $A_n = \mathcal{O}_n - \mathcal{O}_{n+1}$. Fix a Haar measure μ on K so that $\mu(\mathcal{O}) = 1$.

From [4, Remark 4.7], locally constant functions on K with respect to valuation are never maximal spanning. Hence we divide A_n into two parts and define A_n^+ and A_n^- by

$$A_n^+ = \pi^n + \mathcal{O}_{n+2}, \quad A_n^- = \pi^n + \pi^{n+1} + \mathcal{O}_{n+2}.$$

Then $\mu(A_n^+) = \mu(A_n^-) = \mu(A_{n+1}) = \mu(\mathcal{O}_{n+2}) = 2^{-(n+2)}$. The following table is straightforward, and as we will use it repeatedly in the computation, we state it here for the convenience of readers.

$b \in A_m^+$, $\mathbb{Q}_2$-case	$b \in A_m^-$, $\mathbb{Q}_2$-case	$b \in A_m^+$, $\mathbb{F}_2((T))$-case	$b \in A_m^-$, $\mathbb{F}_2((T))$-case
$b + A_{m-1}^+ = A_{m-1}^-$	$b + A_{m-1}^+ = A_{m-1}^-$	$b + A_{m-1}^+ = A_{m-1}^-$	$b + A_{m-1}^+ = A_{m-1}^-$
$b + A_{m-1}^- = A_{m-1}^+$	$b + A_{m-1}^- = A_{m-1}^+$	$b + A_{m-1}^- = A_{m-1}^+$	$b + A_{m-1}^- = A_{m-1}^+$
$b + A_m^+ = A_{m+1}$	$b + A_m^+ = \mathcal{O}_{m+2}$	$b + A_m^+ = \mathcal{O}_{m+2}$	$b + A_m^+ = A_{m+1}$
$b + A_m^- = \mathcal{O}_{m+2}$	$b + A_m^- = A_{m+1}^+$	$b + A_m^- = A_{m+1}^+$	$b + A_m^- = \mathcal{O}_{m+2}$
$b + A_{m+1}^+ \subset A_m^-$	$b + A_{m+1}^+ \subset A_m^+$	$b + A_{m+1}^+ \subset A_m^-$	$b + A_{m+1}^+ \subset A_m^+$
$b + A_{m+1}^- \subset A_m^-$	$b + A_{m+1}^- \subset A_m^+$	$b + A_{m+1}^- \subset A_m^-$	$b + A_{m+1}^- \subset A_m^+$
$b + A_{m+1} = A_m^-$	$b + A_{m+1} = A_m^+$	$b + A_{m+1} = A_m^-$	$b + A_{m+1} = A_m^+$

2.2. Characters

Let $\mathbb{T}$ be the unit circle and $\psi : K \to \mathbb{T}$ be the non-trivial character of $(K, +)$ as in Tate's thesis [12]. Recall that ψ is given as follows.

(1) If $K = \mathbb{Q}_2$, then ψ is the composition

$$\mathbb{Q}_2 \to \mathbb{Q}_2/\mathbb{Z}_2 \sim \mathbb{Z}[\frac{1}{2}]/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z} \cong \mathbb{T},$$

which is characterized by $\psi|_{\mathbb{Z}_2} = 1$ and $\psi(2^{-n}) = e^{2\pi i/2^n}$ for all $n \geq 1$.

(2) If $K = \mathbb{F}_2((T))$, then $\psi(\sum a_n T^n) := e^{\pi i a_{-1}}$. Here we lift $a_{-1} \in \mathbb{F}_2$ to $\mathbb{Z}$ to make sense of the definition.

We identity K with $\widehat{K}$ via $a \mapsto (\psi_a : x \mapsto \psi(ax))$. The *conductor* of a character $\phi : K \to \mathbb{T}$ is the integer l such that $\phi|_{\mathcal{O}_l}$ is not trivial and $\phi|_{\mathcal{O}_{l+1}}$ is trivial. Define $\operatorname{cond}(a)$ to be the conductor of ψ_a for $a \in K$. One easily sees that $\operatorname{cond}(a) + \operatorname{ord}(a) = -1$.

Lemma 2.1. *Let $\phi : K \to \mathbb{T}$ be a character of conductor l.*

(1) *If $K = \mathbb{Q}_2$, then*

$$\int_{A_n^+} \phi(x)\,\mathrm{d}\,x = \begin{cases} 0 & \textit{if } n \leq l-2, \\ \pm i\mu(A_n^+) & \textit{if } n = l-1, \\ -\mu(A_n^+) & \textit{if } n = l, \\ \mu(A_n^+) & \textit{if } n \geq l+1; \end{cases} \qquad \int_{A_n^-} \phi(x)\,\mathrm{d}\,x = \begin{cases} 0 & \textit{if } n \leq l-2, \\ \mp i\mu(A_n^-) & \textit{if } n = l-1, \\ -\mu(A_n^-) & \textit{if } n = l, \\ \mu(A_n^-) & \textit{if } n \geq l+1. \end{cases}$$

(2) *If $K = \mathbb{F}_2((T))$, then*

$$\int_{A_n^+} \phi(x)\,\mathrm{d}\,x = \begin{cases} 0 & \textit{if } n \leq l-2, \\ \pm \mu(A_n^+) & \textit{if } n = l-1, \\ -\mu(A_n^+) & \textit{if } n = l, \\ \mu(A_n^+) & \textit{if } n \geq l+1; \end{cases} \qquad \int_{A_n^-} \phi(x)\,\mathrm{d}\,x = \begin{cases} 0 & \textit{if } n \leq l-2, \\ \mp \mu(A_n^-) & \textit{if } n = l-1, \\ -\mu(A_n^-) & \textit{if } n = l, \\ \mu(A_n^-) & \textit{if } n \geq l+1. \end{cases}$$

Proof. As ϕ has conductor l, $\phi|_{\mathcal{O}_{l+1}} = 1$ and $\phi|_{A_l} = -1$. Hence the identities hold for $n \geq l$. Assume that $n \leq l-1$. Note that $A_n^- = A_n^+ + \pi^{n+1}$, so we have

$$\int\limits_{A_n^-} \phi(x)\,\mathrm{d}\,x = \phi(\pi^{n+1}) \int\limits_{A_n^+} \phi(x)\,\mathrm{d}\,x,$$

and

$$\begin{aligned}\int\limits_{A_n^-} \phi(x)\,\mathrm{d}\,x + \int\limits_{A_n^+} \phi(x)\,\mathrm{d}\,x &= \int\limits_{A_n} \phi(x)\,\mathrm{d}\,x \\ &= \int\limits_{\mathcal{O}_n} \phi(x)\,\mathrm{d}\,x - \int\limits_{\mathcal{O}_{n+1}} \phi(x)\,\mathrm{d}\,x = 0 - 0 = 0.\end{aligned}$$

In the $\mathbb{Q}_2$-case, if $n \leq l-2$, $\phi(\pi^{n+1}) \neq -1$; if $n = l-1$, then $\phi|_{A_n}$ is a constant function with value i or $-i$. The identities in part (1) hold.

In the $\mathbb{F}_2((T))$-case, since $\operatorname{ord}(x) + \operatorname{cond}(x) = -1$, we assume that $\phi = \psi_a$ with $a = T^{-l-1}(1 + \sum_{j=1}^{\infty} a_j T^j)$. Then

$$aA_n^+ = T^{n-l-1}(1 + \sum_{j=1}^{\infty} a_j T^j)(1 + T^2\mathbb{F}_2[[T]]).$$

If $n \leq l-2$ and $x \in A_n$, then $\phi(x) + \phi(x + T^l) = 0$. Hence

$$\mu\{x \in A_n^+ \mid \phi(x) = 1\} = \mu\{x \in A_n^+ \mid \phi(x) = -1\} = \frac{1}{2}\mu(A_n^+),$$

and

$$\mu\{x \in A_n^- \mid \phi(x) = 1\} = \mu\{x \in A_n^- \mid \phi(x) = -1\} = \frac{1}{2}\mu(A_n^-).$$

If $n = l-1$, then $\phi|_{A_n^+}$ is a constant function and

$$\phi|_{A_n^+} = \begin{cases} 1 & \text{if } a_1 = 0, \\ -1 & \text{if } a_1 = 1. \end{cases}$$

The identities in part (2) follow easily. □

2.3. The numbers $\delta(n)$

For $n \in \mathbb{Z}$, as in [4], define $\delta(n) = \begin{cases} 2^{2n} & \text{if } n < 0, \\ 2^{-n} & \text{if } n \geq 0. \end{cases}$ Then

$$\begin{aligned} \delta(n)\mu(A_n) &= \begin{cases} 2^{n-1} & \text{if } n < 0, \\ 2^{-2n-1} & \text{if } n \geq 0. \end{cases} \\ \delta(n)^2\mu(A_n) &= \begin{cases} 2^{3n-1} & \text{if } n < 0, \\ 2^{-3n-1} & \text{if } n \geq 0. \end{cases} \end{aligned} \tag{2.1}$$

Lemma 2.2. *With the above notation, the following claims hold.*

(1) *For any $m \in \mathbb{Z}$, $\sum_{n=m}^{+\infty} \delta(n)\mu(A_n)$ and $\sum_{n=m}^{+\infty} \delta(n)^2\mu(A_n)$ are positive rational numbers.*
(2) *For any $m \in \mathbb{Z}$, define $D_m := -\delta(m)\mu(A_m) + \sum_{n=m+1}^{+\infty} \delta(n)\mu(A_n)$. Then*

$$D_m = \begin{cases} \frac{7}{6} - 3 \cdot 2^{m-1} & \text{if } m \leq -1, \\ -\frac{1}{3}2^{-2m} & \text{if } m \geq 0. \end{cases}$$

Proof. Part (1) is obvious by equation (2.1). Part (2) follows from direct computation. Assume first that $m \leq -1$, then

$$\begin{aligned} D_m &= -2\delta(m)\mu(A_m) + \sum_{n=m}^{0} \delta(n)\mu(A_n) + \sum_{n=1}^{\infty} \delta(n)\mu(A_n) \\ &= -2^m + (1 - 2^{m-1}) + \frac{1}{6} = \frac{7}{6} - 3 \cdot 2^{m-1}. \end{aligned}$$

The case $m \geq 0$ is straightforward and the lemma follows. □

The numbers $\delta(n)$ are for the construction of maximal spanning vectors $f \in L^2(K)$ in Sections 3 and 4. There are infinitely many other choices for $\delta(n)$ and we choose the numbers as above since they are suitable for the verification of certain properties. For example, note that the prime 3 appears in the denominator of D_m, so a finite combination of $\delta(n)\mu(A_n)$ with coefficients in $\{\pm 2^n : n \in \mathbb{Z}\}$ never equals D_m.

3. A maximal spanning vector in $L^2(\mathbb{Q}_2)$

In this section, let $f \in L^2(\mathbb{Q}_2)$ be the function defined by

$$f(x) = \sum_{n \in \mathbb{Z}} \delta(n) \mathbf{1}_{A_n^+}(x),$$

where $\mathbf{1}_{A_n^+}$ is the characteristic function of the set A_n^+. The matrix coefficient of (f,f) is

$$c(a,b) = \langle \pi(a,b)f, f\rangle = \sum_{n\in\mathbb{Z}} \int_{A_n} \psi(ax)f(x+b)\overline{f(x)}\,\mathrm{d}\,x$$

$$= \sum_{n\in\mathbb{Z}} \delta(n) \int_{A_n^+} \psi(ax)f(x+b)\,\mathrm{d}\,x.$$

We have the following result.

Proposition 3.1. *With the above notation, $c(a,b)\neq 0$ for all $(a,b)\in\mathbb{Q}_2\times\mathbb{Q}_2$. Therefore, $f\in L^2(\mathbb{Q}_2)$ is a maximal spanning vector.*

Proof. Let $m = \operatorname{ord}(b)$ and $l = \operatorname{cond}(a)$. Then

$$c(a,b) = \sum_{n=-\infty}^{m-2} \delta(n)^2 \int_{A_n^+} \psi(ax)\,\mathrm{d}\,x + \delta(m-1) \int_{A_{m-1}^+} \psi(ax)f(x+b)\,\mathrm{d}\,x$$

$$+ \delta(m) \int_{A_m^+} \psi(ax)f(x+b)\,\mathrm{d}\,x + \delta(m+1) \int_{A_{m+1}^+} \psi(ax)f(x+b)\,\mathrm{d}\,x$$

$$+ \sum_{n=m+2}^{+\infty} \delta(n) \int_{A_n^+} \psi(ax)f(x+b)\,\mathrm{d}\,x.$$

Denote by (Term 1), (Term 2), (Term 3), (Term 4), (Term 5) the five terms on the right hand side of the above equation respectively. Note that $b + A_{m-1}^+ = A_{m-1}^-$ for any $b\in A_m$, so (Term 2) is always 0.

(1) Case $b\in A_m^+$. In this case, (Term 4) is always zero as $b + A_{m+1}^+ \subset A_m^-$.

- Case $l \le m-1$. We have

$$(\text{Term } 1) = \sum_{n=l-1}^{m-2} \delta(n)^2 \int_{A_n^+} \psi(ax)\,\mathrm{d}\,x,$$

$$(\text{Term } 3) = \delta(m) \int_{A_m^+} \psi(ax)f(x+b)\,\mathrm{d}\,x = \delta(m) \int_{A_m^+} f(x+b)\,\mathrm{d}\,x$$

$$= \delta(m) \int_{A_{m+1}} f(x)\,\mathrm{d}\,x = \delta(m)\delta(m+1)\mu(A_{m+1}^+) \in \mathbb{Q}_{>0},$$

and

$$(\text{Term } 5) = \delta(m) \sum_{n=m+2}^{\infty} \delta(n)\mu(A_n^+) \in \mathbb{Q}_{>0}.$$

The first term of (Term 1) is $\delta(l-1)^2 \int_{A_{l-1}^+} \psi(ax)\,\mathrm{d}\,x$, which is nonzero and pure imaginary by Lemma 2.1. The other terms of (Term 1) are rational numbers. Therefore $c(a,b) \neq 0$ as (Term 3) and (Term 5) are rational numbers.

- Case $l \geq m$. In this case (Term 1) is zero,

$$\begin{aligned}(\text{Term } 3) &= \delta(m) \int\limits_{A_m^+} \psi(ax) f(x+b)\,\mathrm{d}\,x \\ &= \delta(m)\psi(-ab) \int\limits_{A_{m+1}} \psi(ax) f(x)\,\mathrm{d}\,x \\ &= \delta(m)\delta(m+1)\psi(-ab) \int\limits_{A_{m+1}^+} \psi(ax)\,\mathrm{d}\,x,\end{aligned}$$

and

$$(\text{Term } 5) = \delta(m) \sum_{n=m+2}^{\infty} \delta(n) \int\limits_{A_n^+} \psi(ax)\,\mathrm{d}\,x.$$

By Lemma 2.1, we have

$$(\text{Term } 3) = \begin{cases} \delta(m)\delta(m+1)\psi(-ab)\mu(A_{m+1}^+) & \text{if } l = m, \\ -\delta(m)\delta(m+1)\psi(-ab)\mu(A_{m+1}^+) & \text{if } l = m+1, \\ \pm i\delta(m)\delta(m+1)\psi(-ab)\mu(A_{m+1}^+) & \text{if } l = m+2, \\ 0 & \text{if } l \geq m+3; \end{cases}$$

and

$$(\text{Term } 5) = \begin{cases} \delta(m) \sum_{n=m+2}^{\infty} \delta(n)\mu(A_n^+) & \text{if } l = m, \\ \delta(m) \sum_{n=m+2}^{\infty} \delta(n)\mu(A_n^+) & \text{if } l = m+1, \\ -\delta(m)\delta(m+2)\mu(A_{m+2}^+) + \delta(m) \sum_{n=m+3}^{\infty} \delta(n)\mu(A_n^+) & \text{if } l = m+2, \\ \delta(m) \sum_{n=l-1}^{\infty} \delta(n) \int_{A_n^+} \psi(ax)\,\mathrm{d}\,x & \text{if } l \geq m+3. \end{cases}$$

It is easy to verify that $c(a,b) = (\text{Term } 3) + (\text{Term } 5) \neq 0$.

(2) Case $b \in A_m^-$. In this case (Term 5) is zero as $f(x+b) = 0$ for $x \in A_n^+$ with $n \geq m+2$.

- Case $l \leq m-1$. We have

$$(\text{Term } 1) = \sum_{n=l-1}^{m-2} \delta(n)^2 \int\limits_{A_n^+} \psi(ax)\,\mathrm{d}\,x,$$

$$\begin{aligned}(\text{Term } 3) &= \delta(m) \int_{A_m^+} \psi(ax) f(x+b)\, \mathrm{d}\, x = \delta(m) \int_{A_m^+} f(x+b)\, \mathrm{d}\, x \\ &= \delta(m) \int_{\mathcal{O}_{m+2}} f(x)\, \mathrm{d}\, x = \delta(m) \sum_{n=m+2}^{\infty} \delta(n) \mu(A_n^+) \in \mathbb{Q}_{>0},\end{aligned}$$

and

$$\begin{aligned}(\text{Term } 4) &= \delta(m+1)\delta(m) \int_{A_{m+1}^+} \psi(ax)\, \mathrm{d}\, x \\ &= \delta(m)\delta(m+1)\mu(A_{m+1}^+) \in \mathbb{Q}_{>0}.\end{aligned}$$

Then $c(a,b) \neq 0$ as it has nontrivial imaginary part.

- Case $l \geq m$. We have

$$\begin{aligned}(\text{Term } 3) &= \delta(m) \int_{A_m^+} \psi(ax) f(x+b)\, \mathrm{d}\, x \\ &= \delta(m)\psi(-ab) \int_{\mathcal{O}_{m+2}} \psi(ax) f(x)\, \mathrm{d}\, x \\ &= \delta(m)\psi(-ab) \sum_{n=m+2}^{\infty} \int_{A_n^+} \psi(ax) f(x)\, \mathrm{d}\, x \\ &= \delta(m)\psi(-ab) \sum_{n=m+2}^{\infty} \delta(n) \int_{A_n^+} \psi(ax)\, \mathrm{d}\, x,\end{aligned}$$

and

$$(\text{Term } 4) = \delta(m)\delta(m+1) \int_{A_{m+1}^+} \psi(ax)\, \mathrm{d}\, x.$$

By Lemma 2.1, we have

$$(\text{Term } 3) = \begin{cases} \delta(m)\psi(-ab) \sum_{n=m+2}^{\infty} \delta(n)\mu(A_n^+) & \text{if } l = m, \\ \delta(m)\psi(-ab) \sum_{n=m+2}^{\infty} \delta(n)\mu(A_n^+) & \text{if } l = m+1, \\ \delta(m)\psi(-ab)(-\delta(m+2)\mu(A_{m+2}^+) + \sum_{n=m+3}^{\infty} \delta(n)\mu(A_n^+)) & \text{if } l = m+2, \\ \delta(m)\psi(-ab) \sum_{n=l-1}^{\infty} \delta(n) \int_{A_n^+} \psi(ax)\, \mathrm{d}\, x & \text{if } l \geq m+3; \end{cases}$$

and

$$
(\text{Term } 4) = \begin{cases} \delta(m)\delta(m+1)\mu(A_{m+1}^{+}) & \text{if } l = m, \\ -\delta(m)\delta(m+1)\mu(A_{m+1}^{+}) & \text{if } l = m+1, \\ \pm i\delta(m)\delta(m+1)\mu(A_{m+1}^{+}) & \text{if } l = m+2, \\ 0 & \text{if } l \geq m+3. \end{cases}
$$

It is easy to verify that $c(a,b) = (\text{Term } 3) + (\text{Term } 4) \neq 0$.

From the above computation, $c(a,b) \neq 0$ for all $(a,b) \in \mathbb{Q}_2 \times \mathbb{Q}_2$ and the proposition follows. □

As $\widehat{\mathbb{Z}_2} \cong \mathbb{Q}_2/\mathbb{Z}_2$, the following result holds by a similar argument.

Proposition 3.2. *Let $g = f|_{\mathbb{Z}_2}$. Then $c_{g,g}(a,b) \neq 0$ for all $(a,b) \in \widehat{\mathbb{Z}_2} \times \mathbb{Z}_2$. Hence $g \in L^2(\mathbb{Z}_2)$ is a maximal spanning vector.*

4. A maximal spanning vector in $L^2(\mathbb{F}_2((T)))$

The construction in Section 3 does not work for case $G = \mathbb{F}_2((T))$. For example, if $\operatorname{ord}(b) = m$, then

$$
\begin{aligned}
c(a,b) = \sum_{n=-\infty}^{m-1} \int_{A_n} \psi(ax) f(x+b) \overline{f(x)} \, \mathrm{d}\, x \\
+ \int_{A_m} \psi(ax) f(x+b) \overline{f(x)} \, \mathrm{d}\, x + \int_{\mathcal{O}_{m+1}} \psi(ax) f(x+b) \overline{f(x)} \, \mathrm{d}\, x.
\end{aligned}
$$

In characteristic 2 we have $b = -b$. Since $b + A_m = \mathcal{O}_{m+1}$ and $\psi(x) = \pm 1$, by change of variables,

$$
\begin{aligned}
\int_{A_m} \psi(ax) f(x+b) \overline{f(x)} \, \mathrm{d}\, x &= \psi(-ab) \int_{A_m} \psi(a(x+b)) f(x+b) \overline{f(x)} \, \mathrm{d}\, x \\
&= \psi(-ab) \int_{\mathcal{O}_{m+1}} \psi(ax) f(x) \overline{f(x+b)} \, \mathrm{d}\, x \\
&= \psi(-ab) \overline{\int_{\mathcal{O}_{m+1}} \psi(ax) f(x+b) \overline{f(x)} \, \mathrm{d}\, x}.
\end{aligned} \tag{4.1}
$$

Hence if $\sum_{n=-\infty}^{m-1} \int_{A_n} \psi(ax) f(x+b) \overline{f(x)} \, \mathrm{d}\, x = 0$, the function f is real, and $\psi(-ab) = -1$, then $c(a,b) = 0$. This observation motivates the following construction.

Let $f \in L^2(\mathbb{F}_2((T)))$ be the function defined by

$$f(x) = \sum_{n\in\mathbb{Z}} \delta(n)\mathbf{1}_{A_n^+}(x) + i\delta(n)\mathbf{1}_{A_n^-}(x),$$

where $\mathbf{1}_{A_n^+}$ and $\mathbf{1}_{A_n^-}$ are the characteristic functions of the sets A_n^+ and A_n^- respectively. Let $c(a,b) \in L^2(\mathbb{F}_2((T)) \times \mathbb{F}_2((T)))$ be the matrix coefficient of (f,f), i.e.

$$c(a,b) = \langle \pi(a,b)f, f\rangle = \sum_{n\in\mathbb{Z}} \int_{A_n} \psi(ax)f(x+b)\overline{f(x)}\,\mathrm{d}\,x.$$

We have the following result.

Proposition 4.1. *With the above notation, $c(a,b) \neq 0$ for all $(a,b) \in \mathbb{F}_2((T)) \times \mathbb{F}_2((T))$. Hence $f \in L^2(\mathbb{F}_2((T)))$ is a maximal spanning vector.*

Proof. Let $m = \mathrm{ord}(b)$ and $l = \mathrm{cond}(a)$. Then

$$\begin{aligned}
c(a,b) &= \sum_{n\in\mathbb{Z}} \int_{A_n} \psi(ax)f(x+b)\overline{f(x)}\,\mathrm{d}\,x \\
&= \sum_{n=-\infty}^{m-2} \delta(n)^2 \int_{A_n^+} \psi(ax)\,\mathrm{d}\,x + \sum_{n=-\infty}^{m-2} \delta(n)^2 \int_{A_n^-} \psi(ax)\,\mathrm{d}\,x \\
&\quad + \delta(m-1) \int_{A_{m-1}^+} \psi(ax)f(x+b)\,\mathrm{d}\,x - i\delta(m-1) \int_{A_{m-1}^-} \psi(ax)f(x+b)\,\mathrm{d}\,x \\
&\quad + \delta(m) \int_{A_m^+} \psi(ax)f(x+b)\,\mathrm{d}\,x - i\delta(m) \int_{A_m^-} \psi(ax)f(x+b)\,\mathrm{d}\,x \\
&\quad + \delta(m+1) \int_{A_{m+1}^+} \psi(ax)f(x+b)\,\mathrm{d}\,x - i\delta(m+1) \int_{A_{m+1}^-} \psi(ax)f(x+b)\,\mathrm{d}\,x \\
&\quad + \sum_{n=m+2} \delta(n) \int_{A_n^+} \psi(ax)f(x+b)\,\mathrm{d}\,x - \sum_{n=m+2} i\delta(n) \int_{A_n^-} \psi(ax)f(x+b)\,\mathrm{d}\,x.
\end{aligned}$$

Denote the terms in last 5 lines by (Term 1), (Term 2), (Term 3), (Term 4), (Term 5) respectively. Note that they are just the integration of $\psi(ax)f(x+b)\overline{f(x)}$ over $(\mathbb{F}_2((T)) - \mathcal{O}_{m-1})$, A_{m-1}, A_m, A_{m+1}, $\mathcal{O}_{m+2}$ respectively.

(1) Case $b \in A_m^+$.

- Case $l \leq m-1$. We have

$$(\text{Term } 1) = \sum_{n=l-1}^{m-2} \delta(n)^2 \int_{A_n} \psi(ax)\,\mathrm{d}\,x \in \mathbb{Q},$$

$$(\text{Term } 2) = i\delta(m-1)^2 \Big(\int_{A_{m-1}^+} \psi(ax)\,\mathrm{d}\,x - \int_{A_{m-1}^-} \psi(ax)\,\mathrm{d}\,x \Big) = 0,$$

$$(\text{Term } 4) = i\delta(m)(\delta(m+1)\mu(A_{m+1}^+) - i\delta(m+1)\mu(A_{m+1}^-)),$$

$$(\text{Term } 5) = \delta(m) \sum_{n=m+2}^{\infty} (\delta(n)\mu(A_n^+) - i\delta(n)\mu(A_n^-)).$$

Moreover, since $\psi(-ab) = 1$, by equation (4.1),

$$(\text{Term } 3) = \overline{(\text{Term } 4) + (\text{Term } 5)}.$$

Therefore

$$(\text{Term } 3) + (\text{Term } 4) + (\text{Term } 5) = 2\delta(m) \sum_{n=m+1}^{\infty} \delta(n)\mu(A_n^+) \in \mathbb{Q}_{>0}.$$

If $l = m-1$, then (Term 1) is zero and $c(a,b) \in \mathbb{Q}_{>0}$.
Assume that $l \leq m-2$. Then we have the following three cases.
– If $m-1 < 0$, then $l < 0$ and

$$c(a,b) \geq 2\delta(m) \sum_{n=1}^{\infty} \delta(n)\mu(A_n^+) - \delta(l)^2\mu(A_l) > 0.$$

– If $m-1 \geq 0$ and $l < 0$, then

$$c(a,b) > \delta(1)^2\mu(A_1) - \delta(l)^2\mu(A_l) \geq 0.$$

– If $m-1 \geq 0$ and $l \geq 0$, then $\delta(l) > \delta(l+1) > \cdots$ and

$$\begin{aligned} c(a,b) &= -\delta(l)^2\mu(A_l) + \sum_{n=l+1}^{m-2} \delta(n)^2\mu(A_n) + 2\delta(m) \sum_{n=m+1}^{\infty} \delta(n)\mu(A_n^+) \\ &< -\delta(l)^2\mu(A_l) + \delta(l) \sum_{n=l+1}^{\infty} \delta(n)\mu(A_n) < 0. \end{aligned}$$

- Case $l = m$. In this case (Term 1) is zero.

$$\text{(Term 2)} = i\delta(m-1)^2(\int\limits_{A^+_{m-1}} \psi(ax)\,\mathrm{d}\,x - \int\limits_{A^-_{m-1}} \psi(ax)\,\mathrm{d}\,x)$$
$$= \pm i\delta(m-1)^2\mu(A_{m-1}).$$

Since $\psi(ax) = -1$ for all $x \in A_m$, by equation (4.1),

$$\text{(Term3)} = -\overline{\text{(Term 4)} + \text{(Term 5)}}.$$

Note that (Term 4) and (Term 5) are the same as in case $l \leq m-1$, we have

$$\begin{aligned}
-ic(a,b) &= \pm\,\delta(m-1)^2\mu(A_{m-1}) + 2\mathbf{Im}(\int\limits_{\mathcal{O}_{m+1}} f(x+b)\overline{f(x)}\,\mathrm{d}\,x) \\
&= \pm\,\delta(m-1)^2\mu(A_{m-1}) \\
&\quad + \delta(m)\delta(m+1)\mu(A_{m+1}) - \delta(m)\sum_{n=m+2}^{\infty}\delta(n)\mu(A_n) \\
&= \pm\,\delta(m-1)^2\mu(A_{m-1}) - \delta(m)D_{m+1}.
\end{aligned}$$

By Lemma 2.2, $c(a,b) \neq 0$.

- Case $l \geq m+1$. In this case (Term 1) and (Term 2) are zero. By equation (4.1),

$$\text{(Term 3)} = \psi(-ab)\overline{(\text{(Term 4)} + \text{(Term 5)})}.$$

To check that $c(a,b) \neq 0$, it suffices to check that (Term 4)+(Term 5) is neither real nor pure imaginary. We have

$$\begin{aligned}
&\text{(Term 4)} + \text{(Term 5)} \\
=&\delta(m+1)\int\limits_{A^+_{m+1}} \psi(ax)f(x+b)\,\mathrm{d}\,x - i\delta(m+1)\int\limits_{A^-_{m+1}} \psi(ax)f(x+b)\,\mathrm{d}\,x \\
&+ \delta(m)\sum_{n=m+2}^{\infty}(\delta(n)\int\limits_{A^+_n} \psi(ax)\,\mathrm{d}\,x - i\delta(n)\int\limits_{A^-_n} \psi(ax)\,\mathrm{d}\,x) \\
=&\delta(m)(\delta(m+1)\int\limits_{A^-_{m+1}} \psi(ax)\,\mathrm{d}\,x + \sum_{n=m+2}^{\infty}\delta(n)\int\limits_{A^+_n} \psi(ax)\,\mathrm{d}\,x) \\
&+ i\delta(m)(\delta(m+1)\int\limits_{A^+_{m+1}} \psi(ax)\,\mathrm{d}\,x - \sum_{n=m+2}^{\infty}\delta(n)\int\limits_{A^-_n} \psi(ax)\,\mathrm{d}\,x).
\end{aligned}$$

Therefore, it suffices to check that

$$\delta(m+1)\int_{A_{m+1}^-}\psi(ax)\,\mathrm{d}\,x+\sum_{n=m+2}^{\infty}\delta(n)\int_{A_n^+}\psi(ax)\,\mathrm{d}\,x$$

and

$$\delta(m+1)\int_{A_{m+1}^+}\psi(ax)\,\mathrm{d}\,x-\sum_{n=m+2}^{\infty}\delta(n)\int_{A_n^-}\psi(ax)\,\mathrm{d}\,x$$

are nonzero.

Indeed, if $l=m+1$, these two numbers are

$$\frac{1}{2}D_{m+1}\text{ and }-\sum_{n=m+1}\delta(n)\mu(A_n^+),$$

which are not zero by Lemma 2.2;

if $l\geq m+2$, these two numbers are

$$\frac{1}{2}(\pm\delta(l-1)\mu(A_{l-1})+D_l)\text{ and }\frac{1}{2}(\pm\delta(l-1)\mu(A_{l-1})-D_l),$$

which are not zero by Lemma 2.2 again.

(2) Case $b\in A_m^-$. The situation is *symmetric* to the case $b\in A_m^+$. We compute the corresponding terms accordingly and the verification is similar.

- Case $l\leq m-1$.

$$(\text{Term }1)=\sum_{n=l-1}^{m-2}\delta(n)^2\int_{A_n}\psi(ax)\,\mathrm{d}\,x\in\mathbb{Q},$$

$$(\text{Term }2)=i\delta(m-1)^2(\int_{A_{m-1}^+}\psi(ax)\,\mathrm{d}\,x-\int_{A_{m-1}^-}\psi(ax)\,\mathrm{d}\,x)=0,$$

$$(\text{Term }4)=\delta(m)\delta(m+1)\mu(A_{m+1}^+)-i\delta(m)\delta(m+1)\mu(A_{m+1}^-),$$

$$(\text{Term }5)=\delta(m)\sum_{n=m+2}^{\infty}(\delta(n)\mu(A_n^-)+i\delta(n)\mu(A_n^+)),$$

$$(\text{Term}3)=\overline{(\text{Term }4)+(\text{Term }5)}.$$

- Case $l = m$. (Term 1) is zero,

$$\text{(Term 2)} = i\delta(m-1)^2\Big(\int\limits_{A_{m-1}^{+}} \psi(ax)\,\mathrm{d}\,x - \int\limits_{A_{m-1}^{-}} \psi(ax)\,\mathrm{d}\,x\Big)$$
$$= \pm i\delta(m-1)^2\mu(A_{m-1}),$$
$$\text{(Term3)} = -\overline{\text{(Term 4)} + \text{(Term 5)}},$$
$$-ic(a,b) = \pm\,\delta(m-1)^2\mu(A_{m-1}) + 2\mathbf{Im}\Big(\int\limits_{\mathcal{O}_{m+1}} f(x+b)\overline{f(x)}\,\mathrm{d}\,x\Big)$$
$$= \pm\,\delta(m-1)^2\mu(A_{m-1})$$
$$- \delta(m)\delta(m+1)\mu(A_{m+1}) + \delta(m)\sum_{n=m+2}^{\infty}\delta(n)\mu(A_n).$$

- Case $l > m$. (Term 1) and (Term 2) are zero.

$$\text{(Term 3)} = \psi(-ab)\overline{(\text{(Term 4)} + \text{(Term 5)})}.$$

The real part and the imaginary part of (Term 4)+(Term 5) are

$$\delta(m)\delta(m+1)\int\limits_{A_{m+1}^{+}} \psi(ax)\,\mathrm{d}\,x + \delta(m)\sum_{n=m+2}^{\infty}\delta(n)\int\limits_{A_n^{-}} \psi(ax)\,\mathrm{d}\,x$$

and

$$-\delta(m)\delta(m+1)\int\limits_{A_{m+1}^{-}} \psi(ax)\,\mathrm{d}\,x + \delta(m)\sum_{n=m+2}^{\infty}\delta(n)\int\limits_{A_n^{+}} \psi(ax)\,\mathrm{d}\,x$$

respectively.

From the above computation, $c(a,b) \neq 0$ for all $(a,b) \in \mathbb{F}_2((T)) \times \mathbb{F}_2((T))$ and the proposition follows. □

As $\widehat{\mathbb{F}_2[[T]]} \cong \mathbb{F}_2((T))/\mathbb{F}_2[[T]]$, the following result holds by a similar argument.

Proposition 4.2. *Let* $g = f|_{\mathbb{F}_2[[T]]}$. *Then* $c_{g,g}(a,b) \neq 0$ *for all* $(a,b) \in \widehat{\mathbb{F}_2[[T]]} \times \mathbb{F}_2[[T]]$. *Hence* $g \in L^2(\mathbb{F}_2[[T]])$ *is a maximal spanning vector.*

Data availability

No data was used for the research described in the article.

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