ON DRINFELD MODULES WITH GLOBAL MINIMAL EQUATIONS

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Abstract. Let K be a finite separable extension of $\mathbb{F}_q(T)$, and let S be a finite set of places of K containing all places above infinity. In this paper, we investigate the existence and asymptotic distribution of global minimal equations of Drinfeld modules over K. To a rank-r Drinfeld module ϕ over K, we attach an S-Drinfeld class $\overline{\mathfrak{a}}_{\phi/K}$ in the S-class group of K and show that the triviality of $\overline{\mathfrak{a}}_{\phi/K}$ is equivalent to the existence of an S-global minimal equation for ϕ/K. If $(q,r) \neq (2,1)$, we define an asymptotic density of rank r Drinfeld modules admitting S-global minimal equation and show that the density is

$$\frac{\zeta_{K,S}(\mathfrak{C}_0, B_r)}{\zeta_{K,S}(B_r)}, \qquad B_r = \sum_{i=1}^{r}(q^i - 1),$$

where $\zeta_{K,S}(s)$ is the S-zeta function and $\zeta_{K,S}(\mathfrak{C}, s)$ is the partial S-zeta function of the trivial ideal class $\mathfrak{C}_0$. For Drinfeld modules of rank $r \geq 2$, we establish asymptotic equidistribution of S-Drinfeld classes among K-isomorphism classes with respect to a weighted-projective height, subject to a natural congruence condition on the height. Each admissible class has density $\delta/h_{K,S}$, where δ is the greatest common divisor of the degrees of places in S and $h_{K,S}$ is the S-class number of K.

1. Introduction

In this paper we study global minimal equations of Drinfeld modules over global function fields and prove results analogous to those for elliptic curves over number fields. Let F be a number field. Each elliptic curve E/F is associated with a certain ideal class of F whose triviality is a necessary and sufficient condition for E to have a global minimal Weierstrass equation. In particular, if F has class number 1, then there exists a global minimal Weierstrass equation for every elliptic curve E/F (cf. [12, 15]). In [13], motivated by Fröhlich's results [4, 5] on the integral basis problem for finite extensions of number fields, Silverman showed that if F has class number greater than 1, then there exists an elliptic curve E/F which does not have a global minimal Weierstrass equation. The existence result has been strengthened and the asymptotic density of elliptic curves having a global minimal Weierstrass equation has been computed by Bekyel [1, 2].

Let $A = \mathbb{F}_q[T]$, K be a finite separable extension of $\mathbb{F}_q(T)$. In [6], Jeon defined the notion of global minimal equations for Drinfeld modules of rank 2 over K and obtained results similar to Silverman's by introducing the minimal discriminant of a Drinfeld module. The main results of [6] hold true for general rank r Drinfeld modules with small modifications. In this paper, we study S-global minimal equations of Drinfeld modules of arbitrary rank over K, and compute an asymptotic density of rank r Drinfeld modules having an S-global

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minimal equation and an asymptotic density of isomorphism classes of rank r Drinfeld modules having an S-global minimal equation. In what follows, we set up the notation for this paper, recap the basic concepts of Drinfeld modules, and present the main results in detail.

1.1. **Notation and convention.** In this paper, q is a power of a prime number p, $\mathbb{F}_q$ is the finite field with q elements, $A = \mathbb{F}_q[T]$ is the polynomial ring, $k = \mathbb{F}_q(T)$ is the fraction field of A. Let ∞ denote the place of k defined by the valuation at $1/T$.

Let K/k be a finite separable extension with constant field F_K and denote by Q the cardinality of F_K. Let X be the smooth projective curve over F_K with function field K; denote by g the genus of X. We identify a place P with its associated prime divisor, and to each place P we associate its normalized discrete valuation

$$v_P : K^\times \to \mathbb{Z} \cup \{\infty\}.$$

Let $\mathcal{S}_K$ be the set of places of K. For $P \in \mathcal{S}_K$, let K_P denote the completion of K at P and $\mathcal{O}_P$ denote its valuation ring. Denote by $\kappa(P)$ the residue field and let $d_P := [\kappa(P) : F_K]$ be the degree of P.

Let $S \subset \mathcal{S}_K$ be a finite set which is always assumed to contain all places of K lying above ∞. Let

$$\mathcal{O}_{K,S} = \{x \in K : \forall P \in \mathcal{S}_K - S, v_P(x) \geq 0\}$$

denote the ring of S-integers.

A nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_{K,S}$ is called an *S-integral ideal*. A divisor $D = \sum_{P \in \mathcal{S}_K - S} a_P P$ supported outside S is called *S-effective* if $a_P \geq 0$ for all $P \in \mathcal{S}_K - S$. Let $\deg D := \sum_{P \in \mathcal{S}_K} a_P d_P$ denote the degree of D.

For $a \in K^\times$, define its principal S-divisor to be $(a)_S = \sum_{P \in \mathcal{S}_K - S} v_P(a) P$. For a divisor supported outside S, its class in the quotient by principal S-divisors is called its *S-divisor class*. Denote by $\mathrm{Cl}_S(K)$ the S-class group of K and by $h_{K,S}$ the S-class number of K.

For $\mathfrak{C} \in \mathrm{Cl}_S(K)$, the *partial S-zeta function* is defined by

$$\zeta_{K,S}(\mathfrak{C}, s) := \sum_{\substack{D \ S\text{-effective} \\ [D] = \mathfrak{C}}} Q^{-s \deg D}.$$

The *S-zeta function* of $\mathcal{O}_{K,S}$ is defined by

$$\zeta_{K,S}(s) := \sum_{\mathfrak{C} \in \mathrm{Cl}_S(K)} \zeta_{K,S}(\mathfrak{C}, s) = \prod_{P \in \mathcal{S}_K - S} (1 - Q^{-s d_P})^{-1}.$$

1.2. **Drinfeld modules.** We recall the basic definitions and facts about Drinfeld modules used throughout the paper. For more information we refer to [7, 10].

Let R be an A-algebra and $R\{\tau\}$ be the noncommutative ring in one variable τ with commutation relation $\tau u = u^q \tau$ for all $u \in R$. Let $\iota : A \to R$ be the structure morphism.

Definition 1.1. A *Drinfeld module over R* is a homomorphism $\phi : A \to R\{\tau\}$ $(a \mapsto \phi_a)$ of $\mathbb{F}_q$-algebras, such that

(1) the constant term of ϕ_a is $\iota(a)$;
(2) for some $a \in A$, $\phi_a \neq \iota(a)$;
(3) for all nonzero $a \in A$, the leading coefficient of ϕ_a is invertible in R.

If R is a field, then there exists a positive integer r with

$$\deg_\tau \phi_a = r \cdot \deg a, \text{ for all nonzero } a \in A.$$

This integer r is called the *rank* of ϕ.

Let ϕ and ψ be two Drinfeld modules over R. By a *homomorphism* $u : \phi \to \psi$, we mean an element $u \in R\{\tau\}$ such that

$$u\phi_a = \psi_a u, \quad \text{for all } a \in A.$$

A homomorphism u is called an *isomorphism* if $u \in R^\times$.

From the definition, a rank r Drinfeld module ϕ is determined by ϕ_T, which is a polynomial of degree r in $R\{\tau\}$. Write ϕ_T as

$$\phi_T = \iota(T) + g_1\tau + g_2\tau^2 + \cdots + g_r\tau^r, \quad g_i \in R,\ g_r \neq 0.$$

If $u : \phi \to \psi$ is an isomorphism of Drinfeld modules over R, then

$$\psi_T = \iota(T) + u^{1-q}g_1\tau + u^{1-q^2}g_2\tau^2 + \cdots + u^{1-q^r}g_r\tau^r, \tag{1.1}$$

and we write $\psi = u\phi u^{-1}$.

1.3. **Main results.** We start with the following definition.

Definition 1.2. Let ϕ be a rank r Drinfeld module over K.

(1) Let $P \in \mathcal{S}_K$. For a Drinfeld module ψ which is isomorphic to ϕ over K, the equation $\psi_T = \iota(T) + h_1\tau + \cdots + h_r\tau^r$ is called a *minimal equation for ϕ at P* if $h_1, \ldots, h_r \in \mathcal{O}_{K,P}$ and $v_P(h_r)$ is minimal subject to this condition.

(2) An equation as in (1) is called an *S-global minimal equation* for ϕ over K if for all $P \in \mathcal{S}_K - S$, it is a minimal equation for ϕ at P.

Let ϕ be a rank r Drinfeld module over K with $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$. For each $P \in \mathcal{S}_K - S$, choose $u_P \in K_P^\times$ such that

$$\iota(T) + u_P^{1-q}g_1\tau + \cdots + u_P^{1-q^r}g_r\tau^r$$

is a minimal equation for ϕ at P. The exponent $v_P(u_P)$ is independent of the choices (see Remark 2.3), and u_P can be taken to be 1 for all but finitely many P. The *S-Drinfeld divisor* of ϕ over K is the divisor

$$\mathfrak{a}_{\phi/K} := -\sum_{P \in \mathcal{S}_K - S} v_P(u_P)P,$$

and the *S-Drinfeld class* $\overline{\mathfrak{a}}_{\phi/K}$ of ϕ over K is the class of $\mathfrak{a}_{\phi/K}$ in $\mathrm{Cl}_S(K)$. The class $\overline{\mathfrak{a}}_{\phi/K}$ depends only on the K-isomorphism class of ϕ. Moreover, $\overline{\mathfrak{a}}_{\phi/K}$ is trivial if and only if ϕ admits an S-global minimal equation over K (cf. Theorem 3.3).

1.3.1. *The asymptotic equation density.* For each $P \in \mathcal{S}_K$, normalize the absolute value by $|a|_P = Q^{-d_P v_P(a)}$. For $g_1, \ldots, g_r \in \mathcal{O}_{K,S}$, let

$$H([g_1, \ldots, g_r, 1]) = \prod_{P \in \mathcal{S}_K} \max\{|g_1|_P, \ldots, |g_r|_P, 1\}$$

be the height function. For $\mathfrak{C} \in \mathrm{Cl}_S(K)$ and $t \in \mathbb{R}$, let $N_r(\mathfrak{C}, t)$ be the number of r-tuples $(g_1, \ldots, g_r) \in \mathcal{O}_{K,S}^r$ such that $g_r \neq 0$, $H([g_1, \ldots, g_r, 1]) \leq t$, and the rank r Drinfeld module ϕ over K determined by $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$ has S-Drinfeld class $\mathfrak{C}$. Let

$D_r(t) = \sum_{\mathfrak{C}\in \mathrm{Cl}_S(K)} N_r(\mathfrak{C}, t)$. The *asymptotic equation density* of the class $\mathfrak{C}$ is defined to be

$$\rho_r^{\mathrm{eq}}(\mathfrak{C}) := \lim_{t\to\infty} \frac{N_r(\mathfrak{C}, t)}{D_r(t)}$$

if the limit exists. We have the following result.

Theorem 1.1. *Let K be a finite separable extension of $\mathbb{F}_q(T)$ and S be a finite set of places of K containing all places above ∞. Fix an integer $r \geq 1$.*

(1) *Assume that $(q, r) \neq (2, 1)$. Then every rank r Drinfeld module over K has an S-global minimal equation if and only if the S-class number of K is one. In the exceptional case $(q, r) = (2, 1)$, every rank-one Drinfeld module over K has an S-global minimal equation.*

(2) *Assume that $(q, r) \neq (2, 1)$. For every S-divisor class $\mathfrak{C}$ of K, the asymptotic equation density $\rho_r^{\mathrm{eq}}(\mathfrak{C})$ exists and*

$$\rho_r^{\mathrm{eq}}(\mathfrak{C}) = \frac{\zeta_{K,S}(\mathfrak{C}^{-1}, B_r)}{\zeta_{K,S}(B_r)},$$

where $B_r = \sum_{i=1}^{r}(q^i - 1)$. In particular, if $\mathfrak{C}_0$ is the trivial S-divisor class, then the asymptotic density of rank r Drinfeld module equations with S-integral coefficients that admit an S-global minimal equation is

$$\rho_r^{\mathrm{eq}}(\mathfrak{C}_0) = \frac{\zeta_{K,S}(\mathfrak{C}_0, B_r)}{\zeta_{K,S}(B_r)}.$$

The distribution of the asymptotic density appearing in the formula of Theorem 1.1 is not uniform in general (cf. Remark 4.8). The reason is that in Theorem 1.1 (2), we are counting the number of Drinfeld module equations parameterized by r-tuples. However, after passing from equations to isomorphism classes and counting inside the stack of weighted projective spaces by using the idea of [8], the contribution of the different classes becomes asymptotically equidistributed (cf. Theorem 1.2 below).

1.3.2. *The asymptotic isomorphism-class density.* Let $r \geq 2$. Fix a place $P_\infty \in S$ above ∞, and put $v_\infty = v_{P_\infty}$, $d_\infty = d_{P_\infty}$. For a rank r Drinfeld module ϕ over K determined by $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$, the *weighted height* of ϕ is

$$\mathrm{ht}_{\mathbf{w}_r}(\phi) = -\sum_{P\in\mathcal{S}_K} d_P \min_{1\leq i\leq r}\left\lfloor \frac{v_P(g_i)}{q^i - 1}\right\rfloor, \qquad \mathbf{w}_r = (q-1, \ldots, q^r - 1).$$

This is a special case of [8, Definition 2.3.1]. For a number b, let $N_r^{\mathrm{iso}}(b)$ be the number of K-isomorphism classes of rank r Drinfeld modules over K with $\mathrm{ht}_{\mathbf{w}_r}(\phi) = b$, and let $N_{r,S}^{\mathrm{iso}}(\mathfrak{C}, b)$ be the number of such classes with S-Drinfeld class $\mathfrak{C}$. Put $\delta = \gcd\{d_v : v \in S\}$, and let $\deg_S : \mathrm{Cl}_S(K) \to \mathbb{Z}/\delta\mathbb{Z}$ be the surjective degree map induced by the degree of divisors (cf. equation (5.5)).

For $\mathfrak{C} \in \mathrm{Cl}_S(K)$, the *asymptotic isomorphism-class density* of the class $\mathfrak{C}$ is defined to be

$$\rho_r^{\mathrm{iso}}(\mathfrak{C}) := \lim_{\substack{b\to\infty \\ b\equiv \deg_S(\mathfrak{C}) \pmod{\delta}}} \frac{N_{r,S}^{\mathrm{iso}}(\mathfrak{C}, b)}{N_r^{\mathrm{iso}}(b)}$$

if the limit exists. We have the following result.

Theorem 1.2. *Let $r \geq 2$ and $\mathfrak{C} \in \mathrm{Cl}_S(K)$. If $b \not\equiv \deg_S(\mathfrak{C}) \pmod{\delta}$, then $N^{\mathrm{iso}}_{r,S}(\mathfrak{C}, b) = 0$. If $b \equiv \deg_S(\mathfrak{C}) \pmod{\delta}$, the asymptotic isomorphism-class density of $\mathfrak{C}$ exists and*

$$\rho^{\mathrm{iso}}_r(\mathfrak{C}) = \frac{\delta}{h_{K,S}}.$$

The content of the paper is as follows. In Section 2 we study local minimal equations. In Section 3 we introduce S-Drinfeld class, study S-global minimal equations, and prove part (1) of Theorem 1.1. In Sections 4 and 5, we prove Theorem 1.1(2) and Theorem 1.2 respectively.

2. Minimal equation: the local case

In this part, we fix the following setting:

- $\mathcal{O}$ is an A-algebra, which is also a discrete valuation ring. Denote by v the normalized valuation, ϖ a fixed uniformizer.
- $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$ and $F = \mathcal{O}/\mathfrak{m}$.
- L is the fraction field of $\mathcal{O}$.
- ϕ is a Drinfeld module over L of rank $r \geq 1$.

Definition 2.1.
(1) We say that ϕ has *good reduction* if ϕ is isomorphic over L to a Drinfeld module over $\mathcal{O}$, i.e. if after replacing ϕ by an isomorphic Drinfeld module over L, for all nonzero $a \in A$ the coefficients of ϕ_a are in $\mathcal{O}$ and the leading coefficient of ϕ_a is in $\mathcal{O}^\times$.
(2) We say that ϕ has *stable reduction* if after replacing ϕ by an isomorphic Drinfeld module over L, all the ϕ_a have coefficients in $\mathcal{O}$, and $a \mapsto (\phi_a \mod \mathfrak{m})$ is a Drinfeld module over F of positive rank.
(3) For a Drinfeld module ψ which is isomorphic to ϕ over L, the equation $\psi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$ is called a *minimal equation* for ϕ over L if
   (a) $g_1, \ldots, g_r \in \mathcal{O}$,
   (b) $v(g_r)$ is minimal subject to the above condition.

Remark 2.1. From the definition, it is easy to see that ϕ has good reduction if and only if it has a minimal equation such that the leading coefficient has valuation zero.

From equation (1.1), one may show that every Drinfeld module ϕ over L of rank $r \geq 1$ has potentially stable reduction, i.e. there exists a finite ramified extension L' of L such that ϕ over L' has stable reduction. See for example [10, Theorem 4.2].

Remark 2.2. Since v is a discrete valuation, every Drinfeld module over L has a minimal equation. Let $u \in L^\times$ be an isomorphism from ϕ to ψ. If

$$\phi_T = \iota(T) + g_1\tau + g_2\tau^2 + \cdots + g_r\tau^r,$$

then

$$\psi_T = \iota(T) + u^{1-q}g_1\tau + u^{1-q^2}g_2\tau^2 + \cdots + u^{1-q^r}g_r\tau^r.$$

We conclude that if $g_1, \ldots, g_r \in \mathcal{O}$, then ϕ_T is a minimal equation for ϕ if and only if

$$v(g_i) < q^i - 1, \quad \text{for some } i \in \{1, \ldots, r\}.$$

Remark 2.3. Let $\phi_T = \iota(T) + g_1\tau + \cdots + g_{r-1}\tau^{r-1} + g_r\tau^r$ and $\psi_T = \iota(T) + h_1\tau + \cdots + h_{r-1}\tau^{r-1} + h_r\tau^r$. If ϕ_T and ψ_T are minimal equations for the same Drinfeld module over L, then there exists $u \in \mathcal{O}^\times$ such that $g_i = u^{q^i-1}h_i$ for all $1 \leq i \leq r$. In particular, the leading coefficient of the minimal equations for a rank r Drinfeld module over L is unique up to $(\mathcal{O}^\times)^{q^r-1}$.

Let Ω_L be the subset of $\mathcal{O}^r$ consisting of tuples $(g_1,\ ...,\ g_r)$ with $g_r \neq 0$. Each $(g_1,\ ...,\ g_r) \in \Omega_L$ determines a rank r Drinfeld module ϕ over L with $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$. Let $u \in L^\times$ such that $u\phi_T u^{-1} = \iota(T) + u^{1-q}g_1\tau + \cdots + u^{1-q^r}g_r\tau^r$ is minimal for ϕ over L. Then $v(u)$ is a non-negative integer independent of the choice of the minimalizing scalar u for the given equation. Define map

$$\begin{aligned} f : \Omega_L &\to \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ (g_1,\ \ldots,\ g_r) &\mapsto v(u), \end{aligned}$$

and consider f as a map on $\mathcal{O}^r$ by setting $f(g_1,\ \ldots,\ g_{r-1},\ 0) = \infty$. For each $b \in \mathbb{Z}_{\geq 0}$, let π_b be the natural projection

$$\mathcal{O}^r = \mathcal{O} \times \mathcal{O} \times \cdots \times \mathcal{O} \to \prod_{i=1}^{r} \mathcal{O}/\mathfrak{m}^{(b+1)(q^i-1)}.$$

Let C_b be the image of $f^{-1}(b)$ under the map π_b.

Proposition 2.4. *Let $(g_1,\ \ldots,\ g_r) \in \Omega_L$. Then $f(g_1,\ \ldots,\ g_r) = b$ if and only if $\pi_b(g_1,\ \ldots,\ g_r) \in C_b$.*

Proof. The only if part is trivial. For the other direction, we show that for $(h_1, \ldots, h_r) \in \Omega_L$, if $f(g_1,\ \ldots,\ g_r) = b$, and $g_i \equiv h_i \pmod{\mathfrak{m}^{(b+1)(q^i-1)}}$ for all $1 \leq i \leq r$, then $f(h_1,\ \ldots,\ h_r) = b$.

Let $u \in L^\times$ such that $u\phi_T u^{-1} = \iota(T) + u^{1-q}g_1\tau + \cdots + u^{1-q^r}g_r\tau^r$ is minimal for ϕ over L. Then $v(u) = b$ and $v(g_i) \geq b(q^i - 1)$ for all $1 \leq i \leq r$. The same u transfers $(h_1,\ \ldots,\ h_r)$ to $(u^{1-q}h_1,\ \ldots,\ u^{1-q^r}h_r)$. As $g_i \equiv h_i \pmod{\mathfrak{m}^{(b+1)(q^i-1)}}$ for all $1 \leq i \leq r$, $(u^{1-q}h_1,\ \ldots,\ u^{1-q^r}h_r) \in \Omega_L$.

Suppose that $\iota(T) + u^{1-q}h_1\tau + \cdots + u^{1-q^r}h_r\tau^r$ is not minimal. Choose $u' \in L^\times$ such that $\iota(T) + (u'u)^{1-q}h_1\tau + \cdots + (u'u)^{1-q^r}h_r\tau^r$ is minimal. Then $v(u') = b' \geq 1$ and $v(h_i) \geq (b + b')(q^i - 1)$ for all $1 \leq i \leq r$. Therefore

$$v(g_i) \geq (b+1)(q^i - 1) \text{ for all } 1 \leq i \leq r.$$

The polynomial $(\varpi u)\phi_T(\varpi u)^{-1}$ has integral coefficients and the valuation of its leading coefficient is strictly less than the valuation of $u^{1-q^r}g_r$. This contradicts the assumption. Therefore $\iota(T) + u^{1-q}h_1\tau + \cdots + u^{1-q^r}h_r\tau^r$ is minimal and $f(h_1,\ \ldots,\ h_r) = b$. □

Assume that $F = \mathcal{O}/\mathfrak{m}$ is a finite field. Let $N_\mathfrak{m}$ be the cardinality of F.

Proposition 2.5. *Let c_b be the cardinality of C_b for $b \in \mathbb{Z}_{\geq 0}$. Then*

$$c_b = N_\mathfrak{m}^{B_r} - 1, \text{ where } B_r := \sum_{i=1}^{r}(q^i - 1) = q\frac{1-q^r}{1-q} - r.$$

Proof. Let $(g_1, \ldots, g_r) \in \Omega_L$. Then $f(g_1, \ldots, g_r) = 0$ if and only if $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$ is minimal, i.e.

$$v(g_i) \geq 0 \text{ for all } i, \quad v(g_i) < q^i - 1 \text{ for at least one } i.$$

Then one sees that the image of π_0 misses only one point $(0, \ldots, 0)$. Therefore $c_0 = N_{\mathfrak{m}}^{B_r} - 1$.

For $b \geq 1$, let $(g_1, \ldots, g_r) \in \Omega_L$. Then $f(g_1, \ldots, g_r) = b$ if and only if

$$f(\varpi^{b(1-q)}g_1, \ldots, \varpi^{b(1-q^r)}g_r) = 0.$$

This induces a bijection between C_0 and C_b and the proposition follows. □

3. Minimal equation: the global case

In this part, we fix the following setting:

- K is a finite separable extension of k with genus g, $\mathcal{O}_K$ is the integral closure of A in K, $\iota : A \to K$ is the natural embedding.
- F_K is the constant field of K, $Q = \#F_K$.
- $\mathcal{S}_K$ is a complete set of places of K, $S \subset \mathcal{S}_K$ is a finite subset containing all places of K lying over ∞ of A. $\mathcal{O}_{K,S}$ is the ring of S-integers.
- For each $P \in \mathcal{S}_K$, denote by K_P the completion of K at P, denote the valuation ring of K_P by $\mathcal{O}_{K,P}$. Let $\kappa(P)$ denote the residue field and $d_P = [\kappa(P) : F_K]$ denote the degree of P. Choose a uniformizer π_P of $\mathcal{O}_{K,P}$. Let $U_P = \mathcal{O}_{K,P}^{\times}$.
- ϕ is a Drinfeld module over K of rank $r \geq 1$.

Definition 3.1. Let $P \in \mathcal{S}_K$ be a place of K.

(1) We say that ϕ has *good reduction at P* (resp. *stable reduction at P*) if ϕ has good reduction (resp. stable reduction) as a Drinfeld module over K_P.

(2) For a Drinfeld module ψ which is isomorphic to ϕ over K, the equation $\psi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$ is called a *minimal equation at P* for ϕ over K if
   (a) $g_1, \ldots, g_r \in \mathcal{O}_{K,P}$,
   (b) $v_P(g_r)$ is minimal subject to the above condition.

   An equation as above is called an *S-global minimal equation* for ϕ over K if it is simultaneously minimal for every $P \in \mathcal{S}_K - S$.

Proposition 3.1. *If $(q, r) = (2, 1)$, every rank one Drinfeld module over K has an S-global minimal equation.*

Proof. Write $\phi_T = \iota(T) + g\tau$, $g \in K^{\times}$. Since $q = 2$, conjugation by $u = g$ gives

$$(g\phi g^{-1})_T = \iota(T) + g^{1-2}g\tau = \iota(T) + \tau.$$

This equation is minimal at every place. □

In the rest of this paper, we assume that $(q, r) \neq (2, 1)$.

Let $g_{r,P}$ be the leading coefficient of a minimal equation for ϕ over K at P. Define the *S-minimal discriminant divisor* $\mathcal{D}_{\phi/K}$ of ϕ over K by the formula

$$\mathcal{D}_{\phi/K} = \sum_{P \in \mathcal{S}_K - S} v_P(g_{r,P})P. \tag{3.1}$$

Note that if $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$, then for $P \in \mathcal{S}_K - S$ with $v_P(g_i) = 0$ for all $1 \leq i \leq r$, ϕ_T is already a minimal equation for ϕ at P. Hence for almost all P, $v_P(g_{r,P}) = 0$ and the sum is a finite sum. A Drinfeld module ϕ over K has an S-global minimal equation over K if and only if there is a Drinfeld module ψ which is isomorphic to ϕ over K, such that the coefficients of ψ_T are in $\mathcal{O}_{K,S}$ and the S-divisor of the leading coefficient of ψ_T equals $\mathcal{D}_{\phi/K}$.

Let $J = J_K(S)$ be the group of K-ideles without S-components and $U \subset J$ be the subgroup with P-component $\mathcal{O}_{K,P}^\times$ for all $P \in \mathcal{S}_K - S$. The *S-minimal discriminant* of ϕ over K is the element $\Delta(\phi/K) \in J/U^{q^r-1}$ with the property that, for all $P \in \mathcal{S}_K - S$, the local component $\Delta(\phi/K)_P \in K_P^\times/U_P^{q^r-1}$ contains the leading coefficient of any minimal equation for ϕ over K at P. It is well-defined by Remark 2.3.

Proposition 3.2. *Let ϕ be a rank r Drinfeld module over K. The S-minimal discriminant $\Delta(\phi/K)$ lies in $K^\times J^{q^r-1}/U^{q^r-1}$.*

Proof. Let g_r be the leading coefficient of ϕ_T. From the definition, for each P, there exists $u_P \in K_P^\times$ such that $\Delta(\phi/K)_P = u_P^{q^r-1} g_r$. Moreover, $u_P = 1$ for almost all $P \in \mathcal{S}_K$. Let $u \in J$ be the idele with local components u_P for $P \in \mathcal{S}_K - S$. Then $\Delta(\phi/K) \equiv g_r u^{q^r-1} \pmod{U^{q^r-1}}$. The proposition follows. □

The S-minimal discriminant divisor is precisely the S-divisor associated with $\Delta(\phi/K)$. From Proposition 3.2, the minimal discriminant divisor class is a (q^r-1)-th power in the S-divisor class group $\mathrm{Cl}_S(K)$. We could be more precise by introducing the Drinfeld class.

For the Drinfeld module ϕ over K with $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$ and $P \in \mathcal{S}_K - S$, suppose that a minimal equation for ϕ at P is given by

$$\iota(T) + u_P^{1-q} g_1 \tau + \cdots + u_P^{1-q^r} g_r \tau^r,$$

for some $u_P \in K_P^\times$.

Definition 3.2. The *S-Drinfeld divisor of ϕ over K* is the divisor

$$\mathfrak{a}_{\phi/K} := -\sum_{P \in \mathcal{S}_K - S} v_P(u_P) P.$$

The *S-Drinfeld class of ϕ over K* is the class of $\mathfrak{a}_{\phi/K}$ in $\mathrm{Cl}_S(K)$ and we denote it by $\overline{\mathfrak{a}}_{\phi/K}$.

From the definition, we have

$$\mathcal{D}_{\phi/K} = (g_r)_S + (q^r - 1)\mathfrak{a}_{\phi/K}. \tag{3.2}$$

The element $\overline{\mathfrak{a}}_{\phi/K}$ depends only on the isomorphism class of ϕ since

$$\mathfrak{a}_{\phi/K} = \mathfrak{a}_{u\phi u^{-1}/K} - (u)_S.$$

From equation (3.2), the minimal discriminant divisor class is the (q^r-1)-th power of $\overline{\mathfrak{a}}_{\phi/K}$. In terms of ideles, consider the following natural maps

$$\mu : \frac{K^\times J^{q^r-1}}{U^{q^r-1}} \twoheadrightarrow \frac{K^\times J^{q^r-1}}{K^\times U^{q^r-1}} \cong \frac{J^{q^r-1}}{(K^\times)^{q^r-1} U^{q^r-1}} \twoheadleftarrow \frac{J}{K^\times U}. \tag{3.3}$$

Then it is easy to see that $\mu(\Delta_{\phi/K}) = \overline{\mathfrak{a}}_{\phi/K}$.

Theorem 3.3. *Let ϕ be a rank r Drinfeld module over K. The following conditions are equivalent.*

(1) $\Delta_{\phi/K} \in K^\times U^{q^r-1}/U^{q^r-1}$;
(2) $\overline{\mathfrak{a}}_{\phi/K}$ *is the trivial* S*-divisor class;*
(3) ϕ *has an* S*-global minimal equation over* K*.*

Proof. The equivalence between (1) and (2) follows from $\mu(\Delta_{\phi/K}) = \overline{\mathfrak{a}}_{\phi/K}$.

(3)⇒(2). Let $\psi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$ be an S-global minimal equation for ϕ over K. Then for each $P \in \mathcal{S}_K - S$, we may take $\Delta(\phi/K)_P = g_r$, hence $\mu(\Delta(\phi/K))$ is the trivial S-divisor class.

(2)⇒(3). Let $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$. For each $P \in \mathcal{S}_K - S$, there exists $u_P \in K_P^\times$ such that $u_P\phi_T u_P^{-1}$ is minimal for ϕ at P. Then

$$\mathfrak{a}_{\phi/K} = -\sum_{P\in\mathcal{S}_K-S} v_P(u_P)P.$$

Assume that $\mathfrak{a}_{\phi/K} = (u)_S$ for some $u \in K^\times$. Then $v_P(u_P) = -v_P(u)$ for all $P \in \mathcal{S}_K - S$. Let $\psi = u^{-1}\phi u$. It is easy to check that

$$\psi_T = \iota(T) + u^{q-1}g_1\tau + \cdots + u^{q^r-1}g_r\tau^r$$

is a minimal equation for ϕ at any $P \in \mathcal{S}_K - S$, i.e. it is an S-global minimal equation. □

Remark 3.4. The phenomenon in Proposition 3.1 is special to $q = 2$. For example, let $q = 3$, and $E : y^2 = x^3 - x$ be an elliptic curve with function field $K = \mathbb{F}_3(x, y)$. Let the A-structure be given by $T \mapsto x$, and let S consist of the unique point at infinity. The curve has four $\mathbb{F}_3$-points $\infty,\ (0,0),\ (1,0),\ (2,0)$, and

$$\mathrm{Cl}_S(K) \simeq E(\mathbb{F}_3) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

If $P = (0,0)$, then $\mathrm{div}(x) = 2P - 2\infty$. For the rank one Drinfeld module $\phi_T = x + x\tau$, the local minimalization exponent at P is 1, while it is 0 at every other finite place. Hence

$$\overline{\mathfrak{a}}_{\phi/K} = [P] \neq 0,$$

and ϕ has no S-global minimal equation.

Corollary 3.5. *If the* S*-class number of* K *is* 1*, then any Drinfeld module over* K *has an* S*-global minimal equation over* K*.*

Corollary 3.6 (Setzer-type result)**.** *Let* r *be a positive integer. If the* S*-class number of* K *is prime to* $q^r - 1$*, then any rank* r *Drinfeld module over* K *with good reduction outside* S *has an* S*-global minimal equation.*

Proof. Let ϕ/K be a rank r Drinfeld module. The fact that ϕ has good reduction outside S implies that every leading coefficient of a local minimal equation is a unit. Hence we have $\mathcal{D}_{\phi/K} = 0$. Taking divisor classes in equation (3.2) gives

$$(q^r - 1)\overline{\mathfrak{a}}_{\phi/K} = 0.$$

Since the S-class number of K is prime to $q^r - 1$, the class $\overline{\mathfrak{a}}_{\phi/K}$ is trivial. Theorem 3.3 now applies. □

Theorem 3.7. *Assume that* $(q, r) \neq (2, 1)$*. Let* $\mathfrak{C}$ *be an* S*-divisor class of* K*. There exists a rank* r *Drinfeld module* ϕ *over* K *with* S*-Drinfeld class* $\overline{\mathfrak{a}}_{\phi/K} = \mathfrak{C}$*.*

Proof. By the Tchebotarev Density Theorem (cf. [11, Theorem 9.13]), there exist infinitely many prime divisors P in $\mathfrak{C}$. Let P_0, P_1, ..., P_r be distinct prime ideals in $\mathfrak{C}$ and v_0, v_1, ..., v_r be the associated normalized valuations. The fractional ideals $P_0P_i^{-1}$ $(1 \leq i \leq r)$ are principal. Let $a_i \in K^\times$ be such that

$$P_0P_i^{-1} = (a_i), \quad 1 \leq i \leq r.$$

Set $g_i = a_i^{-1}$ for $1 \leq i \leq r$ and consider the rank r Drinfeld module ϕ over K with

$$\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r.$$

We claim that ϕ is a rank r Drinfeld module over K with S-Drinfeld class $\overline{\mathfrak{a}}_{\phi/K} = \mathfrak{C}$.

Indeed, for $P \notin S \cup \{P_0, P_1, ..., P_r\}$, $v_P(g_i) = 0$ for $1 \leq i \leq r$ and ϕ_T is minimal for ϕ/K at P.

For $P = P_i$ and $i \neq 0$, we have

$$v_P(g_i) = v_i(g_i) = v_i(a_i^{-1}) = 1, \quad v_P(g_j) = 0 \text{ for } j \neq i.$$

Hence ϕ_T is minimal at $P \in \{P_1,\ ...,\ P_r\}$.

For $P = P_0$, let $\psi = \pi_{P_0}^{-1}\phi\pi_{P_0}$. Then $\psi_T = \iota(T) + \pi_{P_0}^{q-1}g_1\tau + \cdots + \pi_{P_0}^{q^r-1}g_r\tau^r$. Since

$$v_0(\pi_{P_0}^{q^i-1}g_i) = q^i - 1 + v_0(g_i) = q^i - 2 \text{ for } 1 \leq i \leq r,$$

ψ_T is minimal at P_0 for ϕ/K. Note that we have excluded the case $(q, r) = (2, 1)$, thus $q^r - 2 > 0$.

It is then easy to check that $\overline{\mathfrak{a}}_{\phi/K} = [P_0] = \mathfrak{C}$ and the theorem follows. □

Corollary 3.8. *Let r be a positive integer and $(q, r) \neq (2, 1)$. If the S-class number of K is greater than 1, then there exists a rank r Drinfeld module over K which does not have an S-global minimal equation over K.*

4. Counting Drinfeld modules

In this section, we use the notation from Section 3 and fix the following setting:

- For each place $P \in \mathcal{S}_K$ we normalize the absolute value by $|a|_P = Q^{-d_P v_P(a)}$.
- For g_1, g_2, ..., $g_r \in \mathcal{O}_{K,S}$, define the height function by

$$H([g_1,\ ...,\ g_r,\ 1]) = \prod_{P \in \mathcal{S}_K} \max\{|g_1|_P,\ ...,\ |g_r|_P,\ 1\}.$$

- For two functions $f(x)$ and $g(x)$ with $g(x) \geq 0$, we write $f(x) = O(g(x))$ if there exists a constant $c \in \mathbb{R}_+$ such that $|f(x)| \leq cg(x)$.

For a given S-divisor class of K, we compute the asymptotic equation density of rank r Drinfeld modules over K whose Drinfeld S-class is the given class. Let $\mathfrak{C}$ be an S-divisor class of K and let $N_r(\mathfrak{C}, t)$ be the number of tuples $(g_1,\ ...,\ g_r) \in \mathcal{O}_{K,S}^r$ such that $g_r \neq 0$, $H([g_1,\ ...,\ g_r,\ 1]) \leq t$, and the rank r Drinfeld module ϕ over K determined by $\phi_T = \iota(T) + g_1\tau + \cdots + g_r\tau^r$ has S-Drinfeld class equal to $\mathfrak{C}$. Let $D_r(t) = \sum_{\mathfrak{C}} N_r(\mathfrak{C}, t)$. We prove the following result in this section.

Theorem 4.1. *Let K be a finite separable extension of $\mathbb{F}_q(T)$, S be a finite set of places of K containing all places above ∞, and fix an integer $r \geq 1$. Let $B_r = q\frac{1-q^r}{1-q} - r$. For*

$(q,r) \neq (2,1)$ and a given divisor class $\mathfrak{C} \in \mathrm{Cl}_S(K)$, the asymptotic equation density of $\mathfrak{C}$ is

$$\rho_r^{\mathrm{eq}}(\mathfrak{C}) = \lim_{t\to\infty} \frac{N_r(\mathfrak{C},t)}{D_r(t)} = \frac{\zeta_{K,S}(\mathfrak{C}^{-1}, B_r)}{\zeta_{K,S}(B_r)}.$$

4.1. **Counting height shells.** We count r-tuples with given height. Let X/F_K be the smooth projective curve with function field K. Consider the diagonal embedding

$$\mathcal{O}_{K,S} \hookrightarrow K_S := \prod_{P\in S} K_P$$
$$g_i \mapsto (g_{i,P})_{P\in S}.$$

For $\mathbf{g} = (g_1, \cdots, g_r) \in K^r$, define

$$m_P(\mathbf{g}) = \max\{0, -v_P(g_1), \cdots, -v_P(g_r)\}.$$

The height of $\mathbf{g} \in \mathcal{O}_{K,S}^r$ will be

$$H([g_1, \cdots, g_r, 1]) = \prod_{P\in S} Q^{d_P m_P(\mathbf{g})}.$$

We consider regions in K_S^r. Define

$$\widetilde{R}_{r,D} := \{\mathbf{g} \in \mathcal{O}_{K,S}^r : H([g_1, \cdots, g_r, 1]) = Q^D\}$$

and $R_{r,D} := \{\mathbf{g} \in \widetilde{R}_{r,D} : g_r \neq 0\}$. Take $\delta = \gcd\{d_P : P \in S\}$. It is easy to see that if $\delta \nmid D$, then $R_{r,D} = \emptyset$. For the rest of this section, we always assume that $\delta \mid D$.

For each $P \in S$, choose a Haar measure μ_P on K_P such that $\mu_P(\mathcal{O}_{K,P}) = 1$, and let μ be the product measure of μ_P on K_S and $\mu^{\otimes r}$ on K_S^r. We calculate the covolume of $\mathcal{O}_{K,S}$ in K_S.

Lemma 4.2. *With notation as above, the image of $\mathcal{O}_{K,S}$ under the diagonal embedding has covolume $\mathrm{Covol}(\mathcal{O}_{K,S}) = Q^{g-1}$, where g is the genus of X.*

Proof. Let $E = \sum_{P\in S} m_P P$ be an effective S-supported divisor, with $\deg E > 2g-2$. Let $B(E) := \prod_{P\in S} \pi_P^{-m_P} \mathcal{O}_P \subset K_S$ be the associated open compact box, we have $\mathrm{vol}(B(E)) = \prod_{P\in S} Q^{m_P d_P} = Q^{\deg E}$.

We claim that $K_S = \mathcal{O}_{K,S} + B(E)$. Indeed, for any $x = (x_P)_{P\in S} \in K_S$, we can find another effective S-supported divisor $E' \geq E$ such that $x \in B(E')$. Let $\mathcal{O}_X(E)$ denote the invertible sheaf associated to E and $H^0(X, \mathcal{O}_X(E)) = L(E) = \{f \in K^\times : \mathrm{div}(f) + E \geq 0\} \cup \{0\}$, with F_K-dimension $l(E)$. By the Riemann-Roch Theorem we have $H^1(X, \mathcal{O}_X(E)) = 0$ and $l(E) = \deg E + 1 - g$. Taking cohomology of the exact sequence

$$0 \to \mathcal{O}_X(E) \to \mathcal{O}_X(E') \to \mathcal{O}_X(E')/\mathcal{O}_X(E) \to 0,$$

we obtain a surjection

$$L(E') \twoheadrightarrow H^0(X, \mathcal{O}_X(E')/\mathcal{O}_X(E)) \cong B(E')/B(E),$$

which means that there exists $f \in L(E') \subset \mathcal{O}_{K,S}$ such that $x - f \in B(E)$, the claim follows.

Therefore $\mathcal{O}_{K,S}$ is cocompact in K_S, and its covolume is

$$\mathrm{Covol}(\mathcal{O}_{K,S}) = \frac{\mathrm{vol}(B(E))}{\#(B(E) \cap \mathcal{O}_{K,S})} = \frac{\mathrm{vol}(B(E))}{\#L(E)}$$

$$= \frac{Q^{\deg E}}{Q^{\deg E+1-g}} = Q^{g-1}.$$

The lemma follows. □

For $\mathbf{x}_P = (x_1, \cdots, x_r) \in K_P^r$, define $m_P(\mathbf{x}_P) = \max\{0, -v_P(x_1), \cdots, -v_P(x_r)\}$. The volume of the local shell $\{\mathbf{x}_P \in K_P^r : m_P(\mathbf{x}_P) = m\}$ is

$$\omega_P(m) = \begin{cases} 1, & m = 0, \\ Q^{d_P rm} - Q^{d_P r(m-1)}, & m \geq 1. \end{cases}$$

For $D \in \mathbb{Z}_{\geq 0}$, define

$$W_{r,D} := \mathrm{vol}\{\mathbf{x} \in K_S^r : \prod_{P \in S} \max\{1, |x_{1,P}|_P, \cdots, |x_{r,P}|_P\} = Q^D\},$$

where $x_{i,P}$ is the P-component of the i-th coordinate of $\mathbf{x}$. Then

$$W_{r,D} = \sum_{\substack{m_P \geq 0 \\ \sum_{P \in S} d_P m_P = D}} \prod_{P \in S} \omega_P(m_P).$$

Its generating function is

$$\begin{aligned} F(z) = \sum_{D \geq 0} W_{r,D} z^D &= \prod_{P \in S} \sum_{m \geq 0} \omega_P(m) z^{d_P m} \\ &= \prod_{P \in S} (1 + \sum_{m \geq 1} (Q^{d_P rm} - Q^{d_P r(m-1)}) z^{d_P m}) \\ &= \prod_{P \in S} \frac{1 - z^{d_P}}{1 - Q^{rd_P} z^{d_P}}. \end{aligned}$$

Furthermore, the following proposition counts the number of points in $\widetilde{R}_{r,D}$.

Proposition 4.3. *For D large enough,*

$$\#\widetilde{R}_{r,D} = Q^{r(1-g)} W_{r,D}.$$

If $s = |S| \geq 2$, then

$$W_{r,D} = \delta \frac{Q^{rD} D^{s-1}}{(s-1)!} \prod_{P \in S} \frac{1 - Q^{-rd_P}}{d_P} + O(Q^{rD} D^{s-2}). \tag{4.1}$$

If $s = |S| = 1$, then

$$W_{r,D} = \delta Q^{rD} \frac{1 - Q^{-rd_\infty}}{d_\infty} + O(1).$$

Proof. For $\mathbf{g} \in \mathcal{O}_{K,S}^r$, define $\mathcal{D}(\mathbf{g}) = \sum_{P \in S} \max\{0, -v_P(g_1), \cdots, -v_P(g_r)\}P$. Suppose $E = \sum_{P \in S} m_P P$ is an effective S-supported divisor. First we count $\mathbf{g}$ with $\mathcal{D}(\mathbf{g}) = E$. Let $S_+(E) = \{P \in S : m_P > 0\}$. The number of $\mathbf{g}$ with $\mathcal{D}(\mathbf{g}) \leq E$ when $\deg E > 2g - 2$ is exactly $\#L(E)^r = Q^{rl(E)}$. By the inclusion-exclusion principle we have that if for any $T \subset S_+(E)$, $E_T = E - \sum_{P \in T} P$ satisfies $\deg E_T > 2g - 2$, then

$$\#\{\mathbf{g} \in \mathcal{O}_{K,S}^r : \mathcal{D}(\mathbf{g}) = E\} = \sum_{T \subset S_+(E)} (-1)^{|T|} Q^{rl(E_T)}$$

$$= Q^{r(\deg E+1-g)} \sum_{T\subset S_+(E)} (-1)^{|T|} Q^{-r\deg T}$$
$$= Q^{r(1-g)} Q^{r\deg E} \prod_{P\in S_+(E)} (1-Q^{-rd_P})$$
$$= Q^{r(1-g)} \prod_{P\in S} \omega_P(m_P),$$

which is exactly the ratio of the volume of the local shell to $\mathrm{Covol}(\mathcal{O}_{K,S}^r)$. Thus we get

$$\#\widetilde{R}_{r,D} = \sum_{\deg E=D} \#\{\mathbf{g}\in\mathcal{O}_{K,S}^r : \mathcal{D}(\mathbf{g})=E\} = Q^{r(1-g)} W_{r,D}.$$

The asymptotic estimate for $W_{r,D}$ follows from the standard coefficient extraction for rational generating functions (cf. [3, Theorem IV.9]). We check the case $\delta = 1$. In this case $z = Q^{-r}$ is the only pole of order s of the generating function and all the poles lie on the circle $|z| = Q^{-r}$. Thus $z = Q^{-r}$ gives the leading asymptotic term. Near $z = Q^{-r}$,

$$1 - Q^{rd_P} z^{d_P} = d_P(1 - \frac{z}{Q^{-r}}) + O((1-\frac{z}{Q^{-r}})^2).$$

Hence we have

$$F(z) \sim \prod_{P\in S} \frac{1-Q^{-rd_P}}{d_P} (1-\frac{z}{Q^{-r}})^{-s}.$$

Note that $\binom{D+s-1}{s-1} = \frac{D^{s-1}}{(s-1)!} + O(D^{s-2})$, whence the asymptotic form follows. □

Recall that we always assume that $\delta | D$.

Lemma 4.4. *For $D = k\delta$ large enough with $k\in\mathbb{Z}$, we have*

$$\#(\widetilde{R}_{r,D}\backslash R_{r,D}) = O(Q^{(r-1)D} D^{s-1}),$$

hence

$$\lim_{D\to\infty} \frac{\#R_{r,D}}{\#\widetilde{R}_{r,D}} = 1.$$

Proof. For $r \geq 2$, the condition $g_r = 0$ leaves an $(r-1)$-tuple with the same height condition, the estimate in Proposition 4.3 gives the lemma. For $r = 1$ the rank-deficient set consists only of the zero tuple, which does not occur in a positive height shell. □

4.2. **Congruence conditions.** Given r ideals $I_1, \cdots, I_r$, for $a_1, \cdots, a_r \in \mathcal{O}_{K,S}$, define

$$R_{r,D}(a_1,\cdots,a_r) := \{\mathbf{g}\in\widetilde{R}_{r,D} : g_i \equiv a_i \mod I_i, 1\leq i\leq r\}.$$

Let E_i denote the associated effective divisor of I_i. We count the number of points in $R_{r,D}(a_1,\cdots,a_r)$.

Proposition 4.5. *For $D \geq D_0 = 2g-1+\sum_{v\in S} d_v + \max\{\deg E_i : 1\leq i\leq r\}$, $\#R_{r,D}(a_1,\cdots,a_r) = \#\widetilde{R}_{r,D}/\prod_{i=1}^r N_S(I_i)$.*

Proof. In the inclusion-exclusion formula in the proof of Proposition 4.3, the summands are the spaces $L(E_T)$. For D sufficiently large, uniformly for every E and T occurring there, $\deg(E_T - E_i) > 2g-2$. The exact sequence

$$0 \to \mathcal{O}_X(E_T - E_i) \to \mathcal{O}_X(E_T) \to \mathcal{O}_X(E_T)|_{E_i} \to 0$$

therefore gives a surjective reduction map

$$L(E_T) \twoheadrightarrow \mathcal{O}_{K,S}/I_i.$$

Every fiber has cardinality

$$Q^{l(E_T-E_i)} = Q^{l(E_T)-\deg E_i} = \frac{Q^{l(E_T)}}{N_S(I_i)}.$$

Thus every inclusion-exclusion summand is multiplied by the same factor $\prod_{i=1}^{r} N_S(I_i)^{-1}$. The result follows. □

In the next proposition we count r-tuples with prescribed local minimalization exponents. For a place $P \in \mathcal{S}_K - S$, we write C_b^P, c_b^P, π_b^P and f_P for the local objects C_b, c_b, π_b and f from Section 2. For $\mathbf{g} \in R_{r,D}$, $f_P(\mathbf{g}) = 0$ for almost all $P \in \mathcal{S}_K - S$. Hence

$$I(\mathbf{g}) := \prod_{P \in \mathcal{S}_K - S} P^{f_P(\mathbf{g})} \tag{4.2}$$

is a well-defined S-integral ideal. The associated divisor of $I(\mathbf{g})$ is the inverse of the S-Drinfeld divisor of the Drinfeld module determined by $\mathbf{g}$.

We first record a uniform estimate which will be used both for the infinite sieve and for the later summation over S-integral ideals.

Lemma 4.6. *There are constants $C > 0$ and D_1, depending only on K, S, r, such that for every S-integral ideal J and every $D = k\delta \geq D_1$,*

$$\#\{\mathbf{g} \in R_{r,D} : J \mid I(\mathbf{g})\} \leq C\,|R_{r,D}|\,N_S(J)^{-B_r}.$$

Consequently, for $D = k\delta$,

$$\lim_{Z\to\infty} \limsup_{D\to\infty} \frac{\#\{\mathbf{g} \in R_{r,D} : f_P(\mathbf{g}) \geq 1 \text{ for some } P \in \mathcal{S}_K - S, d_P > Z\}}{|R_{r,D}|} = 0. \tag{4.3}$$

Proof. Let E_J be the effective divisor associated to J and write $a = \deg E_J$. Let $E = \mathcal{D}(\mathbf{g})$ be the effective S-supported divisor as in the proof of Proposition 4.3. If $J \mid I(\mathbf{g})$, then

$$g_i \in L\big(E - (q^i - 1)E_J\big), \qquad 1 \leq i \leq r.$$

Since $g_r \neq 0$, this set can be nonempty only if $(q^r - 1)a \leq D$. Hence every divisor $E - (q^i - 1)E_J$ occurring above has nonnegative degree, and the elementary bound $l(B) \leq \deg B + 1$ gives

$$\prod_{i=1}^{r} Q^{l(E-(q^i-1)E_J)} \leq Q^{r(D+1)-B_r a}.$$

The number of effective S-supported divisors of degree D is $O((D+1)^{s-1})$. Thus Proposition 4.3 and Lemma 4.4 give

$$\#\{\mathbf{g} \in R_{r,D} : J \mid I(\mathbf{g})\} \leq CQ^{r(D+1)-B_r a}(D+1)^{s-1} \leq C'|R_{r,D}|N_S(J)^{-B_r},$$

where $C, C' > 0$, depending only on K, S, r, uniformly in J and D.

Taking $J = P \in \mathcal{S}_K - S$ and using the union bound, we have

$$\frac{\#\{\mathbf{g} \in R_{r,D} : f_P(\mathbf{g}) \geq 1 \text{ for some } P \in \mathcal{S}_K - S, d_P > Z\}}{|R_{r,D}|} \leq C \sum_{\substack{P \in \mathcal{S}_K - S \\ d_P > Z}} N_P^{-B_r}.$$

Note that $(q, r) \neq (2, 1)$, so $B_r > 1$. The series $\sum_{P \in \mathcal{S}_K - S} N_P^{-B_r}$ converges and its tail tends to zero, proving equation (4.3). □

Proposition 4.7. *For any S-integral ideal $I \subset \mathcal{O}_{K,S}$, let $E = \sum_{P \in \mathcal{S}_K - S} v_P(I)P$ be the associated effective divisor. Let $m_P = v_P(I)$, $T = \{P : m_P > 0\}$ and let*

$$G_r(I, D) := \#\{\mathbf{g} \in R_{r,D} : \pi_{m_P}^P(\mathbf{g}) \in C_{m_P}^P \text{ for all } P \in \mathcal{S}_K - S\}.$$

Let $\alpha(I) = Q^{-B_r \deg I}$. Then for $D = k\delta$ with $k \in \mathbb{Z}$,

$$\lim_{D \to \infty} \frac{G_r(I, D)}{\#R_{r,D}} = \zeta_{K,S}(B_r)^{-1} \alpha(I). \tag{4.4}$$

Proof. Fix $Z \geq 1$ and let $\mathcal{P}_Z := \{P \in \mathcal{S}_K - (S \cup T) : d_P \leq Z\}$. This is a finite set. Let $G_{r,Z}(I, D)$ be the number of $\mathbf{g} \in R_{r,D}$ such that

$$f_P(\mathbf{g}) = m_P \quad (P \in T), \qquad f_P(\mathbf{g}) = 0 \quad (P \in \mathcal{P}_Z).$$

For a subset $T' \subset \mathcal{P}_Z$, let $F_r(T', I, D)$ be the number of tuples in $R_{r,D}$ satisfying

$$f_P(\mathbf{g}) = m_P \quad (P \in T), \qquad f_P(\mathbf{g}) \geq 1 \quad (P \in T').$$

Then finite inclusion-exclusion gives

$$G_{r,Z}(I, D) = \sum_{T' \subset \mathcal{P}_Z} (-1)^{\#T'} F_r(T', I, D). \tag{4.5}$$

For $P \in T$, Proposition 2.4 says that the condition $f_P(\mathbf{g}) = m_P$ is represented by the $c_{m_P}^P = N_P^{B_r} - 1$ residue classes in $C_{m_P}^P$, modulo $P^{(m_P+1)(q^i-1)}$ in the i-th coordinate. For $P \in T'$, the condition $f_P(\mathbf{g}) \geq 1$ is the single residue condition $g_i \equiv 0 \pmod{P^{q^i-1}}$, $1 \leq i \leq r$. For fixed $T \cup T'$, the Chinese remainder theorem, Lemma 4.4 and Proposition 4.5 apply for all sufficiently large D. Therefore

$$\lim_{D \to \infty} \frac{F_r(T', I, D)}{\#R_{r,D}} = \prod_{P \in T} \frac{N_P^{B_r} - 1}{N_P^{(m_P+1)B_r}} \prod_{P \in T'} N_P^{-B_r}.$$

Taking the limit in the finite sum in equation (4.5) yields

$$\lim_{D \to \infty} \frac{G_{r,Z}(I, D)}{\#R_{r,D}} = \prod_{P \in T} \frac{N_P^{B_r} - 1}{N_P^{(m_P+1)B_r}} \prod_{P \in \mathcal{P}_Z} (1 - N_P^{-B_r}).$$

It remains to remove the cutoff Z. If an r-tuple is counted by $G_{r,Z}(I, D)$ but not by $G_r(I, D)$, then $f_P(\mathbf{g}) \geq 1$ for some $P \in \mathcal{S}_K - (S \cup T)$ with $d_P > Z$. Hence Lemma 4.6 gives

$$0 \leq \frac{G_{r,Z}(I, D) - G_r(I, D)}{\#R_{r,D}} \leq C \sum_{\substack{P \in \mathcal{S}_K - (S \cup T) \\ d_P > Z}} N_P^{-B_r}.$$

The right-hand side is independent of D and tends to zero as $Z \to \infty$. Thus we obtain

$$\begin{aligned}\lim_{D\to\infty} \frac{G_r(I,D)}{\#R_{r,D}} &= \prod_{P\in T} \frac{N_P^{B_r}-1}{N_P^{(m_P+1)B_r}} \prod_{P\in\mathcal{S}_K-(S\cup T)} (1-N_P^{-B_r}) \\ &= \prod_{P\in T} N_P^{-m_P B_r} \prod_{P\in\mathcal{S}_K-S} (1-N_P^{-B_r}) \\ &= \zeta_{K,S}(B_r)^{-1} N_S(I)^{-B_r}.\end{aligned}$$

The proposition follows. □

4.3. **Proof of Theorem 4.1.** We can rewrite $G_r(I,D)$ from Proposition 4.7 as

$$G_r(I,D) = \#\{\mathbf{g} \in R_{r,D} : I(\mathbf{g}) = I\}.$$

For a shell with height $D = k\delta$, define $n_r(\mathfrak{C},D) := \#\{\mathbf{g} \in R_{r,D} : [I(\mathbf{g})] = \mathfrak{C}^{-1}\}$ and $d_r(D) := \#R_{r,D}$. By this definition we find that

$$n_r(\mathfrak{C},D) = \sum_{I\in\mathfrak{C}^{-1}} G_r(I,D).$$

Since the associated divisor of $I(\mathbf{g})$ is the inverse of the S-Drinfeld divisor of the Drinfeld module determined by $\mathbf{g}$, from Lemma 4.6 and Proposition 4.7, we have

$$\lim_{D\to\infty} \frac{n_r(\mathfrak{C},D)}{d_r(D)} = \frac{1}{\zeta_{K,S}(B_r)} \sum_{\substack{I \ S\text{-integral ideal} \\ [I]=\mathfrak{C}^{-1}}} N_S(I)^{-B_r} = \frac{\zeta_{K,S}(\mathfrak{C}^{-1},B_r)}{\zeta_{K,S}(B_r)}.$$

It remains to pass from shell counts to the cumulative height counts in the statement of the theorem. Taking the sum over D we obtain

$$\lim_{t\to\infty} \frac{N_r(\mathfrak{C},t)}{D_r(t)} = \frac{\zeta_{K,S}(\mathfrak{C}^{-1},B_r)}{\zeta_{K,S}(B_r)},$$

as the denominator tends to infinity by Proposition 4.3. This proves Theorem 4.1.

Remark 4.8. The class distribution is in general not uniform. Consider the field in Remark 3.4. Its affine zeta function is

$$\zeta_{K,S}(u) = \frac{1+3u^2}{1-3u}, \qquad u = 3^{-s}.$$

Here $h_{K,S} = 4$. For the rank 2 case, $B_2 = (3-1)+(9-1) = 10$. Since the trivial partial zeta function contains the unit ideal,

$$\frac{\zeta_{K,S}(\mathfrak{C}_0,10)}{\zeta_{K,S}(10)} \geq \frac{1}{\zeta_{K,S}(10)} > 0.9999.$$

Now let $k = \mathbb{F}_3(x)$. Choose distinct positive integers $d_1 < \cdots < d_m$, $3 \nmid d_i$, and set Artin-Schreier extensions

$$K_m = \mathbb{F}_3(x, y_1, \ldots, y_m), \qquad y_i^3 - y_i = x^{d_i}(x^3 - x).$$

Hence the Artin-Schreier classes are linearly independent and $[K_m : k] = 3^m$. We refer to [14, Proposition 3.7.8] for more information on Artin-Schreier extensions, including ramification, different exponents, and genus formulas.

One finds that infinity is totally ramified and there is a unique degree-one place P_∞ above it. Each of the places $x = 0, 1, 2$ splits completely in K_m. Consequently K_m has at least 3^{m+1} finite degree-one places. Set $S = \{P_\infty\}$, P_∞ has degree one. For rank $r = 1$, $B_1 = 3 - 1 = 2$. Let $\mathfrak{C}_0$ denote the trivial S-class, and define

$$a_n = \#\{D \geq 0 : \mathrm{supp}(D) \cap S = \emptyset,\, \deg D = n,\, [D] = \mathfrak{C}_0\}.$$

Since S consists of one degree-one place, using $l((n-1)P_\infty) \leq n$ we obtain

$$a_n = \frac{3^{l(nP_\infty)} - 3^{l((n-1)P_\infty)}}{2} \leq 3^{n+1}.$$

Hence $\zeta_{K_m,S}(\mathfrak{C}_0, 2) \leq \sum_{n\geq 0} 3^{-n+1} < 5$, while the 3^{m+1} finite degree-one places give $\zeta_{K_m,S}(2) \geq (1 - 3^{-2})^{-3^{m+1}}$. The density $\zeta_{K_m,S}(\mathfrak{C}_0, 2)/\zeta_{K_m,S}(2)$ can be arbitrarily small as $m \to \infty$.

5. Counting isomorphism classes

In this section, we count isomorphism classes of Drinfeld modules using the weighted-projective counting theorem [8, Theorem 4.1.1]. For the convenience of readers, we adopt the notation of [8]. First we recall the definition of the weighted projective stack (cf. [8, Definition 2.2.1]).

Definition 5.1. Let $\mathbb{A}^r$ denote the affine r-space, and let $\mathbb{G}_m$ denote the multiplicative group scheme. Given $\mathbf{w}_r = (q-1, \ldots, q^r - 1)$, the *weighted projective stack* $\mathcal{P}(\mathbf{w}_r)$ is defined to be the quotient stack

$$\mathcal{P}(\mathbf{w}_r) := [(\mathbb{A}^r - \{0\})/\mathbb{G}_m]$$

with respect to the action

$$\begin{aligned} \mathbb{G}_m \times (\mathbb{A}^r - \{0\}) &\to (\mathbb{A}^r - \{0\}) \\ (\lambda, (x_0, \ldots, x_{r-1})) &\mapsto (\lambda^{q-1}x_0, \ldots, \lambda^{q^r-1}x_{r-1}). \end{aligned}$$

Choose one finite place $P_0 \in \mathcal{S}_K - S$. At P_0 impose the projective local condition

$$\Omega_{P_0} := \{[x_1 : \cdots : x_r] \in \mathcal{P}(\mathbf{w}_r)(K_{P_0}) : x_r \neq 0\},$$

where $\mathbf{w}_r = (q-1, \ldots, q^r - 1)$. For every other finite place there is no restriction. The affine cone of Ω_{P_0}, intersected with the integral vectors occurring in the local factor, is

$$\Omega_{P_0}^{\mathrm{aff}} \cap \mathcal{O}_{K,P_0}^r = \left\{(x_1, \ldots, x_r) \in \mathcal{O}_{K,P_0}^r : x_r \neq 0\right\}.$$

Its complement in $\mathcal{O}_{K,P_0}^r$ is $\mathcal{O}_{K,P_0}^{r-1} \times \{0\}$, which has product measure 0. Consequently, $\Omega_{P_0}^{\mathrm{aff}} \cap \mathcal{O}_{K,P_0}^r$ has measure 1. In terms of the notation in [8], $m_{P_0}(\Omega_{P_0}^{\mathrm{aff}} \cap \mathcal{O}_{K,P_0}^r) = 1$.

The boundary of the above affine condition is contained in the hyperplane $x_r = 0$, hence it also has measure 0, as required in [8, Theorem 4.1.1]. Notice that for a K-rational point the condition $x_r \neq 0$ at this one completion is equivalent to $g_r \neq 0$ in K; thus this local condition cuts out exactly the rank r locus.

At P_∞ we impose no projective restriction, i.e. $\Omega_\infty = \mathcal{P}(\mathbf{w}_r)(K_\infty)$. For this trivial condition, the compact affine set is its height-zero fundamental slice

$$\mathcal{F}_\infty(0) := \{x = (x_1, \ldots, x_r) \in K_\infty^r - \{0\} : \mathrm{ht}_{\mathbf{w}_r,\infty}(x) = 0\},$$

where

$$\mathrm{ht}_{\mathbf{w}_r,\infty}(x) = -d_\infty \min_{1\le i\le r}\left\lfloor \frac{v_\infty(x_i)}{w_i}\right\rfloor.$$

Write $h_K := \#\operatorname{Pic}^0(X)$ and use the full zeta function $\zeta_K(s) := \prod_{P\in\mathcal{S}_K}(1-Q^{-d_P s})^{-1}$. With $\Sigma = \{P_0\}$, [8, Theorem 4.1.1] gives

$$\begin{aligned}\kappa_r &= \frac{h_K \gcd(Q-1,w_1,\dots,w_r)}{\zeta_K(B_r)\,Q^{r(g-1)}(Q-1)}\left(\frac{m_\infty(\{x\in\Omega_\infty^{\mathrm{aff}} : \mathrm{ht}_{\mathbf{w}_r,\infty}(x)=0\})}{m_\infty(\{x\in K_\infty^r : \mathrm{ht}_{\mathbf{w}_r,\infty}(x)=0\})}\right)\prod_{P\in\Sigma} m_P(\Omega_P^{\mathrm{aff}}\cap\mathcal{O}_{K,P}^r)\\ &= \frac{h_K(q-1)}{\zeta_K(B_r)\,Q^{r(g-1)}(Q-1)}.\end{aligned}$$

In particular, for $r\ge 2$, the number of isomorphism classes of rank r Drinfeld modules with weighted height b (cf. Section 1.3.2) is

$$N_r^{\mathrm{iso}}(b) = \kappa_r Q^{B_r b} + \begin{cases} O(Q^b b), & r=2 \text{ and } d_\infty = 1,\\ O\big(Q^{\,b(B_r-(q-1)/d_\infty)}\big), & \text{otherwise.}\end{cases} \tag{5.1}$$

For the S-Drinfeld class condition, we use the divisor-class decomposition

$$M(D,b) := \#\{x\in\mathcal{P}(\mathbf{w}_r)(K) : \mathrm{ht}_{\mathbf{w}_r}(x)=b, [\mathfrak{D}_0(x)]=[D], x\in\Omega_P, \forall P\in\mathcal{S}_K\},$$

where the finite scaling divisor $\mathfrak{D}_0(x)$ is defined by

$$\mathfrak{D}_0(x) = \sum_{P\neq P_\infty}\min_{1\le i\le r}\left\lfloor\frac{v_P(g_i)}{q^i-1}\right\rfloor P. \tag{5.2}$$

Let $G := \operatorname{Pic}(\mathcal{O}_{K,P_\infty})$. The image of $[\mathfrak{D}_0(x)]\in G$ in $\mathrm{Cl}_S(K)$ is the inverse of the S-Drinfeld class $\overline{\mathfrak{a}}_{\phi/K}$.

For a fixed class $[D]\in G$, the proof of [8, Theorem 4.1.1] gives $M(D,b)=0$ unless $b\equiv -\deg(D) \pmod{d_\infty}$. When the congruence is satisfied, the same proof gives the class-by-class main term

$$M(D,b)\sim \frac{\kappa_r}{h_K}Q^{B_r b} = \frac{q-1}{(Q-1)Q^{r(g-1)}\zeta_K(B_r)}Q^{B_r b}. \tag{5.3}$$

We now pass from G to the S-class group. Recall that our distinguished P_∞ is a chosen place above ∞. Thus $S-\{P_\infty\}$ contains both the remaining places above ∞, if there are any, and any additional finite places included in S. Let

$$H := \langle [P] : P\in S-\{P_\infty\}\rangle \subseteq G.$$

Then $G/H\simeq \mathrm{Cl}_S(K)$, $|G| = h_K d_\infty$, and $|H| = h_K d_\infty/h_{K,S}$. Put $\delta = \gcd\{d_P : P\in S\}$. Under the degree map $G\to\mathbb{Z}/d_\infty\mathbb{Z}$,

$$\deg(H) = \langle d_P \bmod d_\infty : P\in S-\{P_\infty\}\rangle = \delta\mathbb{Z}/d_\infty\mathbb{Z}.$$

Therefore

$$\big|\ker\big(\deg|_H\big)\big| = \frac{|H|}{d_\infty/\delta} = \frac{h_K\delta}{h_{K,S}}. \tag{5.4}$$

Indeed, quotienting the degree map on G by H gives

$$\deg_S : \mathrm{Cl}_S(K) = G/H \longrightarrow \frac{\mathbb{Z}/d_\infty\mathbb{Z}}{\deg(H)}\simeq \mathbb{Z}/\delta\mathbb{Z}. \tag{5.5}$$

Since $\deg : G \to \mathbb{Z}/d_\infty\mathbb{Z}$ is surjective, $\deg_S$ is also surjective. Hence $\delta \mid h_{K,S}$.

Fix $\mathfrak{C} \in \mathrm{Cl}_S(K)$. The fiber of $G \to \mathrm{Cl}_S(K)$ above $\mathfrak{C}^{-1}$ is a coset of H. By equation (5.4), the number of classes in this fiber whose degree is congruent to $-b$ modulo d_∞ is

$$\begin{cases} \dfrac{h_K\delta}{h_{K,S}}, & b \equiv \deg_S(\mathfrak{C}) \pmod{\delta}, \\ 0, & b \not\equiv \deg_S(\mathfrak{C}) \pmod{\delta}. \end{cases}$$

Moreover, for a shell with fixed height b large enough, the S-classes that occur are exactly the fiber $\deg_S^{-1}(b \bmod \delta)$, which has cardinality

$$\#\{\mathfrak{C} \in \mathrm{Cl}_S(K) : b \equiv \deg_S(\mathfrak{C}) \pmod{\delta}\} = \frac{h_{K,S}}{\delta}. \tag{5.6}$$

Therefore, one should not distribute a fixed height shell uniformly among all $h_{K,S}$ classes, since only $h_{K,S}/\delta$ of them actually occur in that shell.

Let $N_{r,S}^{\mathrm{iso}}(\mathfrak{C}, b) := \#\{[\phi]_K : \mathrm{rank}(\phi) = r, \mathrm{ht}_{\mathbf{w}_r}(\phi) = b, \overline{\mathfrak{a}}_{\phi/K} = \mathfrak{C}\}$. Combining equations (5.3) and (5.4), for $b \equiv \deg_S(\mathfrak{C}) \pmod{\delta}$ we obtain

$$N_{r,S}^{\mathrm{iso}}(\mathfrak{C}, b) \sim \frac{h_K\delta}{h_{K,S}} \frac{\kappa_r}{h_K} Q^{B_r b}$$

If $b \not\equiv \deg_S(\mathfrak{C}) \pmod{\delta}$, then $N_{r,S}^{\mathrm{iso}}(\mathfrak{C}, b) = 0$. Summing up the preceding discussion, we arrive at the following theorem.

Theorem 5.1. *Retain the setting of this section, let $r \geq 2$, put $B_r = \sum_{i=1}^{r}(q^i - 1)$, and let $\delta = \gcd\{d_P : P \in S\}$. Let $\deg_S : \mathrm{Cl}_S(K) \to \mathbb{Z}/\delta\mathbb{Z}$ be the surjective degree map defined in* (5.5)*. Then the following assertions hold.*

(1) *The number of K-isomorphism classes of rank r Drinfeld modules of weighted height b satisfies*

$$N_r^{\mathrm{iso}}(b) = \kappa_r Q^{B_r b} + \begin{cases} O(Q^b b), & r = 2 \text{ and } d_\infty = 1, \\ O\big(Q^{\,b(B_r - (q-1)/d_\infty)}\big), & \text{otherwise}, \end{cases}$$

where

$$\kappa_r = \frac{h_K(q-1)}{\zeta_K(B_r)\, Q^{r(g-1)}(Q-1)}.$$

(2) *Fix $\mathfrak{C} \in \mathrm{Cl}_S(K)$. If $b \not\equiv \deg_S(\mathfrak{C}) \pmod{\delta}$, then $N_{r,S}^{\mathrm{iso}}(\mathfrak{C}, b) = 0$. If $b \equiv \deg_S(\mathfrak{C}) \pmod{\delta}$, then*

$$N_{r,S}^{\mathrm{iso}}(\mathfrak{C}, b) \sim \frac{\delta}{h_{K,S}}\, \kappa_r Q^{B_r b}.$$

Consequently, the asymptotic isomorphism-class density of the class $\mathfrak{C}$ is $\delta/h_{K,S}$.

Remark 5.2. Using the idea of [9] and counting points in the weighted projective stack over number fields, for a number field F, we can also count the number of isomorphism classes of elliptic curves over F with bounded height and compute the asymptotic density of isomorphism classes of elliptic curves over F admitting global minimal Weierstrass equation. In particular, we could obtain a uniform distribution in this case, different from the one obtained by Bekyel (cf. [2, Theorem 1, Section 4]). We defer the precise formulation to the interested reader.

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