MULTIPLICITY ONE OF REGULAR SERRE WEIGHTS

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Abstract. We consider a potentially Barsotti-Tate deformation problem of a modular Galois representation. By constructing a Diamond-Taylor-Wiles system, we prove an $R = T$ theorem and a multiplicity one result in characteristic 0. Applying this result, we then prove a multiplicity one result in characteristic p, which provides certain evidence for a conjecture of Breuil.

1. Introduction

In this paper, we prove a multiplicity one result for Galois representations in cohomology groups of Shimura curves with certain non-trivial coefficients. Namely, we show that, under some technical conditions, the localized mod p cohomology group of a Shimura curve is free of rank two over the localized Hecke algebra. In the modular curves case, this is well-understood by [24], [27], and [37] (and so we exclude this case in this paper). The main tool we use is Diamond's refined Taylor-Wiles construction. In this approach, the freeness part becomes a consequence of the construction.

The novel part of this paper is that, by using a lifting result of Toby Gee [20] and transferring the problem in characteristic p to a problem in characteristic 0, we are able to deal with non-trivial coefficients associated to regular Serre weights (see Definition 2.6). In particular, we do not need a parity condition on the weights. Our results in this paper deal primarily with the minimally ramified case because of the lack of a general Ihara's lemma for Shimura curves.

To explain our results in more detail, we introduce some notation. Let F be a totally real field with degree $d = [F : \mathbb{Q}]$. Let $p > 3$ be a prime number which is unramified in F. The unramifiedness of p is crucial for Corollary 2.8 and crucial for the deformation problem we consider in this paper. (In particular, we use the identity $d = \sum_{v|p}[k_v : \mathbb{F}_p]$, where k_v is the residue field of F_v.) Let $\mathcal{G} = \prod_{v|p} GL_2(k_v)$. Fix an archimedean prime τ_1 of F and a finite set S_D of non-archimedean primes such that

$$| S_D | \equiv d - 1 \pmod 2. \tag{1.1}$$

Let D be the quaternion algebra over F which is ramified at the primes in $S_D \cup \{v \mid v|\infty, v \neq \tau_1\}$. We also use S_D to denote the ideal which is the product of the primes in S_D. Fix $\mathcal{O}_D$ a maximal order of D. Let $G = Res_{F/\mathbb{Q}}D^\times$ be the algebraic group over $\mathbb{Q}$ associated to $D^\times$. Then $G(\mathbb{Q}) = D^\times$, $G(\mathbb{R}) \simeq GL_2(\mathbb{R}) \times (\mathbf{H}^\times)^{d-1}$, where in the second term there appear $d - 1$ copies of the multiplicative group of non-zero elements of the classical Hamiltonian quaternion $\mathbf{H}$.

Let K_0 denote $\prod_v (\mathcal{O}_D \otimes \mathcal{O}_{F_v})^\times$, where v runs through all finite places of F. For $v \nmid S_D$, fix isomorphisms $D \otimes F_v \cong M_2(F_v)$ and $(\mathcal{O}_D \otimes \mathcal{O}_{F_v})^\times \cong GL_2(\mathcal{O}_{F_v})$. Define $K_1(N)$, for an ideal N prime to pS_D, to be the subgroup of K_0 consisting of those u for which u_v congruent to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\mathrm{mod}(v^{ord_v(N)})$ for every $v \mid N$.

Let $\bar{\rho} : G_F \to GL_2(\bar{\mathbb{F}}_p)$ be a continuous, irreducible Galois representation with conductor N', where $(N', p) = 1$ and N' is square free. Assume that $\bar{\rho}$ comes from the Shimura curve $M_{K_1(N)}$ associated to the quaternion algebra D, i.e., $\bar{\rho}$ is a subquotient of $H^1(M_{K_1(N)} \otimes \bar{F}, \mathcal{F}_{\sigma^{op}})_{\mathfrak{m}}$, where $\sigma : \mathcal{G} \to Aut(W_{/\bar{\mathbb{F}}_p})$ is a Serre weight, $\mathcal{F}_{\sigma^{op}}$ is the sheaf associated to σ^{op} (see section 2.1), $\mathfrak{m}$ is a maximal ideal of the corresponding Hecke algebra, N is square free and $N' \mid NS_D$. Let $a = \dim_{\bar{\mathbb{F}}_p} Hom_{G_F}(\bar{\rho}, H^1(M_{K_1(N)} \otimes \bar{F}, \mathcal{F}_{\sigma^{op}})_{\mathfrak{m}})$. We call a the *multiplicity* of $\bar{\rho}$ in $H^1(M_{K_1(N)} \otimes \bar{F}, \mathcal{F}_{\sigma^{op}})_{\mathfrak{m}}$.

In the case of modular curves, we have multiplicity one (i.e. $a = 1$) except some very special cases (see for example [27] Theorem 5.2). But from [28] Theorem 3 and [14] Theorem 3.6, we may have higher multiplicities in the case of Shimura curves. In this paper, we consider the case when σ is regular (see Definition 2.6 below). We show that, if $\bar{\rho}$ satisfies some technical conditions (see equation (3.2) below), then we have multiplicity one. The main result is Theorem 5.3. It follows from Theorem 4.1, which proves a multiplicity one result in characteristic 0.

The paper is organized as follows. In section 2, we recall some basic properties of Shimura curves and Serre weights. Specially, we introduce the lifting result of Toby Gee, which plays an important role in this paper. In section 3 and section 4, we consider a potentially Barsotti-Tate deformation problem and construct a Taylor-Wiles system. Then applying Diamond's result, we prove an $R = T$ theorem as well as a freeness result in characteristic 0 (Theorem 4.1). In this case, we can also compute the rank because we have Lemma 4.10, which is proved by the comparison theorem in characteristic 0. In section 5, we show how we can deduce multiplicity one in characteristic p from multiplicity one in characteristic 0 and prove our main theorem (Theorem 5.3). We also explain how we can get a stronger result (Theorem 5.14) by considering another deformation problem. To compute the size of local deformation rings at primes dividing p in the potentially Barsotti-Tate deformation problem, we use the theory of Breuil modules with descent data. We introduce this theory and prove Lemma 3.3 in section 6. In section 6.1, 6.2, and 6.3, the notation is not consistent with the notation in other parts of this paper.

1.1. **Notation.** If L is a perfect field we will let $\bar{L}$ denote the algebraic closure of L and G_L its absolute Galois group $Gal(\bar{L}/L)$. If L is a number field, we let $\mathbb{A}_L$ denote the ring of adeles over L, and $\mathbb{A}_L^\infty$ denote the ring of finite adeles over L. If $L = \mathbb{Q}$, we write $\mathbb{A}$ and $\mathbb{A}^\infty$ for $\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_{\mathbb{Q}}^\infty$, respectively.

Let F, p, D, G, $\bar{\rho}$, N be as above. For any prime v of F, let F_v be the completion of F at v, $\mathcal{O}_{F_v}$ the ring of integers of F_v, k_v the residue field of $\mathcal{O}_{F_v}$, ϖ_v a uniformizer of $\mathcal{O}_{F_v}$, and $Frob_v \in Gal(\bar{F}_v/F_v)$ an arithmetic Frobenious element. Write $I_v \subset G_{F_v}$ for the inertia group at prime v.

Let Σ be a set of primes of F. If a group U has the form $U = \prod_{v\in\Sigma} U_v$, and J is an ideal which is a product of some elements in Σ, we will write U^J for the subgroup of U given by $U^J = \prod_{v\in\Sigma, v\nmid J} U_v$ and U_J for the subgroup of U given by $U_J = \prod_{v\in\Sigma, v\mid J} U_v$.

Let K_0 and $K_1(N)$, for N prime to pS_D, be the subgroups of $G(\mathbb{A})$ as defined before. We then write $K = K_1(N)$, $K' = K_1(N)^p \times (1 + pM_2(\mathcal{O}_p)) \subset K$. Suppose that $\mathfrak{n}$ is an ideal of $\mathcal{O}_F$ such that $(\mathfrak{n}, pNS_D) = 1$, and for each finite place v of F dividing $\mathfrak{n}$, H_v is a quotient of $(\mathcal{O}_{F_v}/\mathfrak{n}_v)^\times$. Then we will write H for $\prod_{v|\mathfrak{n}} H_v$. We will let $K_H(\mathfrak{n}) = \prod_v K_H(\mathfrak{n})_v$ denote the open subgroup of K defined by setting $K_H(\mathfrak{n})_v$ to be the subgroup of $GL_2(\mathcal{O}_{F_v})$ consisting of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \in \mathfrak{n}_v$ and, in the case $v \mid \mathfrak{n}$, with ad^{-1} mapping to 1 in H_v.

For $v \mid \mathfrak{n}$ we have the decompositions

$$K_H(\mathfrak{n})_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} K_H(\mathfrak{n})_v = \coprod_{a \in k_v} \begin{pmatrix} \varpi_v & \tilde{a} \\ 0 & 1 \end{pmatrix} K_H(\mathfrak{n})_v,$$

$$K_H(\mathfrak{n})_v \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} K_H(\mathfrak{n})_v = \coprod_{a \in k_v} \begin{pmatrix} \varpi_v & 0 \\ \varpi_v \tilde{a} & 1 \end{pmatrix} K_H(\mathfrak{n})_v,$$

and

$$K_H(\mathfrak{n})_v \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} K_H(\mathfrak{n})_v = \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} K_H(\mathfrak{n})_v,$$

and for $v \nmid \mathfrak{n}S_D$ we have the decomposition

$$K_H(\mathfrak{n})_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} K_H(\mathfrak{n})_v = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} K_H(\mathfrak{n})_v \coprod (\coprod_{a \in k_v} \begin{pmatrix} \varpi_v & \tilde{a} \\ 0 & 1 \end{pmatrix} K_H(\mathfrak{n})_v),$$

where $\tilde{a}$ is some lift of a to $\mathcal{O}_{F_v}$.

In this paper, except the last section, E will denote a sufficiently large finite extension of $\mathbb{Q}_p$. It will serve as the coefficient ring. Let $\mathcal{O}$ and κ be its ring of integers and residue field. Denote by $\mathcal{C}_\mathcal{O}$ the category of local complete Noetherian $\mathcal{O}$-algebras with residue field κ.

2. Shimura curves

Let X be the $G(\mathbb{R})$-conjugacy class of the map

$$h : \mathbb{C}^\times \to G(\mathbb{R}) \simeq GL_2(\mathbb{R}) \times \mathbf{H}^\times \times \cdots \times \mathbf{H}^\times,$$

which maps $a + ib$ to $(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1}, 1, ..., 1)$. The conjugacy class X is naturally identified with the union of the upper and lower half plane by the map $ghg^{-1} \mapsto g(i)$, where $g(i) = \frac{a+ib}{c+id}$ for $g = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \cdots)$.

Let $M = M(G, X) = (M_H)_H$ be the canonical model defined over F of the Shimura variety defined by G and X. (Here H runs through the open compact subgroups of $G(\mathbb{A}^\infty)$.) Each M_H is proper and smooth but not necessarily geometrically connected over F, and

$$M_H(\mathbb{C}) \simeq G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty)/H.$$

For each H and H' sufficiently small (see [7] Lemma 1.4.1.1) and $g \in G(\mathbb{A}^\infty)$ with $g^{-1}H'g \subset H$, there is an etale map $\varrho_g : M_{H'} \to M_H$ which on complex points coincides with the one induced by right multiplication by g in $G(\mathbb{A})$. For a normal subgroup H' of H, the etale cover $\varrho_1 : M_{H'} \to M_H$ is Galois, and the mapping $g^{-1} \mapsto \varrho_g$ defines an isomorphism of H/H' with a group of covering maps.

2.1. **Sheaf cohomology and the Hecke algbra.** Suppose that H is an open compact subgroup of $G(\mathbb{A}^\infty)$. We assume that H is sufficiently small and H is of the form $\prod_v H_v$ with $H_v \subset (\mathcal{O}_D \otimes \mathcal{O}_{F_v})^\times$. Suppose that Σ is a finite set of primes, and that for each $v \in \Sigma$ we are given a finitely generated $\mathcal{O}$-module V_v with a left action of H_v which is continuous with respect to the discrete topology of V_v. We can then associate to the H-module $V = \bigotimes_v V_v$ a locally constant sheaf

$$\begin{array}{c} \mathcal{F}_V = G(\mathbb{Q})\backslash(G(\mathbb{A}) \times V)/H \\ \Big\downarrow{\scriptstyle \pi} \\ M_H \end{array} \tag{2.1}$$

on M_H. In order to define Hecke operators, we assume that for all $v \notin \Sigma$ we have $H_v = (\mathcal{O}_D \otimes \mathcal{O}_{F_v})^\times$ and H_v acts trivially on V. Write $\mathcal{F} = \mathcal{F}_V$.

Suppose that H and $H^{'}$ are sufficiently small open compact subgroups of $G(\mathbb{A}^\infty)$, and $g \in G(\mathbb{A}^\infty)$. There is a natural identification of sheaves on $M_{H\cap gH'g^{-1}}$: $\mathcal{F}^{gH^{'}g^{-1}} |_{M_{H\cap gH'g^{-1}}} = \mathcal{F}^{H\cap gH^{'}g^{-1}}$. Here $\mathcal{F}^R$ means that we consider $\mathcal{F}$ as a sheaf over the curve M_R. Then define

$$\begin{aligned} [HgH^{'}] : H^j(M_{H'} \otimes \bar{F}, \mathcal{F}^{H^{'}}) &\to H^j(M_{H'\cap g^{-1}Hg} \otimes \bar{F}, \mathcal{F}^{H^{'}} |_{M_{H'\cap g^{-1}Hg}}) \\ &\to H^j(M_{gH'g^{-1}\cap H} \otimes \bar{F}, \mathcal{F}^{gH^{'}g^{-1}} |_{M_{gH'g^{-1}\cap H}}) \\ &= H^j(M_{gH'g^{-1}\cap H} \otimes \bar{F}, \mathcal{F}^{H\cap gH^{'}g^{-1}}) \\ &\to H^j(M_H \otimes \bar{F}, \mathcal{F}^H), \end{aligned} \tag{2.2}$$

where the first arrow is the restriction map, the second arrow is induced from ϱ_g : $M_{H'\cap g^{-1}Hg} \to M_{gH'g^{-1}\cap H}$, and the last arrow is the trace map. See section 15 of [22] for more details.

Let $H = H^{'}$. If $\mathfrak{q}$ is a prime of $\mathcal{O}_F$ which is unramified in D and does not divide p, let $\omega_\mathfrak{q} \in \mathbb{A}_F^\infty$ be such that $\omega_\mathfrak{q}$ is a uniformizer at $\mathfrak{q}$ and is 1 at every other place. Then write

$$T_\mathfrak{q} = [H \begin{pmatrix} \omega_\mathfrak{q} & 0 \\ 0 & 1 \end{pmatrix} H].$$

If also $H_\mathfrak{q} = GL_2(\mathcal{O}_\mathfrak{q})$, define

$$S_\mathfrak{q} = [H \begin{pmatrix} \omega_\mathfrak{q} & 0 \\ 0 & \omega_\mathfrak{q} \end{pmatrix} H].$$

If $H = K_1(N)$, denote by $\mathbb{T}_\mathcal{O}(H, V)$ the $\mathcal{O}$-algebra generated by $T_\mathfrak{q}$ for $\mathfrak{q} \nmid NS_D$ and $S_\mathfrak{q}$ for $\mathfrak{q}$ with $H_\mathfrak{q} = GL_2(\mathcal{O}_\mathfrak{q})$. Write $\mathbb{T}_A(H, V) = \mathbb{T}_\mathcal{O}(H, V) \otimes A$ for any $\mathcal{O}$-algebra A. Write $U_\mathfrak{q} = T_\mathfrak{q}$ if $\mathfrak{q} \mid N$.

Definition 2.1. A maximal ideal of $\mathbb{T}_A(H, V)$ is *Eisenstein* if it contains $T_v - 2$ and $S_v - 1$ for all but finitely many primes v of F which split completely in some finite abelian extension of F.

Lemma 2.2. *Suppose that H, V are as above and let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}_\mathcal{O}(H, V)$ with finite residue field. Then*

(1) $H^0(M_H \otimes \bar{F}, \mathcal{F})_{\mathfrak{m}}$ *and* $H^2(M_H \otimes \bar{F}, \mathcal{F})_{\mathfrak{m}}$ *vanish.*
(2) If $0 \to V^{'} \to V \to V^{''} \to 0$ *is an exact sequence of* $\mathcal{O}[H]$*-modules, then the sequence*

$$0 \to H^1(M_H \otimes \bar{F}, \mathcal{F}_{V'})_{\mathfrak{m}} \to H^1(M_H \otimes \bar{F}, \mathcal{F}_V)_{\mathfrak{m}} \to H^1(M_H \otimes \bar{F}, \mathcal{F}_{V''})_{\mathfrak{m}} \to 0$$

is exact.
(3) If V *is free over* $\mathcal{O}$*, then the natural map*

$$H^1(M_H \otimes \bar{F}, \mathcal{F}_V)_{\mathfrak{m}} \otimes_{\mathcal{O}} \kappa \to H^1(M_H \otimes \bar{F}, \mathcal{F}_{V \otimes_{\mathcal{O}} \kappa})_{\mathfrak{m}}$$

is an isomorphism.

Proof. (1) By Lemma 2.2 of [5], the action of G_F on the cohomology groups $H^0(M_H \otimes \bar{F}, \mathcal{F})$ and $H^2(M_H \otimes \bar{F}, \mathcal{F})$ factors through an abelian quotient. Since $\mathfrak{m}$ is non-Eisenstein, the localizations vanish.
(2) Certainly, we have a short exact sequence of sheaves

$$0 \to \mathcal{F}_{V'} \to \mathcal{F}_V \to \mathcal{F}_{V''} \to 0.$$

Write down the long exact sequence of cohomology groups

$$\cdots \to H^0(M_H \otimes \bar{F}, \mathcal{F}_{V''}) \to H^1(M_H \otimes \bar{F}, \mathcal{F}_{V'}) \to H^1(M_H \otimes \bar{F}, \mathcal{F}_V) \to$$
$$H^1(M_H \otimes \bar{F}, \mathcal{F}_{V''}) \to H^2(M_H \otimes \bar{F}, \mathcal{F}_{V'}) \to \cdots$$

By (2), we get the desired short exact sequence after localization.
(3) We have a short exact sequence $0 \to V \to V \to V \otimes_{\mathcal{O}} \kappa \to 0$. Then by (3), we have a short exact sequence

$$0 \to H^1(M_H \otimes \bar{F}, \mathcal{F}_V)_{\mathfrak{m}} \to H^1(M_H \otimes \bar{F}, \mathcal{F}_V)_{\mathfrak{m}} \to H^1(M_H \otimes \bar{F}, \mathcal{F}_{V \otimes_{\mathcal{O}} \kappa})_{\mathfrak{m}} \to 0,$$

which gives the desired isomorphism. □

2.2. **Serre weights.** For our fixed F and p, since p is unramified in F, $\mathcal{G} = \prod_{v|p} GL_2(k_v) \cong GL_2(\mathcal{O}_F/p)$. A *Serre weight* is an isomorphism class of irreducible $\bar{\mathbb{F}}_p$-representations of $\mathcal{G}$. They can be described explicitly as follows. For each prime v of F dividing p, let S_v be the set of embeddings $\lambda : k_v \to \bar{\mathbb{F}}_p$. Then every irreducible $\bar{\mathbb{F}}_p$-representation of $GL_2(k_v)$ is equivalent to one of the form

$$V_{\vec{a},\vec{b}} = \bigotimes_{\lambda \in S_v} (det^{a_\lambda} \otimes_{k_v} Sym^{b_\lambda - 1} k_v^2) \otimes_\lambda \bar{\mathbb{F}}_p,$$

where a_λ, $b_\lambda \in \mathbb{Z}$, $\vec{a} \neq (\overrightarrow{p-1})$, $0 \leq a_\lambda \leq p-1$ and $1 \leq b_\lambda \leq p$ for each $\lambda \in S_v$. The irreducible representations of $\mathcal{G}$ are thus of the form $V = \otimes_{v|p} V_v$, where the tensor product is over $\bar{\mathbb{F}}_p$ and each V_v is of the form $V_{\vec{a},\vec{b}}$ for $(\vec{a}, \vec{b})$ as above. We write V^{op} for the dual of V.

Definition 2.3. Suppose that $\rho : G_F \to GL_2(\bar{\mathbb{F}}_p)$ is a continuous, irreducible representation and V is a finite-dimensional $\bar{\mathbb{F}}_p$ vector space with a left action of $\mathcal{G}$. We say that ρ is modular of weight V if there is a quaternion algebra D over F split at the primes above p, at τ_1 and at no other archimedean places of F, and a sufficiently small open compact subgroup U of $G(\mathbb{A}^\infty)$ of level prime to p, such that ρ is an $\bar{\mathbb{F}}_p G_F$-subquotient of $H^1_{et}(M_U \otimes \bar{F}, \mathcal{F}_{V^{op}}) \simeq Hom_{\mathcal{G}}(V, H^1_{et}(M_{U'} \otimes \bar{F}, \bar{\mathbb{F}}_p))$, where $U^{'} = Ker(U \to \mathcal{G})$.

Remark 2.4. By Lemma 2.4 of [5], we see that the above definition is equivalent to saying that ρ is modular of weight V if there is a quaternion algebra D over F split at the primes above p, at τ_1 and at no other archimedean places of F, and a sufficiently small open compact subgroup U of $G(\mathbb{A}^\infty)$ of level prime to p, such that ρ is an $\bar{\mathbb{F}}_p G_F$-subquotient of $(Pic^0(M_{U'})[p](\bar{F}) \otimes V)^{\mathcal{G}}$. Here $U' = Ker(U \to \mathcal{G})$, $\mathcal{G}$ acts diagonally on the tensor product, and G_F acts trivially on V.

In this paper, we use the cohomology version of the definition. For an irreducible $\rho : G_F \to GL_2(\bar{\mathbb{F}}_p)$, from Corollary 2.12 of [5], the following conditions are equivalent:
(1) $\rho \cong \bar{\rho_\pi}$ for some holomorphic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$;
(2) $\rho \cong \bar{\rho_\pi}$ for some holomorphic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ of weight $(\vec{2}, 0)$;
(3) $\rho \cong \chi\bar{\rho_\pi}$ for some holomorphic cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ of weight $(\vec{2}, 0)$ and level $U = U^p U_1(p)$, where $U_1(p) = \{(\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix})_{v|p} \in \prod_{v|p} GL_2(\mathcal{O}_{F_v}) \mid c_v \equiv 0 \pmod{v} \text{ and } d_v \equiv 1 \pmod{v}\}$;
(4) ρ is modular of weight V for some Serre weight V.

Let $\rho : G_F \to GL_2(\bar{\mathbb{F}}_p)$ be a continuous, irreducible and totally odd representation. In [5], the authors construct a set $W(\rho)$ of Serre weights, and make the following conjecture.

Conjecture 2.5. *If $\rho : G_F \to GL_2(\bar{\mathbb{F}}_p)$ is modular, then*

$$W(\rho) = \{V \mid \rho \textit{ is modular of weight } V\}.$$

For the detailed construction of $W(\rho)$ and related topics, we refer to [5]. In fact, for each $v|p$, they define a set of representations $W_v(\rho)$ of $GL_2(k_v)$ depending on $\rho|_{I_v}$, and then define $W(\rho)$ as the set of Serre weights of the form $\otimes_v V_v$ with $V_v \in W_v(\rho)$. This conjecture has been proved in many cases by Toby Gee [20]. In a recent paper [21], the authors proved the conjecture completely. We give the following definition of regularity as in paper [20].

Definition 2.6. We say that a weight $V_{\vec{a},\vec{b}}$ for $GL_2(k_v)$ is *regular* if $2 \le b \le p-2$ for all b. We say that a Serre weight $V = \otimes_{v|p} V_v$ is *regular* if all the V_v's are regular.

Theorem 2.7 (Toby Gee)**.** *Suppose that $V_{\vec{a},\vec{b}} \in W_v(\rho)$ is regular. Then there is a representation $\tilde{V}_{\vec{a},\vec{b}}$ of $GL_2(k_v)$ over $\bar{\mathbb{Q}}_p$ with a $\bar{\mathbb{Z}}_p$-lattice $I(V_{\vec{a},\vec{b}})$, such that there is precisely one of the Jordan-Holder factors of the reduction mod p of $I(V_{\vec{a},\vec{b}})$ belonging to $W_v(\rho)$, and that factor is isomorphic to $V_{\vec{a},\vec{b}}$.*

Proof. In Proposition 3.5.2 and Proposition 4.1.2 of [20], the author constructed an explicit $I(V_{\vec{a},\vec{b}})$ (for each regular V) which satisfies the conditions in the theorem. In this paper, we will take $I(V_{\vec{a},\vec{b}})$ to be the one constructed in [20]. □

Corollary 2.8. *Let $\rho : G_F \to GL_2(\bar{\mathbb{F}}_p)$ be an irreducible modular representation. If $V \in W(\rho)$ is a regular weight of ρ, then there exists a representation $\tilde{V}$ of $\mathcal{G}$ over $\bar{\mathbb{Q}}_p$ with a $\bar{\mathbb{Z}}_p$-lattice I_V, such that there is precisely one of the Jordan-Holder factors of the reduction mod p of I_V in $W(\rho)$, and the factor is isomorphic to V.*

Proof. Assume that $V = \otimes_{v|p} V_{\vec{a},\vec{b}}$. Then by the above theorem, we can take I_V to be $\otimes_{v|p} I(V_{\vec{a},\vec{b}})$, where $I(V_{\vec{a},\vec{b}})$ is the one constructed in [20]. $\square$

3. The deformation problem

For our fixed field F, quaternion algebra D, and group $K = K_1(N)$, let M_K be the corresponding Shimura curve. Let V be a regular Serre weight such that $V = \bigotimes_{v|p} V_{\vec{a},\vec{b}}$. Let V^{op} denote the dual of V. We assume that V is defined over κ.

The cohomology group $H^1(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})$ is a module over the Hecke algebra $\mathbb{T}_\kappa(K, V^{op})$. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}_\kappa(K, V^{op})$. We have an irreducible Galois representation ([6], [33])

$$r_{\mathfrak{m}} : G_F \to GL_2(\mathbb{T}_\kappa(K, V^{op})_{\mathfrak{m}})$$

such that if $v \nmid S_D N p$, then $r_{\mathfrak{m}}$ is unramified at v and $Trace(r_{\mathfrak{m}}(Frob_v)) = T_v$. Write $\bar{r}$ for $r_{\mathfrak{m}}$ mod $(\mathfrak{m})$, then $\bar{r}$ is a modular representation of weight V.

Let I_V be the lattice attached to $(\bar{r} \otimes \bar{\mathbb{F}}_p, V)$ constructed in the proof of Corollary 2.8. We may assume that $\tilde{V}$ is defined over E and I_V is an $\mathcal{O}$-lattice in $\tilde{V}$. We also have a sheaf $\mathcal{F}_{I_V^{op}}$ in characteristic 0. The projection $I_V \to V$ induces natural maps $H^1(M_K \otimes \bar{F}, \mathcal{F}_{I_V^{op}}) \to H^1(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})$ and $\mathbb{T}_{\mathcal{O}}(K, I_V^{op}) \to \mathbb{T}_\kappa(K, V^{op})$. We also write $\mathfrak{m}$ for the preimage of $\mathfrak{m} \subset \mathbb{T}_\kappa(K, V^{op})$ in $\mathbb{T}_{\mathcal{O}}(K, I_V^{op})$ under the natural map. We have the following diagram.

$$\begin{tikzcd} & GL_2(\mathbb{T}_{\mathcal{O}}(K, I_V^{op})_{\mathfrak{m}}) \arrow[d, two heads] \\ G_F \arrow[ur, "\rho_{\mathfrak{m}}"] \arrow[r, "r_{\mathfrak{m}}"] & GL_2(\mathbb{T}_\kappa(K, V^{op})_{\mathfrak{m}}) \end{tikzcd} \tag{3.1}$$

We will write $\bar{\rho}$ for $\rho_{\mathfrak{m}}$ mod $(\mathfrak{m})$. Notice that it is the same as $r_{\mathfrak{m}}$ (mod $\mathfrak{m}$). We impose the following conditions on $\bar{\rho}$:

$$\begin{aligned} &\bar{\rho} \text{ is absolutely irreducible;} \\ &\text{if } v|N, \text{ then } \bar{\rho} \text{ is ramified at } v; \\ &\text{if } v|S_D, \text{ and } Norm(v)^2 \equiv 1 \bmod p, \text{ then } \bar{\rho} \text{ is ramified at } v; \\ &\text{if } v|p, \text{ then } End_{\bar{\mathbb{F}}_p[G_{F_v}]}((\bar{\rho}|_v) \otimes \bar{\mathbb{F}}_p) = \bar{\mathbb{F}}_p. \end{aligned} \tag{3.2}$$

Let $\epsilon : G_F \to \bar{\mathbb{Q}}_p^\times$ be the cyclotomic character. Let $\chi : \mathbb{A}_F^\times / F^\times \to \mathcal{O}^\times$ be a character such that $\chi|_{F_v^\times}$ is trivial if $v \nmid pN'$, $\epsilon(\chi \circ Art^{-1})$ reduces to $det(\rho_{\mathfrak{m}})$, and therefore reduces to $det(\bar{\rho})$. This is possible since we chose E to be sufficiently large.

The rest of this section is devoted to stating a deformation condition for $\bar{\rho}$ which is a good candidate for having $\mathbb{T}_{\mathcal{O}}(K, I_V^{op})_{\mathfrak{m}}$ as the universal deformation ring.

3.1. **Deformation conditions at $v|p$.** Fix $v|p$. If a p-adic representation ρ_v of G_{F_v} is potentially semistable, one associates to ρ_v a Weil-Deligne representation $WD(\rho_v)$ over $\bar{\mathbb{Q}}_p$ (See for example Appendix B.1 of [11]). Then ρ_v becomes semistable over L if and only if $WD(\rho_v)|_{I_L}$ is trivial. The Galois type $\tau(\rho_v)$ associated to such ρ_v is defined to be the isomorphism class of the representation $WD(\rho_v)|_{I_v}$.

Let $V = \bigotimes_{v|p} V_{\vec{a},\vec{b}}$ be the regular Serre weight and $\bar{\rho}$ be the representation of G_F constructed at the beginning of this section. Let $\bar{\rho}_v = \bar{\rho}|_{G_{F_v}}$. For $A \in \mathcal{C}_{\mathcal{O}}$, we consider the deformations $\rho_v : G_{F_v} \to GL_2(A)$ of $\bar{\rho}_v$ with the following properties:

(1) ρ_v is potentially semistable with Hodge-Tate weights $(0,1)$,

(2) $\tau(\rho_v)$ is isomorphic to τ_v,

(3) $\det(\rho_v) = \epsilon(\chi \circ Art^{-1})$ where ϵ is the cyclotomic character and χ is the character defined as above.

Here if $\bar{\rho}_v$ is reducible, then τ_v is defined by equation (3.4); if $\bar{\rho}_v$ is irreducible, then τ_v is defined by equation (3.5). Define $H^1_f(G_{F_v}, Ad(\bar{\rho}_v)) \subset H^1(G_{F_v}, Ad(\bar{\rho}_v))$ to be the set of extensions of $\bar{\rho}_v$ by itself which are potentially Barsotti-Tate. We will use the theory of Breuil modules with descent data to compute the size of $H^1_f(G_{F_v}, Ad(\bar{\rho}_v))$. For a detail introduction to this theory, please see section 6.

3.1.1. *Define τ_v in the reducible case.* Assume that
$$\bar{\rho}_v \sim \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix} \text{ is reducible and nonsplit.}$$
Let $L = F_v((-p)^{\frac{1}{p^{d_v}-1}})$, where $d_v = [F_v : \mathbb{Q}_p]$. Let $\omega : Gal(L/F_v) \to k_v^\times$ be the map defined by $\omega(g) = g(\pi)/\pi \pmod{\pi}$, where $\pi = (-p)^{\frac{1}{p^{d_v}-1}}$ is a uniformizer of L. Fix $\tau_0 : k_v \to \bar{\mathbb{F}}_p$ and define $\tau_i = \tau_0 \circ Frob^{-i}$ where $Frob$ is given by $(x \mapsto x^p)$. Define $\omega_i = \tau_i \circ \omega : Gal(L/F_v) \to \bar{\mathbb{F}}_p^\times$. We assume that κ is large enough that it contains the image of one, and hence all, ω_i.

Let $S_v = Hom(F_v, \bar{\mathbb{Q}}_p) \cong Hom(k_v, \bar{\mathbb{F}}_p) \cong \mathbb{Z}/d_v$. From section 3.5 of [20], there exists a subset $J_v \subset S_v = Hom(F_v, \bar{\mathbb{Q}}_p)$, such that
$$\psi_1|_{I_{F_v}} = \prod_{i \in S_v} \omega_i^{a_i} \prod_{i \in J_v} \omega_i^{b_i}, \quad \psi_2|_{I_F} = \prod_{i \in S_v} \omega_i^{a_i} \prod_{i \notin J_v} \omega_i^{b_i}.$$
We define
$$c_i = \begin{cases} b_i - \delta_{J_v}(i+1) & \text{if } i \in J_v \\ p - b_i - \delta_{J_v}(i+1) & \text{if } i \notin J_v \end{cases} \tag{3.3}$$
where $\delta_{J_v}(i) = \begin{cases} 1 \text{ if } i \in J_v, \\ 0 \text{ if } i \notin J_v. \end{cases}$ We then define a type τ_{J_v} by
$$\tau_{J_v} = \tilde{\chi}_{J_v} \oplus \tilde{\chi}_{J_v} \prod_{i \in S_v} \tilde{\omega}_i^{c_i},$$
where ˜ means Techmuller lift and
$$\chi_{J_v} = \prod_{i \in S} \omega_i^{a_i} \prod_{i \notin J_v} \omega_i^{b_i - p}.$$
In the reducible case, we define
$$\tau_v = \tau_{J_v}. \tag{3.4}$$
Remark 3.1. Notice that in this case, the lattice constructed by Gee in Theorem 2.7 is a lattice in $I(\tilde{\chi}_{J_v}, \tilde{\chi}_{J_v} \prod_{i \in S_v} \tilde{\omega}_i^{c_i})$, which is a representation of $GL_2(k_v)$ defined in section 1 of [15].

3.1.2. *Define τ_v in the irreducible case.* Assume that $\bar{\rho}_v : G_{F_v} \to GL_2(\kappa)$ is irreducible. Let F'_v be the degree two unramified extension of F_v. Let k'_v be the residue field of F'_v. Then k'_v has degree two over k_v. Let $L = F'_v((-p)^{\frac{1}{p^{2d_v}-1}})$, where $d_v = [F_v : \mathbb{Q}_p]$. Let $\omega : Gal(L/F_v) \to k_v^\times$ be the map defined by $\omega(g) = g(\pi)/\pi \pmod{\pi}$, where $\pi = (-p)^{\frac{1}{p^{2d_v}-1}}$ is a uniformizer of L. Fix $\tau_0 : k'_v \to \bar{\mathbb{F}}_p$ and define $\tau_i = \tau_0 \circ Frob^{-i}$ where $Frob$ is the Frobenius of k'_v given by $(x \mapsto x^p)$. Define $\omega_i = \tau_i \circ \omega : Gal(L/F_v) \to \bar{\mathbb{F}}_p^\times$. We assume that κ contains the image of all ω_i.

Let $S_v = Hom(F_v, \bar{\mathbb{Q}}_p) \cong Hom(k_v, \bar{\mathbb{F}}_p) \cong \mathbb{Z}/d_v$, $S'_v = Hom(F'_v, \bar{\mathbb{Q}}_p) \cong Hom(k'_v, \bar{\mathbb{F}}_p) \cong \mathbb{Z}/2d_v$. We say a subset $J_v \subset S'_v$ is $full$ if the restriction to J_v of the natural projection $\pi : S' \to S$ is a bijection onto S. From section 4.1 of [20], there exists a full subset $J_v \subset S'_v = Hom(F'_v, \bar{\mathbb{Q}}_p)$, such that

$$\bar{\rho}|_{I_{F'_v}} \sim \prod_{\sigma \in S_v} \omega_\sigma^{a_\sigma} \begin{pmatrix} \prod_{\sigma \in J_v} \omega_\sigma^{b_\sigma} & 0 \\ 0 & \prod_{\sigma \notin J_v} \omega_\sigma^{b_\sigma} \end{pmatrix}$$

where we write a_σ and b_σ for $a_{\pi(\sigma)}$ and $b_{\pi(\sigma)}$ respectively.

For the given regular weight $V_{\vec{a},\vec{b}}$ of $GL_2(k_v)$ and the full subset $J_v \subset S'$, we define a representation of $GL_2(k_v)$ and a type as follows. Let $K_{J_v} = \pi(J_v \cap \{1, \cdots, d_v\})$. Then let

$$c_i = \begin{cases} b_i + \delta_{K_{J_v}}(1) - 1 & \text{if } 0 = i \in K_{J_v} \\ p - b_i + \delta_{K_{J_v}}(1) - 1 & \text{if } 0 = i \notin K_{J_v} \\ b_i + \delta_{K_{J_v}}(i+1) & \text{if } 0 \neq i \in K_{J_v} \\ p - b_i - \delta_{K_{J_v}}(i+1) & \text{if } 0 \neq i \notin K_{J_v} \end{cases}$$

Define a character

$$\psi_{J_v} = \omega_0^{-\delta_{K_J}(1)} \prod_{i \in S_v} \omega_i^{a_i} \prod_{i \notin K_{J_v}} \omega_i^{b_i - p}.$$

Then we define

$$I'_{J_v} = \Theta(\tilde{\psi}_{J_v} \tilde{\omega}_{d_v} \prod_{i=1}^{d_v} \tilde{\omega}_i^{c_i})$$

and

$$\tau'_{J_v} = \tilde{\psi}_{J_v} \tilde{\omega}_{d_v} \prod_{i=1}^{d_v} \tilde{\omega}_i^{c_i} \oplus (\tilde{\psi}_{J_v} \tilde{\omega}_{d_v} \prod_{i=1}^{d_v} \tilde{\omega}_i^{c_i})^{p^r}.$$

Here $\tilde{}$ means Techmuller lift, $\Theta(\tilde{\psi}_{J_v} \tilde{\omega}_{d_v} \prod_{i=1}^{d_v} \tilde{\omega}_i^{c_i})$ is a cuspidal representation of $GL_2(k_v)$ defined in section 1 of [15]. We define

$$\tau_v = \tau'_{J_v}. \tag{3.5}$$

Remark 3.2. In this case, the lattice constructed by Gee in Theorem 2.7 is a lattice in $\Theta(\tilde{\psi}_{J_v} \tilde{\omega}_{d_v} \prod_{i=1}^{d_v} \tilde{\omega}_i^{c_i})$.

3.1.3. *Local Selmer groups.* Fix $v|p$. Recall that $H^1_f(G_{F_v}, Ad(\bar{\rho}_v)) \subset H^1(G_{F_v}, Ad(\bar{\rho}_v))$ is the subset of infinitesimal deformations of $\bar{\rho}_v$ which are potentially Barsotti-Tate. We have the following lemma.

Lemma 3.3. *If $\bar{\rho}_v$ is non-split, then*
(1) $\dim_\kappa H^1_f(G_{F_v}, Ad(\bar{\rho}_v)) = 1 + d_v$.
(2) $\dim_\kappa H^1_f(G_{F_v}, Ad^0(\bar{\rho}_v)) = d_v$.

This lemma is proved in section 6.4 by using the theory of Breuil modules with descent data.

3.2. **Deformation conditions at primes dividing NS_D.** (The argument in this section is based on the idea in [35] section 2.2.) Fix a finite prime $v|NS_D$. If $v|N$, then $\bar{\rho}$ is ramified at v, and we consider the deformations ρ_v of $\bar{\rho}|_{G_{F_v}}$ such that $\rho_v(I_v) \subset \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. (See section 2.7 of [12] for more details.)

If $v|S_D$, we need to consider the deformations of $\bar{\rho}$ which are special at v. Let g be a weight $(2, ..., 2)$ Hilbert eigenform such that $\bar{\rho}_g \simeq \bar{\rho}$. Let π_g be the automorphic representation of $GL_2(\mathbb{A}_F)$. Then the local component $\pi_{g,v}$ is special of conductor v if and only if $\rho_g \mid_{I_v} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ with $*$ ramified.

First, if $\bar{\rho}$ is ramified at v, we get a suitable deformation condition at v by requiring the restriction to I_v to be unipotent.

If $\bar{\rho}$ is unramified at v, we have to rule out those deformations of $\bar{\rho}$ arising from Hilbert modular forms which are not special at v. Note in this case, $\epsilon(\chi \circ Art^{-1}) = \epsilon$.

Lemma 3.4. *Let $v \nmid p$ be a finite prime such that $p \nmid (Norm(v)^2 - 1)$. Let $\bar{\rho}_v : G_{F_v} \to GL_2(\kappa)$ be an unramified representation. Assume that $\bar{\rho}_v(Frob_v) = \pm \begin{pmatrix} Norm(v) & 0 \\ 0 & 1 \end{pmatrix}$. Then every deformation of $\bar{\rho}_v$ over an $\mathcal{O}$-algebra $A \in \mathcal{C}_\mathcal{O}$ is strictly equivalent to an upper triangular representation ρ such that $\rho(I_v) \subset \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.*

Proof. Let $\mathfrak{m}_A$ be the maximal ideal of A. Since $\bar{\rho}_v$ is unramified, $\rho(I_v) \subset 1 + \mathfrak{m}_A$, and the wild inertia group acts trivially. Let f be $Frob_v$ in G_{F_v}, and let t be a topological generator of I_v^{tame}. Since $p \nmid (Norm(v)^2 - 1)$, we may assume that $\rho(f) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a \equiv \pm Norm(v)$, $b \equiv \pm 1 \pmod{\mathfrak{m}_A}$. It suffices to prove that $\rho(t)$ has the form $\begin{pmatrix} 1 & \iota \\ 0 & 1 \end{pmatrix}$ for some $\iota \in \mathfrak{m}_A$. By induction on n, we write $\rho(t) = \begin{pmatrix} 1 & \iota_n \\ 0 & 1 \end{pmatrix} + \Theta_n$, with $\Theta_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \equiv 0 (\mathrm{mod}\ \mathfrak{m}_A^n)$. Using the relation $ftf^{-1} = t^{Norm(v)}$, we have

$$\rho(ftf^{-1}) = (\begin{pmatrix} 1 & \iota_n \\ 0 & 1 \end{pmatrix} + \Theta_n)^{Norm(v)} \equiv \begin{pmatrix} 1 & Norm(v)\iota_n \\ 0 & 1 \end{pmatrix} + Norm(v)\Theta_n \ \mathrm{mod}(\mathfrak{m}_A^{n+1}).$$

On the other hand,

$$\rho(ftf^{-1}) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} (\begin{pmatrix} 1 & \iota_n \\ 0 & 1 \end{pmatrix} + \Theta_n) \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

Comparing the entries shows that a_n, c_n, $d_n \in \mathfrak{m}_A^{n+1}$. The desired result follows by the topological nilpotency of $\mathfrak{m}_A$. ∎

Considering the deformations of $\bar{\rho}_v$, by the previous lemma, every class of strict equivalence of deformations ρ over A with cyclotomic determinant is determined by a pair of elements (μ, ι) in $\mathfrak{m}_A$, given by

$$\rho(f) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \rho(t) = \begin{pmatrix} 1 & \iota \\ 0 & 1 \end{pmatrix},$$

where $a = \pm Norm(v) + \mu$, $b = Norm(v)/a$, such that

$$\begin{pmatrix} a & 0 \\ 0 & Norm(v)/a \end{pmatrix} \begin{pmatrix} 1 & \iota \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a/Norm(v) \end{pmatrix} = \begin{pmatrix} 1 & Norm(v)\iota \\ 0 & 1 \end{pmatrix}.$$

The last equation is $(a^2/Norm(v))\iota = Norm(v)\iota$, which is the same as $\mu\iota = 0$. Moreover, two deformations ρ_1, ρ_2 corresponding to the pairs (μ_1, ι_1) and (μ_2, ι_2) respectively are strictly equivalent if and only if there exists $M \in Id_{2\times 2} + M_2(\mathfrak{m}_A)$ such that $M\rho_1(f)M^{-1} = \rho_2(f)$ and $M\rho_1(t)M^{-1} = \rho_2(t)$, if and only if $\mu_1 = \mu_2$ and $\iota_2 \in (1 + \mathfrak{m}_A)\iota_1$. Then the universal deformation ring of $\bar{\rho}_v$ is given by $\mathcal{R}_v = \mathcal{O}[[X, Y]]/(XY)$. If the residue representation $\bar{\rho}$ is suitably diagonalized, then the universal deformation ρ_v^{univ} is given by

$$\rho_v^{univ}(f) = \begin{pmatrix} \pm Norm(v) + X & 0 \\ 0 & Norm(v)/(\pm Norm(v) + X) \end{pmatrix}, \quad \rho_v^{univ}(t) = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}.$$

Definition 3.5. Let v be a finite prime such that $p \nmid (Norm(v)^2 - 1)$ and $\bar{\rho}_v$ is unramified at v. We say that a deformation ρ of $\bar{\rho}_v$ over an $\mathcal{O}$-algebra $A \in \mathcal{C}_{\mathcal{O}}$ satisfies *sp-condition* if the homomorphism $\psi : \mathcal{R}_v \to A$ associated to ρ has $\psi(X) = 0$.

Remark 3.6. From the computation, it is obvious that a deformation ρ satisfies sp-condition if and only if

$$Trace(\rho(f))^2 = (Norm(v) + 1)^2.$$

Remark 3.7. Suppose that $\bar{\rho}$ is unramified at v, where $Norm(v)^2 \not\equiv 1 \bmod p$. Let g be a Hilbert modular form such that $\bar{\rho}_{g,v} \sim \bar{\rho}_v$. If g is special at v, then $\rho_{g,v} \sim \begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix} \otimes \psi$ for some unramified character ψ. Therefore $\rho_{g,v}$ satisfies the sp-condition. On the other hand, if g is not special at v, then by local Langlands correspondence for GL_2, $\rho_{g,v}$ does not satisfy sp-condition. Indeed, if we assume that $\pi_{g,v} = Ind_B^{GL_2}\mu_1 \otimes \mu_2$ such that $\mu_1\mu_2^{-1} \neq |\cdot|^{\pm}$, then $Trace(\rho(f_{g,v}))^2 = (\mu_1(\varpi_v) + \mu_2(\varpi_v))^2 \neq (Norm(v) + 1)^2$, where ϖ_v is a uniformizer of F_v.

If $v | S_D$ such that $p \nmid (Norm(v)^2 - 1)$ and $\bar{\rho}_v$ is unramified at v, we consider the deformations of $\bar{\rho}_v$ which are special. This space includes the representations coming from Hilbert modular forms which are special at v. The corresponding universal deformation ring is

$$\mathcal{R}_{v,sp} = \mathcal{O}[[X, Y]]/(X, XY) = \mathcal{O}[[Y]].$$

If we think of the geometric picture, we just choose an irreducible component from the universal deformation space.

3.3. **The global deformation conditions.**

Definition 3.8. Let Σ be a square free ideal of $\mathcal{O}_F$, prime to pNS_D. We consider the functor $\mathcal{Q}_\Sigma$ from $\mathcal{C}_\mathcal{O}$ to the category of sets which associates to an object A in $\mathcal{C}_\mathcal{O}$ the set of strictly equivalent classes of continuous homomorphisms $\rho : G_F \to GL_2(A)$ lifting $\bar{\rho}$ satisfying the following conditions:
(1) ρ is unramified outside $pN^{'}S_D\Sigma$;
(2) if $v|N^{'}$, i.e. $\bar{\rho}$ is ramified at v, then $\rho|_{I_v}$ is unipotent;
(3) if $v|S_D$, $\bar{\rho}$ is unramified at v, then ρ satisfies sp-condition at v;
(4) if $v|p$, then $\rho_v = \rho|_{G_{F_v}}$ is Barsotti-Tate over L and $WD(\rho_v) \cong \tau_v$;
(5) $det(\rho) = \epsilon(\chi \circ Art^{-1})$.

Proposition 3.9. *The functor $\mathcal{Q}_\Sigma$ is representable.*

Proof. We need to check that being potentially Barsotti-Tate is a deformation condition, i.e., it satisfies conditions (1)-(3) in section 6 of [13]. This is true by the results proved in Appendix B of [11]. □

We say that the functor is represented by the universal deformation

$$\rho_\Sigma^{univ} : G_F \to GL_2(\mathcal{R}_\Sigma).$$

We use the notation $\mathcal{R}$ and ρ^{univ} if $\Sigma = \emptyset$.

4. Taylor-Wiles system

In this section, we construct a Taylor-Wiles system corresponding to our potentially Barsotti-Tate deformation problem. Recall that $K = K_1(N)$ and $K^{'} = K_1(N)^p \times (1 + pM_2(\mathcal{O}_p)) \subset K$. We have a short exact sequence

$$0 \to K^{'} \to K \to \mathcal{G} \to 0.$$

The aim is to prove the following theorem.

Theorem 4.1. *Let $\bar{\rho}$ be the Galois representation constructed at the beginning of section 3. Suppose that $\bar{\rho}$ satisfies the conditions in equation (3.2) and the restriction of $\bar{\rho}$ to the absolute Galois group of $F(\sqrt{(-1)^{(p-1)/2}p})$ is irreducible. Then there is a natural surjection*

$$\mathcal{R} \to \mathbb{T}_\mathcal{O}(K, I_V^{op})_\mathfrak{m}.$$

Furthermore, it is an isomorphism of complete intersections and $H^1(M_K \otimes \bar{F}, \mathcal{F}_{I_V^{op}})_\mathfrak{m}$ is free of rank two as a $\mathbb{T}_\mathcal{O}(K, I_V^{op})_\mathfrak{m}$ module.

4.1. **The construction.** Let Q be a finite set of finite places of F not dividing pNS_D such that if $x \in Q$, then
- $Norm(x) \equiv 1 \pmod p$,
- $\bar{\rho}$ is unramified at x and $\bar{\rho}(Frob_x)$ has distinct eigenvalues $\alpha_x \neq \beta_x$.

By Hensel's Lemma the polynomial $X^2 - T_x X + Norm(x)\chi(\varpi_x)$ splits as $(X - A_x)(X - B_x)$ in $\mathbb{T}_\mathcal{O}(K, I_V^{op})_\mathfrak{m}$.

For $x \in Q$ we will let Δ_x denote the maximal p-power quotient of $(\mathcal{O}_F/x)^\times$. We will let $\mathfrak{n}_Q = \prod_{x\in Q} x$; $\Delta_Q = \prod_{x \in Q} \Delta_x$; $K_{0,Q} = K_{\{1\}}(\mathfrak{n}_Q) \cap K$; and $K_{1,Q} = K_{\Delta_Q}(\mathfrak{n}_Q) \cap K$. Let $\mathfrak{m}_Q$ denote the ideal of either $\mathbb{T}_\mathcal{O}(K_{0,Q}, I_V^{op})$ or $\mathbb{T}_\mathcal{O}(K_{1,Q}, I_V^{op})$ generated by

• p;
• $T_x - Trace(\bar{\rho}(Frob_x))$ for $x \nmid pN\mathfrak{n}_Q$.

Remark 4.2. Let $\mathbb{T}^{red}_{\mathcal{O}}(K_{0,Q}, I_V^{op})$ be the Hecke algebra $\mathbb{T}_{\mathcal{O}}(K_{0,Q}, I_V^{op})[U_x|x \in Q]$, and let $\mathfrak{m}^{red}$ be the maximal ideal of $\mathbb{T}_{\mathcal{O}}(K_{0,Q}, I_V^{op})$ generated by
• p;
• $T_x - Trace(\bar{\rho}(Frob_x))$ for $x \nmid pN\mathfrak{n}_Q$; and
• $U_x - \alpha_x$ for $x \in Q$.
Then we have an isomorphism $\mathbb{T}^{red}_{\mathcal{O}}(K_{0,Q}, I_V^{op})_{\mathfrak{m}^{red}} \cong \mathbb{T}_{\mathcal{O}}(K_{0,Q}, I_V^{op})_{\mathfrak{m}}$.

Lemma 4.3. *Let Q satisfy the assumptions as above, then*
(1) if $x \in Q$, then $\rho_Q^{univ}|_{G_{F_x}} \sim \chi_{\alpha,x} \oplus \chi_{\beta,x}$ where $\chi_{\alpha,x}$ and $\chi_{\beta,x}$ mod $\mathfrak{m}_{\mathcal{R}_Q}$ are unramified and take $Frob_x$ to α_x and β_x.
(2) All $\chi_{\alpha,x} \circ Art\, |_{\mathcal{O}^{\times}_{F,x}}$ factor through Δ_x and these maps make $\mathcal{R}_Q$ into an $\mathcal{O}[\Delta_Q]$-module.
(3) The universal property of $\mathcal{R}_Q$ gives rise to a surjection of $\mathcal{O}[\Delta_Q]$-algebras

$$\mathcal{R}_Q \twoheadrightarrow \mathbb{T}_{\mathcal{O}}(K_{1,Q}, I_V^{op})_{\mathfrak{m}_Q}$$

under which ρ_Q^{univ} pushes forward to $\rho_{\mathfrak{m}_Q}$.

Proof. (1) Since $\bar{\rho}$ is unramified at x, and $\mathcal{R}_Q$ is a p-adic ring, $\rho_Q^{univ}(I_x)$ is a pro-p group. Since $x \nmid p$, ρ_Q^{univ} is tamely ramified at x. Let f and t be the generators of $Gal(F_x^{tr}/F_x)$ such that f restricts to the Frobenius automorphism on the maximal unramified extension F_x^{ur} of F_x, and t fixes F_x^{ur}. Using the fact that $\alpha_x \neq \beta_x$, choose a basis for the space of ρ_Q^{univ} in which $\rho_Q^{univ}(f) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is diagonal. Since $\bar{\rho}$ is unramified at x, $\rho_Q^{univ}(t) \equiv 1$ mod $\mathfrak{m}_{\mathcal{R}_Q}$. Now suppose that $\rho_Q^{univ}(t) = 1 + N$, where $N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$ with $n_{11}, n_{22} \in \mathfrak{m}_{\mathcal{R}_Q}$ and $n_{12}, n_{12} \in \mathfrak{m}^n_{\mathcal{R}_Q}$, and $n > 0$. Using the relation $ftf^{-1} = t^q$ (here $q = Norm(v)$) one gets

$$1 + \begin{pmatrix} n_{11} & ab^{-1}n_{12} \\ a^{-1}bn_{21} & n_{22} \end{pmatrix} \equiv (1 + qN) \pmod{\mathfrak{m}^{n+1}_{\mathcal{R}_Q}}$$

which implies that N is diagonal mod $\mathfrak{m}^{n+1}_{\mathcal{R}_Q}$, since $a \neq b$ and $q \equiv 1$ mod p. The desired result follows by induction.

(2) This is clear since $\mathcal{R}_Q$ is a p-adic ring.

(3) There exists such a map because of the construction of I_V and τ_v. From the definition of Hecke algebra, it is easy to see that it is surjective. It induces the map between representations because $Trace(\rho_Q^{univ}(Frob_x)) \mapsto T_x$ for $x \nmid pN\mathfrak{n}_Q$ and $\chi_{\alpha,x}(Frob_x) \mapsto U_x$ for $x \in Q$. □

4.2. **Basic properties.** In this subsection, we write $H^1(H)$ for the cohomology group $H^1_{et}(M_H, \mathcal{F}_{I_V^{op}})$ and $\mathbb{T}(H)$ for the Hecke algebra $\mathbb{T}_{\mathcal{O}}(H, I_V^{op})$.

Lemma 4.4. *For any $x \in Q$, the map*

$$\eta : H^1(K_{0,Q-x})_{\mathfrak{m}_{Q-x}} \to H^1(K_{0,Q})_{\mathfrak{m}_Q}$$

given by $f \mapsto A_x 1_* f - \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}_* f$ *is an isomorphism which induces an isomorphism*

$$\eta' : \mathbb{T}(K_{0,Q})_{\mathfrak{m}_Q} \to \mathbb{T}(K_{0,Q-x})_{\mathfrak{m}_{Q-x}}.$$

Proof. The map η is well defined because $K_{0,Q} \subset K_{0,Q-x}$, $\begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}^{-1} K_{0,Q} \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix} \subset K_{0,Q-x}$, and $U_x \circ \eta = \eta \circ A_x$. From the natural inclusion $K_{0,Q} \hookrightarrow K_{0,Q-x}$, we have a map $1_* : H^1(K_{Q-x}) \to H^1(K_Q)$. From section 3 of [17], we know that the adjoint of 1_* is 1^*, the composition of η with 1^* is $1^* A_x 1_* - 1^* \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}_* = Norm(x)(A_x) - T_x = (Norm(x) - 1)A_x - B_x \notin \mathfrak{m}_Q$, so η is injective with torsion free cokernel. As $\alpha_x \neq \beta_x$, no lift of $\bar{\rho}$ with the required determinant has conductor at x exactly x. Thus

$$H^1(K_{0,Q})_{\mathfrak{m}_Q} = (1_* H^1(K_{0,Q-x}) + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}_* H^1(K_{0,Q-x}))_{\mathfrak{m}_Q}.$$

Furthermore,

$$U_x(1_* f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}_* f_2) = 1_*(T_x f_1 + (Norm(x))\chi(\varpi_x) f_2) - \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}_* f_1$$

and the matrix $\begin{pmatrix} T_x & Norm(x)\chi(\varpi_x) \\ -1 & 0 \end{pmatrix}$ has eigenvalues A_x and B_x which are distinct mod $\mathfrak{m}_Q$. The lemma follows. □

Remark 4.5. The identities in the proof of the above lemma all come from the double coset decompositions at the end of section 1. See for example chapter one of [26] and Lemma 2.2 of [32].

Lemma 4.6. *Let* Λ *be a finite abelian group. If we consider the group ring* $\mathcal{O}[\Lambda]$ *as a* Λ*-module, then* $\mathcal{O}[\Lambda]^\Lambda \simeq \mathcal{O}[\Lambda]_\Lambda$.

Proof. Let $\mathcal{N} : \mathcal{O}[\Lambda] \to \mathcal{O}[\Lambda]$ be the usual norm map which sends $\xi \in \mathcal{O}[\Lambda]$ to $\sum_{\lambda \in \Lambda} \lambda\xi$. It is easy to see that $Im\mathcal{N} \subset \mathcal{O}[\Lambda]^\Lambda$, and $(1 - \Lambda)\mathcal{O}[\Lambda] \subset Ker\mathcal{N}$. It suffices to prove that the two relations are actually equalities.

If $\sum_\lambda o_\lambda \lambda \in \mathcal{O}[\Lambda]^\Lambda$, then

$$\sum_\lambda o_\lambda \lambda = \lambda_1^{-1} \sum_\lambda o_\lambda \lambda.$$

Comparing the coefficients, we get $o_{\lambda_1} = o_1$. Then all the o_λ are the same, and so $\sum_\lambda o_\lambda \lambda \in Im\mathcal{N}$.

If $\sum_\lambda o_\lambda \lambda \in Ker\mathcal{N}$, then $0 = \mathcal{N} \sum_\lambda o_\lambda \lambda$, and by computing the coefficient of 1, we get $\sum_\lambda o_\lambda = 0$. Therefore $\sum_\lambda o_\lambda \lambda \in (1 - \Lambda)\mathcal{O}[\Lambda]$. □

Lemma 4.7. *(1)* $H^1(K_{1,Q})_{\mathfrak{m}_Q}$ *is a free* $\mathcal{O}[\Delta_Q]$*-module.*
(2) $(H^1(K_{1,Q})_{\Delta_Q})_\mathfrak{m} \to H^1(K)_\mathfrak{m}$ *is an isomorphism compatible with a map* $(\mathbb{T}(K_{1,Q})_{\Delta_Q})_\mathfrak{m} \to \mathbb{T}(K)_\mathfrak{m}$ *sending* T_x *to* T_x *for* $x \nmid pS_D\mathfrak{n}_Q$, $< h >$ *to* 1 *for* $h \in \Delta_Q$, *and* U_x *to* A_x *for* $x \in Q$.

Proof. (1) $H^1(K_{1,Q})_{\mathfrak{m}_Q}$ is certainly free over $\mathcal{O}$. Then to prove the freeness over $\mathcal{O}[\Delta_Q]$, we only need to prove that $H^i(\Delta_Q, H^1(K_{1,Q})_{\mathfrak{m}_Q}) = 0 \ \ \forall i > 0$. We have a short exact sequence

$$0 \to K_{1,Q} \to K_{0,Q} \to \Delta_Q \to 0.$$

By Serre-Hochschild spectral sequence, we have

$$E_2^{p,q} := H^p(\Delta_Q, H^q(K_{1,Q})) \Rightarrow H^{p+q}(K_{0,Q}).$$

Since localization is exact, we get another spectral sequence

$$E_2^{'p,q} := H^p(\Delta_Q, H^q(K_{1,Q})_{\mathfrak{m}_Q}) \Rightarrow H^{p+q}(K_{0,Q})_{\mathfrak{m}_Q}.$$

By Lemma 2.2(1), if $p+q \geq 2$, $H^{p+q}(K_{0,Q})_{\mathfrak{m}_Q} = 0$. $E_2^{'1,0} = 0$ since $H^0(K_{1,Q})_{\mathfrak{m}_Q} = 0$. Therefore

$$H^i(\Delta_Q, H^1(K_{1,Q})_{\mathfrak{m}_Q}) = H^{i+1}(K_{0,Q})_{\mathfrak{m}_Q} = 0 \ \ \forall i > 0.$$

(2)Using the same spectral sequence, we have an exact sequence

$$0 \to H^1(\Delta_Q, H^0(K_{0,Q})_{\mathfrak{m}_Q}) \to H^1(K_{0,Q})_{\mathfrak{m}_Q} \to H^1(K_{1,Q})_{\mathfrak{m}_Q}^{\Delta_Q} \to H^2(\Delta_Q, H^0(K_{0,Q})_{\mathfrak{m}_Q}) \to \dots$$

By Lemma 2.2(1), we get an isomorphism

$$H^1(K_{0,Q})_{\mathfrak{m}_Q} \simeq H^1(K_{1,Q})_{\mathfrak{m}_Q}^{\Delta_Q}.$$

By the last lemma and the freeness of $H^1(K_{1,Q})_{\mathfrak{m}_Q}$, we get the desired isomorphism. □

4.3. **Computing the Selmer groups.** The computation is standard. We sketch it here. For more details about the computation, and the relation between Selmer groups and Galois representations, see [36], [25], [31], and [37].

Let $\bar{\rho}$ be our mod p Galois representation, and define

$$W = ad^0(\bar{\rho}) \quad \text{(trace zero matrices in the adjoint representation of } \bar{\rho}). \tag{4.1}$$

$$W^* = Hom(W, \mu_p) \cong W(1) \cong Sym^2(\bar{\rho}). \tag{4.2}$$

We then define the *local conditions*, which are subgroups L_v of $H^1(G_{F_v}, W)$ for the various decomposition groups G_{F_v}. (It is used to determine global cohomology classes whose restrictions to every G_{F_v} fall into L_v.) Since $p \geq 5$, $H^1(G_{F_v}, W) = 0$ if $v|\infty$, and we only have to define those local conditions at the finite places. Let

(1) $L_v = H^1(G_{F_v}/I_v, W^{I_v})$ if $v \mid N^{'}$;
(2) $L_v = H^1_f(G_{F_v}, W)$ if $v \mid p$;
(3) $L_v = H^1(G_{F_v}, W)$ if $v \mid Q$;
(4) $L_v = H^1_{sp}(G_{F_v}, W)$ if $v|S_D$ and $v \nmid N'$, where $H^1_{sp}(G_{F_v}, W)$ consists of the elements whose corresponding Galois representations satisfying the sp-condition.

Now we compute the sizes of these local terms in our deformation problem.

• $\sharp L_v = \sharp H^0(G_{F_v}, W) \cdot (\sharp\kappa)^{[F_v:\mathbb{Q}_p]}$ if $v \mid p$

This is from Lemma 3.3.

• $\sharp L_v = \sharp H^0(G_{F_v}, W)/\sharp\kappa = 1$ if $v \mid \infty$

Here $L_v = H^1(G_{F_v}, W) = 0$ by definition. Since the eigenvalues of the complex conjugation are ± 1, the eigenvalues on W are $\{-1, -1, 1\}$, so $\sharp H^0(G_{F_v}, W)/\sharp k = 1$.

• $\sharp L_v = \sharp H^0(G_{F_v}, W)$ for $v \mid N'$

There is an exact sequence

$$0 \to W^{G_{F_v}} \to W^{I_v} \to W^{I_v} \to W^{I_v}/(Frob_v - 1)W^{I_v} \to 0,$$

where the middle map is $(Frob_v - 1)$. The exactness at the first W^{I_v} follows from the fact that if $w \in W^{I_v}$ and $(Frob_v - 1)w = 0$, then w is fixed by both I_v and $Frob_v$, which topologically generate G_{F_v}. The first term gives $H^0(G_{F_v}, W)$ and the last term gives L_v.

• $\sharp L_v = \sharp\kappa$ if $v|S_D$ and $v \nmid N'$

This is true because of the formula at the end of section 3.3.

• $\sharp H^0(G_{F_v}, W) = \sharp\kappa$ if $v|S_D$ and $v \nmid N'$

This is true because the eigenvalues of $\bar{\rho}(Frob_v)$ are distinct, by condition (3.2).

• $H^0(F, W) = H^0(F, W^*) = 0$

For W this follows from the irreducibility of $\bar{\rho}$, since by Schur's lemma the only endomorphisms commuting with the Galois action are the scalars, which are missing from W.

For W^* we use equation (4.2). A Galois-invariant vector in W^* is an invariant symmetric bilinear form. If the bilinear form is degenerate, then its kernel is invariant, contradicting the irreducibility of $\bar{\rho}$. If the bilinear form is nondegenerate, this means that the image of $\bar{\rho}$ is contained in some orthogonal group. This contradicts the fact that $det(\bar{\rho}) \neq 1$.

• $\sharp H^0(G_{F_v}, W) = \sharp\kappa$ for $v \mid Q$

By assumption on Q, the eigenvalues of $Frob_v$ on W are $\alpha_v\beta_v = Norm(v) = 1$, α_v^2 and β_v^2. The latter two are not equal to 1 since $\alpha_v \neq \beta_v$.

• $\sharp H^1(G_{F_v}, W) = \sharp\kappa^2$ for $v \mid Q$

First, we have the inflation-restriction exact sequence

$$\begin{aligned} 0 \to H^1(G_{F_v}/I_v, W^{I_v}) \to H^1(G_{F_v}, W) \to H^1(I_v, W)^{G_{F_v}/I_v} \\ \to H^2(G_{F_v}/I_v, W^{I_v}) \to H^2(G_{F_v}, W). \end{aligned}$$

Since $\bar{\rho}$ is unramified at v, $W^{I_v} = W$. By the assumption on α_v and β_v, $H^1(G_{F_v}/I_v, W) = W/(Frob_v - 1)W$ is one-dimensional,

$$\begin{aligned} H^1(I_v, W)^{G_{F_v}/I_v} &= Hom(\mathbb{Z}_p(1), W)^{Frob_v} \\ &= W[Frob_v - Norm(v)] \\ &= W[Frob_v - 1] \\ &= W^{Frob_v} \end{aligned} \tag{4.3}$$

is again one-dimensional, and $H^2(G_{F_v}/I_v, W) = 0$ since $G_{F_v}/I_v \cong \hat{\mathbb{Z}}$.

Lemma 4.8. *Suppose that the restriction of $\bar{\rho}$ to the absolute Galois group of $F(\sqrt{(-1)^{(p-1)/2}p})$ is irreducible. Then for any $m \in \mathbb{Z}_{>0}$ we can find a set Σ_m of primes such that*
(1) $\sharp\Sigma_m = dimH^1_{\varnothing}(G_F, ad^0\bar{\rho}(1))$,
(2) $\mathcal{R}_{\Sigma_m}$ can be topologically generated by $dimH^1_{\varnothing}(G_F, ad^0\bar{\rho}(1))$ elements as an $\mathcal{O}$-algebra,
(3) if $x \in \Sigma_m$ then $Norm(x) \equiv 1$ mod p^m and $\bar{\rho}(Frob_x)$ has distinct eigenvalues α_x and β_x.

Proof. See Lemma 2.5 of [32]. □

Recall the statement of Theorem 2.1 of [14].

Theorem 4.9. *Fix a positive integer r. Let $A = \kappa[[S_1, \ldots, S_r]]$ and $B = \kappa[[X_1, \ldots, X_r]]$ and write $\mathfrak{n}$ for the maximal ideal of A. Suppose that R is a κ-algebra and H is a nonzero R-module, finite dimensional over κ. Suppose that for each positive integer n, there exist κ-algebra homomorphisms $\phi_n : A \to B$ and $\psi_n : B \to R$, a B-module H_n and a B-linear homomorphism $\pi_n : H_n \to H$ such that the following hold:*
(a) ψ_n is surjective and $\psi_n \circ \phi_n(\mathfrak{n}) = 0$;
(b) π_n induces an isomorphism $H_n/\mathfrak{n}H_n \to H$;

(c) $Ann_A H_n = \mathfrak{n}^n$ and H_n is free over $A/\mathfrak{n}^n$.
Then R is a complete intersection of dimension zero, and H is free over R.

Proof of Theorem 4.1. We apply the above theorem to our situation. Let A and B be as in Theorem 4.9, $r = dim H^1_{\varnothing}(G_F, ad^0\bar{\rho}(1))$, Σ_n ($n \in \mathbb{Z}_{>0}$) be sets which satisfy the conditions in Lemma 4.8, G_n be the maximal quotient of $\prod_{x\in\Sigma_n}(\mathcal{O}_F/x)^\times$ of p-power order. By Lemma 4.8, $\mathcal{R}_n = \mathcal{R}_{\Sigma_n}/\lambda\mathcal{R}_{\Sigma_n}$ is topologically generated by r elements. We may define a surjective κ-algebra homomorphism $\theta_n : B \to \mathcal{R}_n$. By Lemma 4.3, we can endow $\mathcal{R}_n$ with the structure of an algebra over the group ring $\kappa[G_n]$. The definition ensures that the image of the maximal ideal of $\kappa[G_n]$ in $\mathcal{R}$ is trivial. We also choose a surjective κ-algebra homomorphism $A \to \kappa[G_n]$. Note that the kernel is contained in $\mathfrak{n}^n$, where $\mathfrak{n}$ is the maximal ideal of A. We then define $\phi_n : A \to B$ so that the diagram

$$\begin{tikzcd} A \arrow[r, "\phi_n"] \arrow[d] & B \arrow[d, "\theta_n"] \\ \kappa[G_n] \arrow[r] & \mathcal{R}_n \end{tikzcd}$$

commutes. Define ψ_n as the composite of θ_n with $\mathcal{R}_n \to \mathcal{R}$. Then ψ_n is surjective and $\psi_n \circ \phi_n(\mathfrak{n}) = 0$.

Take $H = H^1(M_K \otimes \bar{F}, \mathcal{F}_{I_V^{op}})$, and $H_n = H^1(M_{K_{1,\Sigma_n}} \otimes \bar{F}, \mathcal{F}_{I_V^{op}})$. By Lemma 4.7, we may apply Theorem 4.9 to prove the freeness result of Theorem 4.1. The rank is two because of the following lemma. □

Lemma 4.10. *$rank_{\mathcal{O}} H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{I_V^{op}}) = 2 \cdot rank_{\mathcal{O}} \mathbb{T}_{\mathcal{O}}(K, I_V^{op})$.*

Proof. We extend coefficients from E to $\bar{\mathbb{Q}}_p$. We have the following decomposition of the cohomology of Shimura curves from [6]:

$$\varinjlim_U H^1_{et}(M_U \otimes \bar{F}, \bar{\mathbb{Q}}_p) = \bigoplus_{\pi=\sigma_2\otimes\pi^\infty} \pi^\infty \otimes \rho(\pi),$$

where the sum is extended to the set of all automorphic representations of $G(\mathbb{A})$, with infinite component isomorphic to

$$\sigma_2 = \begin{pmatrix} \text{weight 2 holomorphic discrete} \\ \text{series representation of } GL_2(\mathbb{R}) \end{pmatrix} \otimes \begin{pmatrix} \text{trivial representation of} \\ (\mathbf{H}^\times)^{d-1} \end{pmatrix},$$

and $\rho(\pi)$ stands for some two dimensional p-adic representation of G_F. In particular, if we do not consider the Galois action, we have

$$\varinjlim_U H^1_{et}(M_U \otimes \bar{F}, \bar{\mathbb{Q}}_p) = \bigoplus_{\pi=\sigma_2\otimes\pi^\infty} (\pi^\infty)^2.$$

Then

$$
\begin{aligned}
H^1_{et}(M_K\otimes\bar F,\mathcal{F}_{I_V^{op}}) &= Hom_{\mathcal{G}}(I_V,(\varinjlim_U H^1_{et}(M_U\otimes\bar F,\bar{\mathbb{Q}}_p))^{K'}) \\
&= Hom_{\mathcal{G}}(I_V,\bigoplus_{\pi=\sigma_2\otimes\pi^\infty}((\pi^\infty)^{K'})^2) \\
&= \bigoplus_{\pi=\sigma_2\otimes\pi^\infty} Hom_{\mathcal{G}}(I_V,(\pi^\infty)^{K'})^2 \\
&= \bigoplus_{\pi=\sigma_2\otimes\pi^\infty} Hom_{\mathcal{G}}(I_V,(JL(\pi)^\infty)^{K'})^2.
\end{aligned}
\tag{4.4}
$$

In the above equation, JL means the Jacquet-Langlands correspondence. By the Jacquet-Langlands correspondence, the last term is the same as

$$\bigoplus_f Hom_{\mathcal{G}}(I_V,(\tilde\pi_f^\infty)^{K'})^2,$$

where the direct sum is over Hilbert new forms which are of weight $(2,...,2)$ and special at primes $v|S_D$, and $\tilde\pi_f$ is the automorphic representation of $GL_2(\mathbb{A}_F)$ associated with f. By the result on multiplicities of types (see for example the appendix of [2]), we have

$$\dim_{\bar{\mathbb{Q}}_p} Hom_{\mathcal{G}}(I_V,(\tilde\pi_f^\infty)^{K'})\le 1,$$

and

$$
\begin{aligned}
\dim_{\bar{\mathbb{Q}}_p} H^1_{et}(M_K\otimes\bar F,\mathcal{F}_{I_V^{op}}) &= 2\cdot\sharp\{f\mid Hom_{\mathcal{G}}(I_V,(\tilde\pi_f^\infty)^{K'})\neq 0\} \\
&= 2\cdot\dim_{\bar{\mathbb{Q}}_p}\mathbb{T}_{\mathcal{O}}(K,I_V^{op})\otimes\bar{\mathbb{Q}}_p.
\end{aligned}
\tag{4.5}
$$

This proves the lemma.

□

5. From characteristic 0 to characteristic p

5.1. **Computation for the multiplicity.** In the following two lemmas, we write $H^1(K)$ for $H^1(M_K\otimes\bar F,\mathcal{F}_{V^{op}})$.

Lemma 5.1. *The natural inclusion*

$$Hom_{G_F}(\bar\rho,H^1(K)[\mathfrak{m}])\hookrightarrow Hom_{G_F}(\bar\rho,H^1(K)) \tag{5.1}$$

is an equality.

Proof. Consider the evaluation map

$$\bar\rho\otimes_{\bar{\mathbb{F}}_p} Hom_{G_F}(\bar\rho,H^1(K))\to H^1(K). \tag{5.2}$$

This is injective since $\bar\rho$ is irreducible, and is $\mathbb{T}_\kappa(K,V^{op})[G_F]$-linear, if $\mathbb{T}_\kappa(K,V^{op})[G_F]$ acts on the tensor product through the action of G_F on the first factor and the action of $\mathbb{T}_\kappa(K,V^{op})$ on the second factor. By Eichler-Shimura relations on M_K, we have

$$Frob_v^2-T_vFrob_v+Norm(v)S_v=0$$

on $H^1(K)$ for all $v\nmid pS_DN$. Therefore we have

$$\bar\rho(Frob_v)(a)\otimes(T_v-Trace(\bar\rho(Frob_v)))(b)+a\otimes(S_vNorm(v)-det(\bar\rho(Frob_v)))(b)=0 \tag{5.3}$$

for every $a \in \bar{\rho}$, $b \in Hom_{\bar{\mathbb{F}}_p[G_F]}(\bar{\rho}, H^1(K))$, and $v \nmid pS_DN$. If v is such that $\bar{\rho}(Frob_v)$ does not act by a scalar, then it is easy to see that

$$T_v - Trace(\bar{\rho}(Frob_v)) = S_v Norm(v) - det(\bar{\rho}(Frob_v)) = 0. \tag{5.4}$$

The above equality is actually true for all $v \nmid pS_DN$ since those $Frob_v$ for which $\bar{\rho}(Frob_v)$ does not act by a scalar generate a dense subgroup of G_{F,pS_DN}. Then the result follows. □

Lemma 5.2. *The evaluation map* $\bar{\rho} \otimes_{\bar{\mathbb{F}}_p} Hom_{G_F}(\bar{\rho}, H^1(K)) \to H^1(K)$ *induces an isomorphism*

$$\bar{\rho} \otimes_{\bar{\mathbb{F}}_p} Hom_{G_F}(\bar{\rho}, H^1(K)) \xrightarrow{\sim} H^1(K)[\mathfrak{m}] \tag{5.5}$$

Proof. Since $\bar{\rho}$ is irreducible, the evaluation map is injective by Schur's lemma. By the above lemma, the image lies in $H^1(K)[\mathfrak{m}]$. By [1], we know that the semisimplification of $H^1(K)[\mathfrak{m}]$ is a direct sum of copies of $\bar{\rho}$ because of Eichler-Shimura relation. The evaluation map is also surjective. □

Theorem 5.3. *Let $K = K_1(N)$ and M_K be the Shimura curve of level K attached to the totally real field F and the quaternion algebra D. Let $\bar{\rho} : G_F \to GL_2(\kappa)$ be the Galois representation constructed at the beginning of section 3. Assume that $\bar{\rho}$ satisfies conditions in equation (3.2) and the restriction of $\bar{\rho}$ to the absolute Galois group of $F(\sqrt{(-1)^{(p-1)/2}p})$ is irreducible, then*

$$\dim_\kappa Hom_{G_F}(\bar{\rho}, H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})_\mathfrak{m}) = 1.$$

Proof. Let I_V be the lattice constructed in the proof of Corollary 2.8. If V^{op} is a subrepresentation of $I_V^{op} \otimes_\mathcal{O} \kappa$, we may assume that we have a short exact sequence

$$0 \to V^{op} \to I_V^{op} \otimes_\mathcal{O} \kappa \to (V^c)^{op} \to 0, \tag{5.6}$$

where V^c is a representation of $\mathcal{G}$ with no Jordan-Holder factor compatible with $\bar{\rho}$. Then by Lemma 2.2(2), we have a short exact sequence

$$0 \to H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})_\mathfrak{m} \to H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{I_V^{op} \otimes_\mathcal{O} \kappa})_\mathfrak{m} \to H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{(V^c)^{op}})_\mathfrak{m} \to 0. \tag{5.7}$$

From the construction of V^c, we know that $Hom_{G_F}(\bar{\rho}, H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{(V^c)^{op}})_\mathfrak{m}) = 0$. Thus $Hom_{G_F}(\bar{\rho}, H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})_\mathfrak{m}) = Hom_{G_F}(\bar{\rho}, H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{I_V^{op} \otimes_\mathcal{O} \kappa})_\mathfrak{m})$.

If V^{op} is a quotient representation of $I_V^{op} \otimes_\mathcal{O} \kappa$, a similar argument proves $Hom_{G_F}(\bar{\rho}, H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})_\mathfrak{m}) = Hom_{G_F}(\bar{\rho}, H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{I_V^{op} \otimes_\mathcal{O} \kappa})_\mathfrak{m})$. Note that $I_V^{op} \otimes_\mathcal{O} \kappa$ has finite length as a representation of $\mathcal{G}$, the theorem follows by induction on the length of $I_V^{op} \otimes_\mathcal{O} \kappa$. □

Remark 5.4. See [4] for more topics on multiplicities of Serre weights. See also [18] section 3.5.

5.2. **Another $R = T$ theorem.** We explain how we can prove a stronger result from Theorem 5.3, i.e., we prove that $H^1_{et}(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})_\mathfrak{m}$ is free of rank 2 over $\mathbb{T}_\kappa(K, V^{op})_\mathfrak{m}$. The strategy is the same as the proof of Theorem 4.1. We consider a Fontaine-Laffaille deformation problem, construct a Taylor-Wiles system, and apply Theorem 4.9 to prove a multiplicity free result. It follows from Theorem 5.3 that the rank is two.

5.2.1. *Fontaine-Laffaille theory.* Fontaine-Laffaille theory is needed to state the deformation condition at primes dividing p. Suppose that $I/\mathbb{Q}_p$ is a finite unramified extension, that $\mathcal{O}$ is the ring of integers of a finite extension of I with uniformizer λ and residue field κ.

Recall that a filtered Dieudonné $\mathcal{O}_I \otimes \mathcal{O}$-module is an $\mathcal{O}_I \otimes \mathcal{O}$-module $\mathfrak{D}$ furnished with a decreasing, exhaustive, separated filtration $(\mathfrak{D}^i)_{i\in\mathbb{Z}}$ of sub $\mathcal{O}_I \otimes \mathcal{O}$-modules such that for each integer i, we have a $Frob\otimes 1$-linear map $\phi^i : \mathfrak{D}^i \to \mathfrak{D}$. Furthermore, it is required that for $x \in \mathfrak{D}^{i+1}$, $\phi^{i+1}(x) = p\phi^i(x)$. These filtered modules form an $\mathcal{O}_I \otimes \mathcal{O}$-linear additive category $\mathcal{MF}_{I,\mathcal{O}}$.

We denote by $\mathcal{MF}^f_{I,\mathcal{O}}$ the full subcategory of $\mathcal{MF}_{I,\mathcal{O}}$ whose objects $\mathfrak{D}$ have underlying spaces that are $\mathcal{O}_I \otimes \mathcal{O}$-modules of finite length and satisfy $\sum Im\phi^i = \mathfrak{D}$. The category $\mathcal{MF}^{p-1}_{I,\mathcal{O}}$ is the full subcategory of $\mathcal{MF}^f_{I,\mathcal{O}}$ whose objects satisfy $\mathfrak{D}^0 = \mathfrak{D}$, $\mathfrak{D}^{p-1} = 0$. We write κ-$\mathcal{MF}^{p-1}_{I,\mathcal{O}}$ for the subcategory of $\mathcal{MF}^{p-1}_{I,\mathcal{O}}$ whose objects are killed by λ.

Theorem 5.5. *There is a fully faithful, $\mathcal{O}$ length preserving, exact, $\mathcal{O}$-additive, covariant functor $\mathbb{M}$ from $\mathcal{MF}^{p-1}_{I,\mathcal{O}}$ to the category of continuous $\mathcal{O}[G_I]$-modules with essential image closed under the formation of sub-objects and quotients.*

Proof. See for example section 2.4.1 of [10] and section 9 of [19]. □

Remark 5.6. (1) If M is an object in $\mathcal{MF}^{p-1}_{I,\mathcal{O}}$, then the $\mathcal{O}$ length of M is $[I : \mathbb{Q}_p]$ times the $\mathcal{O}$ length of $\mathbb{M}(M)$.
(2) For any objects M and N in $\mathcal{MF}^{p-1}_{I,\mathcal{O}}$ (resp. κ-$\mathcal{MF}^{p-1}_{I,\mathcal{O}}$), the map

$$Ext^1_{\mathcal{MF}^{p-1}_{I,\mathcal{O}}}(M,N) \hookrightarrow Ext^1_{\mathcal{O}[G_I]}(\mathbb{M}(M), \mathbb{M}(N))$$

$$(\text{resp. } Ext^1_{\kappa\text{-}\mathcal{MF}^{p-1}_{I,\mathcal{O}}}(M,N) \hookrightarrow Ext^1_{\kappa[G_I]}(\mathbb{M}(M), \mathbb{M}(N)))$$

is an injection.
(3) We also have the following isomorphisms.

$$Ext^1_{\kappa[G_I]}(\mathbb{M}(M), \mathbb{M}(N)) \cong H^1(G_I, Hom_\kappa(\mathbb{M}(M), \mathbb{M}(N))),$$

$$Hom_{\mathcal{MF}^{p-1}_{I,\mathcal{O}}}(M,N) \cong H^0(G_I, Hom_{\mathcal{O}}(\mathbb{M}(M), \mathbb{M}(N))).$$

Lemma 5.7. *Suppose that M and N are objects of κ-$\mathcal{MF}^{p-1}_{I,\mathcal{O}}$. Then there is an exact sequence*

$$\begin{aligned} 0 \to & Hom_{\kappa\text{-}\mathcal{MF}^{p-1}_{I,\mathcal{O}}}(M,N) \to Fil^0 Hom_{\mathcal{O}_I\otimes\mathcal{O}}(M,N) \to \\ & Hom_{\mathcal{O}_I\otimes\mathcal{O}, Frob\otimes 1}(grM, N) \to Ext^1_{\kappa\text{-}\mathcal{MF}^{p-1}_{I,\mathcal{O}}}(M,N) \to 0; \end{aligned} \tag{5.8}$$

where $Fil^i Hom_{\mathcal{O}_I\otimes\mathcal{O}}(M,N)$ denotes the subset of $Hom_{\mathcal{O}_I\otimes\mathcal{O}}(M,N)$ consisting of elements which take $Fil^j M$ to $Fil^{i+j} N$ for all j and where $grM = \oplus_i gr^i M$. The middle map is given by $\beta \mapsto (\beta\phi^i_M - \phi^i_N\beta)$.

Proof. This is Lemma 2.4.2 of [10]. □

Proposition 5.8. *The representation $\rho_{\mathfrak{m}}$ is Fontaine-Laffaille. i.e., for $v|p$, $\rho_{\mathfrak{m}}|_{G_{F_v}}$ is in the image of $\mathbb{M} : \kappa$-$\mathcal{MF}^{p-1}_{F_v,\mathcal{O}} \to Mod_{\kappa[G_{F_v}]}$.*

We want to apply p-adic Hodge theory to prove this result. First recall the construction of unitary Shimura curves from section 2.2 of [7].

Let $q < 0$ be a rational number such that $\mathbb{Q}(\sqrt{q})$ splits p. Define $L = F(\sqrt{q})$. Let $z \mapsto \bar{z}$ denote the conjugation of L with respect to F. Define $B = D \otimes_F L$ and denote by $l \mapsto \bar{l}$ the product of the canonical involution of D with the conjugation of L over F. Let V denote the underlying $\mathbb{Q}$-vector space of B with left action of B. Choose $\delta \in B$ such that $\bar{\delta} = \delta$ and define an involution on B by $l^* = \delta^{-1}\bar{l}\delta$. Choose $\alpha \in L$ such that $\bar{\alpha} = -\alpha$. One can define a symplectic form Φ on V: for $v, w \in V$, define

$$\Phi(v, w) = Trace_{L/\mathbb{Q}}(\alpha Trace_{B/L}(v\delta w^*)).$$

The symplectic form Φ is an alternating nondegenerate form on V and satisfies

$$\Phi(lv, w) = \Phi(v, l^*w).$$

Let G' be the reductive algebraic group over $\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra R, $G'(R)$ is group of B linear symplectic similitudes of $(V \otimes R, \Phi \otimes R)$. Following section 2.2.4 of [7], we can define a morphism $h' : Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m \to G'_{\mathbb{R}}$, such that the $G'(\mathbb{R})$-conjugacy class X' of h' can be identified with the complex upper half plane, and the composition $Res_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m \to G'_{\mathbb{R}} \to GL(V_{\mathbb{R}})$ defines a Hodge structure on $V_{\mathbb{R}}$ which is of type $\{(-1, 0), (0, -1)\}$.

Now, (G', X') gives us Shimura data. For $K' \subset G'(\mathbb{A}^\infty)$ open compact, we have a compact unitary Shimura curve $M'_{K'}$ defined over L with complex points

$$M'_{K'}(\mathbb{C}) = G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty) \times X'/K'.$$

$M'_{K'}$ is a fine moduli space of certain abelian varieties with additional structures. Let $H \subset G(\mathbb{A}^\infty)^{\mathfrak{p}}$ be sufficiently small and open compact, and $M_{0,H}$ be the Shimura curve with level $GL_2(\mathcal{O}_{\mathfrak{p}})H$. Then we have the following theorem which is proved in [7] section 4.5.4.

Theorem 5.9. *There exists an open compact subgroup $H' \subset G'(\mathbb{A}^\infty)^{\mathfrak{p}}$, such that for any connected component N_H of $M_{0,H}$, there is a connected component $N_{H'}$ of $M_{0,H'}$ with the property that N_H and $N_{H'}$ are isomorphic over $F_{\mathfrak{p}}^{ur}$.*

From [7] section 3.2.3, we know that the Weil group $W(F_{\mathfrak{p}}^{ab}/F_{\mathfrak{p}}) \cong F_{\mathfrak{p}}^\times$ acts on the set of connected components of $\lim_{K'} M'_{K'}$ via the map

$$F_{\mathfrak{p}}^\times \to T'(\mathbb{Q}) = \mathbb{Q}_p^\times \times F_{\mathfrak{p}}^\times \times F_{\mathfrak{p}_2}^\times \times \cdots \times F_{\mathfrak{p}_r}^\times$$

defined by $z \mapsto (Norm_{F_{\mathfrak{p}}/\mathbb{Q}_p} z; z, 1, \cdots, 1)$. This tells us that every connected component of $\lim_{K'} M'_{K'}$ is defined over $F_{\mathfrak{p}}^{ur}$. If we consider the action of $W(F_{\mathfrak{p}}^{ab}/F_{\mathfrak{p}}) \cong F_{\mathfrak{p}}^\times$ on the set of connected components of $M'_{0,H'}$, we see that it is unramified and factors through a finite quotient. (See for example [7] section 3.2.4.) Therefore, every connected component of $M'_{0,H'}$ is defined over a finite unramified extension of $F_{\mathfrak{p}}$.

Similarly, using the description in section 1.3 of [7], we see that every connected component of $M_{0,H}$ is also defined over a finite unramified extension of $F_{\mathfrak{p}}$.

Now we prove Proposition 5.8. I would like to thank Toby Gee for suggesting to transfer the problem to unitary Shimura curve case.

Proof of Proposition 5.8. It suffices to prove that $\rho_{\mathfrak{m}}$ is Fontaine-Laffaille after restricting to some unramified extension. Write $\mathcal{F} = \mathcal{F}_{V^{op}}$. Since $\rho_{\mathfrak{m}}$ is realized in $H^1_{et}(M_K \otimes \bar{F}, \mathcal{F})$, it suffices to prove that $H^1_{et}(M_K \otimes \bar{F}, \mathcal{F})$ is Fontaine-Laffaille. Write $M_K = \coprod M_K^i$ as a finite disjoint union of its connected components. These components are rational over a finite extension of F which is unramified at p. Let F_K be an extension of F which is unramified at p. We may assume that F_K is large enough that each component of M_K is rational over F_K, and is isomorphic over F_K to a component of $M'_{K'}$. Here, $M'_{K'}$ is a unitary Shimura curve with level K'. Since K has no level structure at p, we can choose K' with no level structure at p. Then over F_K,

$$H^1_{et}(M_K,\mathcal{F}) = H^1_{et}(\coprod_i M^i_K,\mathcal{F}) = H^1_{et}(\coprod_i N^i_{K'},\mathcal{F}) \hookrightarrow \bigoplus_i H^1_{et}(M'_{K'},\mathcal{F}),$$

which makes $H^1_{et}(M_K,\mathcal{F})$ a submodule of $\bigoplus_i H^1_{et}(M'_{K'},\mathcal{F})$. It suffices to prove that $H^1_{et}(M'_{K'}\otimes\bar{L},\mathcal{F})$ is Fontaine-Laffaille.

For the curve $M'_{K'}$, we do not need the parity condition on b_λ to lift V^{op} to characteristic zero. In fact, the maximal $\mathbb{Q}$-split torus in the center of G' coincides with the maximal $\mathbb{R}$-split torus in the center of G'. Let $\tilde{V^{op}} = \otimes_\lambda det^{a_\lambda}\otimes Symm^{b_\lambda-1}\mathcal{O}^2$, this is a lift of V^{op}. Then $\tilde{V^{op}}\otimes E$ gives us a well defined smooth sheaf on $M'_{K'}$. (See part I section 1 of [23]. In particular, G' satisfies the requirements in that paper, but G does not.) Since $M'_{K'}$ is a fine moduli space of certain abelian varieties, the sheaf $\mathcal{F}_{\tilde{V^{op}}}$ can be constructed from the universal object over $M'_{K'}$ (see for example [29] section 6.1.) Therefore, since V is regular, $H^1_{et}(M'_{K'}\otimes\bar{L},\mathcal{F}_{\tilde{V^{op}}})$ is crystalline with Hodge-Tate weights in $[0,p-2]$ by p-adic Hodge theory. Because $H^1_{et}(M'_{K'}\otimes\bar{L},\mathcal{F})$ is the reduction of $H^1_{et}(M'_{K'}\otimes\bar{L},\mathcal{F}_{\tilde{V^{op}}})$ (mod p), it is Fontaine-Laffaille. □

5.2.2. *Deformation conditions at primes dividing p.* Fix a finite prime $v|p$ of F, and write F_v for the completion of F at v, S_{F_v} for the set of embeddings $F_v\hookrightarrow\bar{\mathbb{Q}}_p$. We generalize section 2.1 of [16] to the totally real field case.

For $i=1,2$, let V_i be representations of G_{F_v} over E which are from the category $\mathcal{MF}^{p-1}_{F_v,\mathcal{O}}$, i.e., there exists $\mathfrak{D}_i\in\mathcal{MF}^{p-1}_{F_v,\mathcal{O}}$ such that, $\mathbb{M}(\mathfrak{D}_i)\otimes E\cong V_i$. Suppose that L_i is a G_{F_v}-stable $\mathcal{O}$ lattice in V_i with $L_i\cong\mathbb{M}(\mathfrak{D}_i)$ and set

$$V = Hom_E(V_1,V_2),\quad T = Hom_{\mathcal{O}}(L_1,L_2),\quad W = V/T.$$

For $n\geq 1$, put

$$W_n = \{x\in W \mid \lambda^n x = 0\}\cong T/\lambda^n T.$$

Then we have a natural isomorphism

$$H^1(G_{F_v},W_n)\cong Ext^1_{\mathcal{O}/\lambda^n[G_{F_v}]}(L_1/\lambda^n L_1,\lambda^{-n}L_2/L_2). \tag{5.9}$$

Definition 5.10. Let V_i be as above, then $L_i/\lambda^n L_i$ are in the essential image of $\mathbb{M}$. Define

$$H^1_f(G_{F_v},W_n)\subset H^1(G_{F_v},W_n)$$

to be the subset of extensions of $\mathcal{O}/\lambda^n[G_{F_v}]$-modules

$$0\to\lambda^{-n}L_2/L_2\to\mathcal{E}\to L_1/\lambda^n L_1\to 0 \tag{5.10}$$

so that $\mathcal{E}$ is in the essential image of $\mathbb{M}$.

We consider the deformations of $\bar{\rho}|_{G_{F_v}}$ satisfying the following: if $\rho_v : G_{F_v}\to GL_2(R)$ is a deformation of $\bar{\rho}|_{G_{F_v}}$ where R is a complete local Noetherian ring, then for any Artinian quotient R', $\rho_v \otimes R'$ is in the essential image of $\mathbb{M}$. This is a local deformation problem. From the above, we have

$$Ext^1_{\kappa\text{-}\mathcal{MF}^{p-1}_{I,\mathcal{O}}}(\mathbb{M}^{-1}(\bar{\rho}|_{G_{F_v}}), \mathbb{M}^{-1}(\bar{\rho}|_{G_{F_v}})) \cong H^1_f(G_{F_v}, ad(\bar{\rho}|_{G_{F_v}})) \subset H^1(G_{F_v}, ad(\bar{\rho}|_{G_{F_v}})).$$

Now we can prove the following lemma which will be used when we apply the Taylor-Wiles argument.

Lemma 5.11. *Suppose that $\bar{L}$ is a two-dimensional G_F representation over the finite field κ of characteristic $p > 2$ so that $\bar{L} \mid_{G_{F_v}} \cong \mathbb{M}(\bar{\mathfrak{E}})$ for some object $\bar{\mathfrak{E}} \in \kappa\text{-}\mathcal{MF}^{p-1}_{F_v,\mathcal{O}}$. Let $ad^0_\kappa \bar{L} \subset ad_\kappa \bar{L} := Hom_\kappa(\bar{L}, \bar{L})$ be the set of endomorphisms of trace zero. Then*

$$\dim_\kappa H^1_f(G_{F_v}, ad^0_\kappa \bar{L}) = \sharp S_{F_v} + \dim_\kappa H^0(G_{F_v}, ad^0_\kappa \bar{L}). \tag{5.11}$$

Proof. From Lemma 5.7, taking $M = N = \bar{\mathfrak{E}}$ we have

$$\begin{aligned} \dim_\kappa H^1_f(G_{F_v}, ad_\kappa \bar{L}) &= \sharp S_{F_v}(2 \times 2 - 3) + \dim_\kappa H^0(G_{F_v}, ad_\kappa \bar{L}) \\ &= \sharp S_{F_v} + \dim_\kappa H^0(G_{F_v}, ad_\kappa \bar{L}). \end{aligned} \tag{5.12}$$

Taking $M = N = \kappa$ we have $\dim_\kappa H^1_f(G_{F_v}, \kappa) = \dim_\kappa H^0(G_{F_v}, \kappa)$. Since we have $ad^0_\kappa \bar{L} \oplus \kappa = ad_\kappa \bar{L}$, the lemma follows. □

5.2.3. *The Global deformation problem.*

Definition 5.12. Let Σ be a square free ideal of $\mathcal{O}_F$, prime to pNS_D. We consider the functor $\mathcal{P}_\Sigma$ from $\mathcal{C}_\mathcal{O}$ to the category of sets which associates to an object A in $\mathcal{C}_\mathcal{O}$ the set of strictly equivalent classes of continuous homomorphisms $\rho : G_F \to GL_2(A)$ lifting $\bar{\rho}$ satisfying the following conditions:
(1) ρ is unramified outside $pN' S_D \Sigma$;
(2) if $v|N'$, i.e. $\bar{\rho}$ is ramified at v, then $\rho \mid_{I_v}$ is unipotent;
(3) if $v|S_D$, $\bar{\rho}$ is unramified at v, then ρ satisfies sp-condition at v;
(4) if $v|p$, then for each finite length (as an $\mathcal{O}$-module) quotient A/J of A, the $\mathcal{O}[G_{F_v}]$-module $(A/J)^2$ is isomorphic to $\mathbb{M}(\mathfrak{D})$ for some object $\mathfrak{D} \in \mathcal{MF}^{p-1}_{F_v,\mathcal{O}}$;
(5) $det(\rho) = \epsilon(\chi \circ Art^{-1})$.

Proposition 5.13. *The functor $\mathcal{P}_\Sigma$ is representable.*

Proof. The proof is similar to the proof of Proposition 3.9. Condition (4) is a deformation condition by the discussion in Section 2.4.1 of [10]. □

We say that the functor is represented by the universal deformation

$$\rho^{univ}_{FL,\Sigma} : G_F \to GL_2(\mathcal{R}_{FL,\Sigma}).$$

We write $\mathcal{R}_{FL}$ and ρ^{univ}_{FL} if $\Sigma = \emptyset$. We also write $\mathcal{R}'_{FL,\Sigma} = \mathcal{R}_{FL,\Sigma} \otimes_\mathcal{O} \kappa$.

We have the following theorem.

Theorem 5.14. *Let $\bar{\rho}$ be the Galois representation constructed at the beginning of section 3. Suppose that $\bar{\rho}$ satisfies the conditions in equation (3.2) and the restriction of $\bar{\rho}$ to the absolute Galois group of $F(\sqrt{(-1)^{(p-1)/2}p})$ is irreducible. Then there is a natural surjection*

$$\mathcal{R}'_{FL} \to \mathbb{T}_\kappa(K, V^{op})_\mathfrak{m}.$$

Furthermore, it is an isomorphism of complete intersections and $H^1(M_K \otimes \bar{F}, \mathcal{F}_{V^{op}})_{\mathfrak{m}}$ *is free of rank two as a* $\mathbb{T}_\kappa(K, V^{op})_{\mathfrak{m}}$ *module.*

Proof. The proof is essentially the same as the proof of Theorem 4.1. Although we have sheaf $\mathcal{F}_{V^{op}}$, we construct the same Taylor-Wiles system as in section 4 and apply Theorem 4.9 to get $R = T$ and the freeness result. The rank is two by Theorem 5.3. □

6. Computation on Breuil modules

In this section, we review the definition of Breuil modules and prove Lemma 3.3. The notation in subsection 6.1, 6.2, and 6.3 is not consistent with the notation in other parts of this paper. We choose this notation to be consistent with the notation used in the references [20] and [9].

6.1. **Introduction and definitions.** In [9], the author studied the structure of reducible rank two Breuil modules with descent data and computed $Ext^1(\mathcal{M}, \mathcal{M})$ for reducible rank two $\mathcal{M}$ of type J (see Definition 6.4 below). The computation is limited to a special case. In this paper, we introduce an exact sequence of Breuil modules, which can be used to compute $Ext^1(\mathcal{M}, \mathcal{M})$ for more general $\mathcal{M}$. In particular, we reprove Theorem 4.2 of [9]. Furthermore, we compute $Ext^1(\mathcal{M}, \mathcal{M})$ for irreducible rank two Breuil modules of type J (see Definition 6.5).

We first recall the definition of Breuil modules with descent data. Let k be a finite extension of $\mathbb{F}_p$ of degree r, $W(k)$ the ring of Witt vectors. Let $K_0 = W(k)[1/p]$, K be a totally and tamely ramified extension of K_0 of degree e. Fix a subfield F of K_0, and assume that there is a uniformizer π of $\mathcal{O}_K$ such that $\pi^e \in F$. Then K/F is tamely ramified, K_0/F is unramified. Assume that K/F is Galois. Write $G = Gal(K/F)$. Let $S = Hom_{\mathbb{F}_p}(k, \bar{\mathbb{F}}_p) \cong \mathbb{Z}/r\mathbb{Z}$. Fix $\tau_0 \in S$, let $\tau_i = \tau_0 \circ Frob^{-i}$, where $Frob$ is the arithmetic Frobenius. Let E be a finite extension of $\mathbb{F}_p$ such that the image of τ_i is a subset of E. Let $\mathcal{S} = k \otimes_{\mathbb{F}_p} E[u]/u^{ep}$.

Let $\omega : G \to k^\times$ be the map defined by $\omega(g) = g(\pi)/\pi \pmod{\pi}$. We see that $\omega(gh) = g(\omega(h))\omega(g)$. It is a cocycle. It is a character if and only if G acts trivially on $k^\times$, if and only if $K_0 = F$. Let ω_i be the composite of ω with τ_i. Then we have $\omega_i = \omega_{i+1}^p$. For any $g \in G$, we write $[g] : \mathcal{S} \to \mathcal{S}$ to be the endomorphism of $\mathcal{S}$ as $k \otimes E$-algebra such that $[g](u) = (\omega(g) \otimes 1)u$. Let $\phi : \mathcal{S} \to \mathcal{S}$ be the map of $\mathcal{S}$ such that $\phi((a \otimes b)u) = (a^p \otimes b)u^p$.

Definition 6.1. Let $\kappa \in [2, p-1]$ be an integer. The category $BrMod^{\kappa-1}_{dd,K/F}$ consists of quintuples $(\mathcal{M}, Fil^{\kappa-1}\mathcal{M}, \phi_{\kappa-1}, [g], N)$ where:

(1) $\mathcal{M}$ is a finitely generated $\mathcal{S}$ module, free over $k[u]/u^{ep}$.

(2) $Fil^{\kappa-1}\mathcal{M}$ is an $\mathcal{S}$-submodule of $\mathcal{M}$ containing $u^{e(\kappa-1)}\mathcal{M}$.

(3) $\phi_{\kappa-1} : Fil^{\kappa-1}\mathcal{M} \to \mathcal{M}$ is an E-linear and ϕ-semilinear map with image generating $\mathcal{M}$ as an $\mathcal{S}$-module.

(4) $N : \mathcal{M} \to u\mathcal{M}$ is a $k \otimes E$-linear map such that

$$N(ux) = uN(x) - ux \ \ \forall x \in \mathcal{M},$$
$$u^e N(Fil^{\kappa-1}\mathcal{M}) \subset Fil^{\kappa-1}\mathcal{M},$$
$$\phi_{\kappa-1}(u^e N(x)) = (-\pi^e/p)N(\phi_{\kappa-1}(x)) \ \ \forall x \in Fil^{\kappa-1}\mathcal{M}.$$

(5) $[g] : \mathcal{M} \to \mathcal{M}$ are additive bijections for each $g \in G$, preserving $Fil^{\kappa-1}\mathcal{M}$, commuting with the $\phi_{\kappa-1}$-, N-, and E-actions, and satisfying $[g_1] \circ [g_2] = [g_1 g_2]$ for all $g_1, g_2 \in G$, and $[1]$ is the identity map. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} E$, $m \in \mathcal{M}$, then

$$[g](au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i[g](m).$$

If $\kappa = 2$, the category $BrMod^1_{dd,K/F}$ is equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ together with an E-action and descent data on the generic fiber from K to F (this equivalence depends on π). In this case it follows from other axioms that there is always a unique N which satisfies the required properties. See for example Proposition 5.1.3 of [3].

In this paper, we always assume that $\kappa = 2$. Then to give an object in $BrMod^1_{dd,K/F}$ is the same to give an object $(\mathcal{M}, Fil^1\mathcal{M}, \phi_1, [g])$. In the following, we give some examples of Breuil modules with descent data.

Recall that $S = Hom(k, \bar{\mathbb{F}}_p) \cong \mathbb{Z}/r\mathbb{Z}$ and E contains the image of $\tau_i \in S$, so we have a ring isomorphism $k \otimes_{\mathbb{F}_p} E \simeq E^S$ where the action of $x \otimes 1$ on the τ-component coincides with the action of $1 \otimes \tau(x)$ for $\tau \in S$. Therefore we may write $\mathcal{S} = \oplus_S E[u]/u^{ep}$. We also denote ϕ to be the map $\phi : E[u]/u^{ep} \to E[u]/u^{ep}$ which sends u to u^p and acts trivially on E.

If $\mathcal{M}$ is an object of $BrMod^{\kappa-1}_{dd,K/K_0}$, then

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_r,$$

where $\mathcal{M}_i = \mathcal{M} \otimes_{\mathcal{S},\tau_i} E[u]/u^{ep}$ is a free $E[u]/u^{ep}$-module, which is characterized by the fact that the action of $x \otimes 1$ on $\mathcal{M}_i$ coincides with the action of $1 \otimes \tau_i(x)$ for $\tau_i \in S$. In this paper if $\mathcal{M}$ is a Breuil module over $\mathcal{S}$, then $\mathcal{M}_i$ will always denote the τ_i-component of $\mathcal{M}$. By convention, the subscripts i are taken modulo r. Similarly, $Fil^{\kappa-1}\mathcal{M}$ has a decomposition

$$Fil^{\kappa-1}\mathcal{M} = Fil^{\kappa-1}\mathcal{M}_1 \oplus Fil^{\kappa-1}\mathcal{M}_2 \oplus \cdots \oplus Fil^{\kappa-1}\mathcal{M}_r,$$

with $u^{e(\kappa-1)}\mathcal{M}_i \subset Fil^{\kappa-1}\mathcal{M}_i \subset \mathcal{M}_i$. The Frobenius action of $k \otimes_{\mathbb{F}_p} E$ maps E_{τ_i} to $E_{\tau_{i+1}}$, $\phi_{\kappa-1}$ induces $\phi_{\kappa-1} : Fil^{\kappa-1}\mathcal{M}_i \to \mathcal{M}_{i+1}$ for $i \in \mathbb{Z}/r\mathbb{Z}$ and the image generates $\mathcal{M}_{i+1}$, and $N(\mathcal{M}_i) \subset \mathcal{M}_i$.

6.1.1. *Rank one objects.* Assume that $K = K_0((-p)^{1/p^r-1})$ and $F = K_0$. Note that in this case we have $e = p^r - 1$ and K is Galois over K_0 with $Gal(K/K_0) \cong \mathbb{Z}/(p^r - 1)\mathbb{Z}$. The following proposition is Proposition 2.2 of [8] (see also Proposition 2.3 of [9]).

Proposition 6.2. *If $\mathcal{M}$ is a rank one object of $BrMod^{\kappa-1}_{dd,K/K_0}$, then there exist integers $m_i \in [0, e(\kappa-1)]$, $\mu_i \in [0, e-1]$, and $a \in E^\times$, such that we can choose basis e_i for $\mathcal{M}_i$, and*

(1) $Fil^{\kappa-1}\mathcal{M}_i = E[u]/u^{ep}\langle u^{m_i} e_i\rangle$,
(2) $\phi_{\kappa-1}(u^{m_i} e_i) = (a)_{i+1} e_{i+1}$,
(3) $\mu_{i+1} \equiv p(\mu_i + m_i)(mod\ e)$,
(4) $[g] \cdot e_i = \omega_i^{\mu_i}(g) e_i$,
(5) $N(e_i) = 0$.

We will write the Breuil module with these invariants $\mathcal{M}(m_i, \mu_i, a)$.

Definition 6.3. Let $J \subset S$. We say $\mathcal{M}(m_i, \mu_i, a)$ is of *type* J if $m_i = e(\kappa-1)\delta_J(i+1)$, where $\delta_J(i) = \begin{cases} 1 \text{ if } \tau_i \in J, \\ 0 \text{ otherwise.} \end{cases}$

6.1.2. *Reducible rank two objects of type J.* In this subsection, we still assume that $K = K_0((-p)^{1/p^r-1})$ and $F = K_0$.

Definition 6.4. Let J be a subset of S. We say a reducible rank two Breuil module $\mathcal{M}$ is of type J if it is an extension of a rank one Breuil module of type J (in the sense of Definition 6.3) by a rank one Breuil module of type J^c. Here $J^c = S \backslash J$.

Assume that $\mathcal{M}$ is a reducible rank two Breuil module of type J such that the rank one submodule and the rank one quotient of $\mathcal{M}$ have non isomorphic generic fibers. By Theorem 3.9 of [9], we may assume that $\mathcal{M} = \oplus_{i\in S}\mathcal{M}_i$ has the following form.

$$\mathcal{M}_i = E[u]/u^{ep}\langle e_i, f_i\rangle,$$

$$Fil^1\mathcal{M}_i = E[u]/u^{ep}\langle u^{j_i}e_i, u^{e-j_i}f_i + \lambda_i u^{h_i}e_i\rangle,$$

$$\phi_1(u^{j_i}e_i) = (b)_{i+1}e_{i+1}, \quad \phi_1(u^{e-j_i}f_i + \lambda_i u^{h_i}e_i) = (a)_{i+1}f_{i+1},$$

$$[g]e_i = \omega_i^{\beta_i}(g)e_i, \quad [g]f_i = \omega_i^{\alpha_i}(g)f_i,$$

where $\lambda_i \in E$ with $\lambda_i = 0$ if $i+1 \notin J$, $(a)_i = \begin{cases} a & \text{if } i = 1 \\ 1 & \text{otherwise} \end{cases}$, $j_i = \begin{cases} e & i+1 \in J \\ 0 & i+1 \notin J \end{cases}$, and $h_i \in [0, e-1]$ with $h_i \equiv \alpha_i - \beta_i (\bmod e)$. Note that $\mathcal{M}$ is split if and only if all the $\lambda_i's$ are 0.

6.1.3. *Irreducible rank two objects of type J.* Assume now that k is a finite extension of $\mathbb{F}_p$ with even degree $[k : \mathbb{F}_p] = r = 2s$. Let $K_0 = W(k)[1/p]$, $k' \subset k$ be the subfield with $[k : k'] = 2$, and $F = W(k')[1/p]$. Then K_0/F is an unramified extension of degree 2. Let $K = K_0((-p)^{1/p^{2s}-1})$. Then K/K_0 is a totally ramified extension of degree $e = p^{2s}-1$. In this section, we consider Breuil modules over $\mathcal{S} = k \otimes E[u]/u^{ep}$ with descent data from K to F. Let $\omega : Gal(K/F) \to k^\times$ be the map given by $g \mapsto g(\pi)/\pi \pmod{\pi}$ for $\pi = (-p)^{1/p^{2s}-1}$. Note that $\omega_i|_{Gal(K/K_0)}$ is a character.

Let $S' = Hom(k', \bar{\mathbb{F}}_p)$. Fix $g_\phi \in Gal(K/F)$ such that it maps to the nontrivial element of $Gal(K_0/F)$. For $g \in Gal(K/K_0)$, it acts on $\mathcal{S}$ as $[g](u) = (\omega(g) \otimes 1)u$. In the following, we use g to denote the elements in $Gal(K/K_0)$.

Let $\mathcal{M} = \oplus_S \mathcal{M}_i$ in $BrMod^1_{dd,K/F}$ be an irreducible rank two Breuil module that has the the following form

$$\mathcal{M}_i = E[u]/u^{ep}\langle e_i, f_i\rangle$$

$$Fil^1\mathcal{M}_i = E[u]/u^{ep}\langle u^{n_i}e_i, u^{m_i}f_i\rangle$$

$$\phi_1(u^{n_i}e_i) = (a)'_{i+1}e_{i+1}, \quad \phi_1(u^{m_i}f_i) = (a)_{i+1}f_{i+1}$$

$$N(e_i) = 0, \quad N(f_i) = 0$$

$$[g_\phi]e_i = f_{i+s}, \quad [g_\phi]f_i = e_{i+s}$$

$$[g]e_i = \omega_i^{\gamma_i}(g)e_i, \quad [g]f_i = \omega_i^{\mu_i}(g)f_i.$$

Here n_i and m_i are integers in $[0, e]$, γ_i and μ_i are integers in $[0, e-1]$, $(a)_i = \begin{cases} a & \text{if } i = 1 \\ 1 & \text{otherwise} \end{cases}$, $(a)'_i = \begin{cases} a \text{ if } i = s+1 \\ 1 \text{ otherwise} \end{cases}$. To make sure that $\mathcal{M}$ is a well defined object in $BrMod^1_{dd,K/F}$, the m_i, n_i, γ_i, and μ_i satisfy the following equations.

(1) $n_i = m_{i+s}$ because ϕ_1 commutes with $[g_\phi]$.

(2) $\gamma_{i+1} \equiv p(n_i + \gamma_i) \pmod e$ and $\mu_{i+1} \equiv p(m_i + \mu_i) \pmod e$ because ϕ_1 commutes with $[g]$.

(3) $\gamma_i \equiv \mu_{i+s} \pmod e$ and $\mu_i \equiv \gamma_{i+s} \pmod e$ because $gg_\phi = g_\phi g^{p^s}$.

Let $J \subset S = Hom(k, \bar{\mathbb{F}}_p)$ be a subset such that the restriction of the projection $S \to S'$ to $J \to S'$ is a bijection. We define a special type of Breuil modules.

Definition 6.5. Let $\mathcal{M}$ and J be as above, we say that $\mathcal{M}$ is of *type* J if we have

$$n_i = \begin{cases} e & \text{if } i+1 \in J \\ 0 & \text{if } i+1 \notin J, \end{cases}$$

$$m_i = \begin{cases} e & \text{if } i+1 \notin J \\ 0 & \text{if } i+1 \in J. \end{cases}$$

6.2. **The exact sequence.** Let $\mathcal{M}$ and $\mathcal{N}$ be two objects in $BrMod^1_{dd,K/F}$. Let $\mathcal{E} \in Ext^1_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{N})$. We have a short exact sequence

$$0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{M} \to 0.$$

We say that the pair $(\mathcal{M}, \mathcal{N})$ is *simple* if for every $\mathcal{E}$, we can write $\mathcal{E} = \mathcal{N} \oplus \mathcal{M}$ such that $\mathcal{N} \to \mathcal{E}$ is the natural inclusion and $\mathcal{E} \to \mathcal{M}$ is the natural projection, and

(1) $Fil^1\mathcal{E} = Fil^1\mathcal{N} \oplus Fil^1\mathcal{M}$;

(2) the descent data $[g]^{\mathcal{E}} = [g]^{\mathcal{N}} \oplus [g]^{\mathcal{M}}$, i.e., if (n, m) is an element in $\mathcal{E}$, then $[g]^{\mathcal{E}}(n, m) = ([g]^{\mathcal{N}}(n), [g]^{\mathcal{M}}(m))$.

We have the following result.

Lemma 6.6. *Let $\mathcal{M}$, $\mathcal{N}$ be objects in $BrMod^1_{dd,K/F}$ such that $(\mathcal{M}, \mathcal{N})$ is simple, then we have the following exact sequence.*

$$\begin{aligned} 0 &\to Hom_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{N}) \to Hom_{dd,Fil}(\mathcal{M}, \mathcal{N}) \\ &\to Hom_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{N}) \to Ext^1_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{N}) \to 0. \end{aligned} \tag{6.1}$$

Here $Hom_{dd,Fil}$ is the set of $\mathcal{S}$-module homomorphisms which commute with the descent data and map Fil^1 to Fil^1, $Hom_{dd,1\otimes Frob}$ is the set of $\mathcal{S}$-module homomorphisms which commute with the descent data and are $1 \otimes Frob$-linear, the middle arrow is given by $\beta \mapsto (\beta\phi_1^{\mathcal{M}} - \phi_1^{\mathcal{N}}\beta)$.

Proof. This is an analogy of Lemma 2.4.2 of [10], where the result is about Fontaine-Laffaille modules (see also Lemma 5.7). The proof is similar. Assume that we have a short exact sequence in $BrMod^1_{dd,K/F}$

$$0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{M} \to 0.$$

We may write $\mathcal{E} = \mathcal{N} \oplus \mathcal{M}$, $Fil^1\mathcal{E} = Fil^1\mathcal{N} \oplus Fil^1\mathcal{M}$, and $[g]^{\mathcal{E}} = [g]^{\mathcal{N}} \oplus [g]^{\mathcal{M}}$, such that $\mathcal{N} \to \mathcal{E}$ is the natural injection and $\mathcal{E} \to \mathcal{M}$ is the natural projection. Then $\mathcal{E}$ is determined by the map $\phi_1^{\mathcal{E}} : Fil^1\mathcal{E} \to \mathcal{E}$. Write

$$\phi_1^{\mathcal{E}} = \begin{pmatrix} \phi_1^{\mathcal{N}} & \alpha \\ 0 & \phi_1^{\mathcal{M}} \end{pmatrix},$$

where α is an element in $Hom_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{N})$.
Conversely, any element α in $Hom_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{N})$ gives rise to such an extension. Two elements α and α' give rise to isomorphic extensions if there exists an element $\beta \in Hom_{dd,Fil}(\mathcal{M}, \mathcal{N})$ such that

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1^{\mathcal{N}} & \alpha \\ 0 & \phi_1^{\mathcal{M}} \end{pmatrix} = \begin{pmatrix} \phi_1^{\mathcal{N}} & \alpha' \\ 0 & \phi_1^{\mathcal{M}} \end{pmatrix} \begin{pmatrix} 1 & \beta|_{Fil^1\mathcal{M}} \\ 0 & 1 \end{pmatrix}.$$

Therefore we get an exact sequence

$$Hom_{dd,Fil}(\mathcal{M}, \mathcal{N}) \to Hom_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{N}) \to Ext^1_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{N}) \to 0.$$

Here the first arrow is the map $\beta \mapsto (\beta\phi_1^{\mathcal{M}} - \phi_1^{\mathcal{N}}\beta)$. The kernel of this map is clearly $Hom_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{N})$. ☐

Remark 6.7. Unfortunately, most of the pairs $(\mathcal{M}, \mathcal{N})$ are not simple. Even if $(\mathcal{M}, \mathcal{N})$ is simple, we can not get a nice formula as in Lemma 2.4.3 of [10]. Nevertheless, there are certain pairs $(\mathcal{M}, \mathcal{N})$ which are simple and we can use the exact sequence to make useful computations. We give some examples in next section.

6.3. **Computation with the exact sequence.** In this section, we use the exact sequence (6.1) to compute $\dim_E Ext^1(\mathcal{M}, \mathcal{M})$, where $\mathcal{M}$ is a rank two Breuil module of type J in the sense of Definition 6.4 and 6.5.

6.3.1. *Reducible case.* Let $\mathcal{M}$ be a reducible rank two Breuil module with descent data of type J as given in section 6.1.2. By Lemma 4.3 and 4.4 of [9], the pair $(\mathcal{M}, \mathcal{M})$ is simple, we may apply the exact sequence to compute $\dim_E Ext^1(\mathcal{M}, \mathcal{M})$. Write $\mathcal{M} = \oplus_{i\in S}\mathcal{M}_i$, then

$$Hom_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{M}) \cong \oplus_i Hom_{dd}(Fil^1\mathcal{M}_i, \mathcal{M}_{i+1}). \tag{6.2}$$

Assume that $\psi \in Hom_{dd}(Fil^1\mathcal{M}_i, \mathcal{M}_{i+1})$ with

$$\begin{cases} \psi(u^{j_i}e_i) = X_{i+1}e_{i+1} + Y_{i+1}f_{i+1} \\ \psi(u^{e-j_i}f_i + \lambda_i u^{h_i}e_i) = Z_{i+1}e_{i+1} + W_{i+1}f_{i+1}, \end{cases} \tag{6.3}$$

where X, Y, Z, W are elements in $E[u]/u^{ep}$. Notice that G acts trivially on u^{n_i} and u^{e-n_i} since n_i is either 0 or e. Because ψ commutes with the G-action, it is easy to see that

$$\begin{cases} g(X_{i+1}) = X_{i+1} \\ g(Y_{i+1})\omega_{i+1}^{\alpha_{i+1}}(g) = \omega_{i+1}^{p\beta_i}(g)Y_{i+1} \\ g(W_{i+1}) = W_{i+1} \\ g(Z_{i+1})\omega_{i+1}^{\beta_{i+1}}(g) = \omega_{i+1}^{p\alpha_i}(g)Z_{i+1}. \end{cases} \tag{6.4}$$

On the other hand, every matrix $\begin{pmatrix} X_{i+1} & Y_{i+1} \\ Z_{i+1} & W_{i+1} \end{pmatrix}$ satisfying these equations gives rise an element in $Hom_{dd}(Fil^1\mathcal{M}_i, \mathcal{M}_{i+1})$.

The equation $g(X_{i+1}) = X_{i+1}$ means that all nonzero terms of X_{i+1} have degree congruent to $0 \pmod e$. Equation $g(Y_{i+1})\omega_{i+1}^{\alpha_{i+1}}(g) = \omega_{i+1}^{p\beta_i}(g)Y_{i+1}$ means that all nonzero terms of Y_{i+1} have degree congruent to $(p\beta_i - \alpha_{i+1}) \pmod e$. Similar statements hold for Z_{i+1} and W_{i+1}. Because the degrees of these elements are less than ep, we have

$$\begin{cases} \dim_E Hom_{dd}(Fil^1\mathcal{M}_i, \mathcal{M}_{i+1}) = 4p \\ \dim_E Hom_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{M}) = 4pr. \end{cases} \tag{6.5}$$

Similarly, we have

$$Hom_{dd,Fil}(\mathcal{M}, \mathcal{M}) \cong \oplus_i Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i). \tag{6.6}$$

We show that $\dim_E Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i) = 4p-1$. Let $\rho \in Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i)$ with

$$\begin{cases} \rho(e_i) = P_ie_i + Q_if_i \\ \rho(f_i) = R_ie_i + S_if_i. \end{cases} \tag{6.7}$$

First, ρ commutes with the G-action, it is easy to get that

$$g(P_i) = P_i,\ g(Q_i) = \omega_i^{\beta_i - \alpha_i}(g)Q_i,\ g(R_i) = \omega^{\alpha_i - \beta_i}(g)R_i,\ g(S_i) = S_i.$$

Second, ρ maps $Fil^1\mathcal{M}_i$ to $Fil^1\mathcal{M}_i$.
(Case 1) $Fil^1\mathcal{M}_i = \langle e_i, u^ef_i\rangle$.
In this case, $\rho(Fil^1\mathcal{M}_i) = \langle P_ie_i + Q_if_i, u^e(R_ie_i + S_if_i)\rangle$. Therefore $P_ie_i + Q_if_i \in \langle e_i, u^ef_i\rangle$. We get $u^e \mid Q_i$. In this case $\dim_E Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i) = 4p-1$.
(Case 2) $Fil^1\mathcal{M}_i = \langle u^ee_i, f_i + \lambda_iu^{h_i}e_i\rangle$.
In this case, $\rho(Fil^1\mathcal{M}_i) = \langle u^e(P_ie_i + Q_if_i), R_ie_i + S_if_i + \lambda_iu^{h_i}(P_ie_i + Q_if_i)\rangle$. Therefore $R_ie_i + S_if_i + \lambda_iu^{h_i}(P_ie_i + Q_if_i) \in \langle u^ee_i, f_i + \lambda_iu^{h_i}e_i\rangle$. Note that

$$\begin{aligned} R_ie_i + S_if_i + \lambda_iu^{h_i}(P_ie_i + Q_if_i) =& (S_i + \lambda_iu^{h_i}Q_i)(f_i + \lambda_iu^{h_i}e_i) \\ &+ (R_i + \lambda_iu^{h_i}P_i - \lambda_iu^{h_i}S_i - \lambda_i^2u^{2h_i}Q_i)e_i, \end{aligned} \tag{6.8}$$

we have $u^e \mid (R_i + \lambda_iu^{h_i}P_i - \lambda_iu^{h_i}S_i - \lambda_i^2u^{2h_i}Q_i)$.
If $\lambda_i = 0$, then it is the same as $u^e \mid R_i$, so $\dim_E Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i) = 4p-1$.
If $\lambda_i \neq 0$ and $h_i = 0$, then $u^e \mid (R_i + \lambda_iP_i - \lambda_iS_i - \lambda_i^2Q_i)$, we have $\dim_E Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i) = 4p-1$.
If $\lambda_i \neq 0$ and $h_i \neq 0$, then $deg(\lambda_i^2u^{2h_i}Q_i) > e$, and $u^e \mid (R_i + \lambda_iu^{h_i}P_i - \lambda_iu^{h_i}S_i)$, we still have $\dim_E Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i) = 4p-1$.

Then we have

$$\dim_E Hom_{dd,Fil}(\mathcal{M}, \mathcal{M}) = \sum_{i\in S} \dim_E Hom_{dd,Fil}(\mathcal{M}_i, \mathcal{M}_i) = r(4p-1). \tag{6.9}$$

By the above analysis, we conclude that

$$\dim_E Ext^1_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{M}) = \dim_E Hom_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{M}) + r. \tag{6.10}$$

In particular, if $\mathcal{M}$ is split, then $\dim_E Ext^1_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{M}) = 2 + r$. If $\mathcal{M}$ is nonsplit, then $\dim_E Ext^1_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{M}) = 1 + r$.

6.3.2. *Irreducible case.* Let $\mathcal{M}$ be an irreducible rank two Breuil module of type J as in Definition 6.5. First, we show that the pair $(\mathcal{M}, \mathcal{M})$ is simple. Let $\mathcal{N} \in Ext^1(\mathcal{M}, \mathcal{M})$. Write $\mathcal{N} = \oplus_{i\in S}\mathcal{N}_i$. Assume that

$$\mathcal{N}_i = E[u]/u^{ep}\langle e_i, f_i, e'_i, f'_i\rangle,$$

where $e'_i \in \mathcal{N}_i$ (resp. $f'_i \in \mathcal{N}_i$) is a lift of $e_i \in \mathcal{M}_i$ (resp. $f_i \in \mathcal{M}_i$).

Lemma 6.8. *We may assume that*

$$Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle u^{n_i}e_i, u^{e-n_i}f_i, u^{n_i}e'_i, u^{e-n_i}f'_i\rangle.$$

Proof. If $i+1 \in J$, then $n_i = e$. Write $Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle u^e e_i, f_i, u^e e'_i + A_i e_i + B_i f_i, f'_i + C_i e_i + D_i f_i\rangle$. We may assume that $B_i = D_i = 0$. Since $u^e e'_i \in Fil^1\mathcal{N}_i$, we have $u^e|A_i$. Let $A_i/u^e \in E[u]/u^{ep}$ such that $u^e(A_i/u^e) = A_i$. Let $e'' = e'_i + (A_i/u^e)e_i$ and $f''_i = f'_i + C_i e_i$, we may assume that $A_i = C_i = 0$.

If $i+1 \notin J$, the argument is the same. □

Assume that $\mathcal{N} = \oplus\mathcal{N}_i$ has the following form.

$$\begin{aligned}
\mathcal{N}_i &= E[u]/u^{ep}\langle e_i, f_i, e'_i, f'_i\rangle,\\
Fil^1\mathcal{N}_i &= E[u]/u^{ep}\langle u^{n_i}e_i, u^{e-n_i}f_i, u^{n_i}e'_i, u^{e-n_i}f'_i\rangle,\\
\phi_1(u^{n_i}e'_i) &= (a)'_{i+1}e'_{i+1} + X_{i+1}e_{i+1} + Y_{i+1}f_{i+1},\\
\phi_1(u^{e-n_i}f'_i) &= (a)_{i+1}f'_{i+1} + Z_{i+1}e_{i+1} + W_{i+1}f_{i+1},\\
[g](e'_i) &= \omega_i^{\gamma_i}(g)e'_i + A_{i,g}e_i + B_{i,g}f_i,\\
[g](f'_i) &= \omega_i^{\mu_i}(g)f'_i + C_{i,g}e_i + D_{i,g}f_i,\\
[g_\phi](e'_i) &= f'_{i+s} + P_{i+s}e_{i+s} + Q_{i+s}f_{i+s},\\
[g_\phi](f'_i) &= e'_{i+s} + R_{i+s}e_{i+s} + S_{i+s}f_{i+s},
\end{aligned}$$

where the X, Y, Z, W and A, B, C, D are in $E[u]/u^{ep}$.

Lemma 6.9. *We may assume that $A_{i,g} = B_{i,g} = C_{i,g} = D_{i,g} = 0$.*

Proof. Let $\mathcal{M}'$ be $\mathcal{M}$ without the descent data of g_ϕ, then $\mathcal{M}'$ is a direct sum of two rank one objects in $BrMod^1_{dd,K/K_0}$. More precisely, $\mathcal{M}'$ is isomorphic to a reducible rank two object of some type J' in the sense of Definition 6.4. Then the proof is the same as the proof of [9] Lemma 4.4. □

Lemma 6.10. *We may assume that $P_{i+s} = Q_{i+s} = R_{i+s} = S_{i+s} = 0$.*

Proof. If $i+1 \in J$, since $[g_\phi](Fil^1) \subset Fil^1$, we have $u^e|Q_{i+s}, S_{i+s}$. Let $e''_{i+s} = e'_{i+s} + R_{i+s}e_{i+s} + S_{i+s}f_{i+s}$ and $f''_{i+s} = f'_{i+s} + P_{i+s}e_{i+s} + Q_{i+s}f_{i+s}$. This does not change the form of Fil^1 since $E[u]/u^{ep}\langle e_{i+s}, u^e f_{i+s}, e'_{i+s}, u^e f'_{i+s}\rangle = E[u]/u^{ep}\langle e_{i+s}, u^e f_{i+s}, e''_{i+s}, u^e f''_{i+s}\rangle$. We have to check that it does not change the form of $[g]$. By the relation $g_\phi g^{p^s} = gg_\phi$, we have

$$\begin{aligned}
[gg_\phi](e'_i) &= \omega_{i+s}^{\mu_{i+s}}(g)f'_{i+s} + g(P_{i+s})\omega_{i+s}^{\gamma_{i+s}}(g)e_{i+s} + g(Q_{i+s})\omega_{i+s}^{\mu_{i+s}}(g)f_{i+s}\\
&= [g_\phi g^{p^s}](e'_i)\\
&= \omega_{i+s}^{\mu_{i+s}}(g)(f'_{i+s} + P_{i+s}e_{i+s} + Q_{i+s}f_{i+s}).
\end{aligned}$$

Therefore, $[g]f''_{i+s} = \omega_{i+s}^{\mu_{i+s}}(g)f''_{i+s}$. Similarly, $[g]e''_{i+s} = \omega_{i+s}^{\gamma_{i+s}}(g)e''_{i+s}$

If $i+1 \notin J$, the argument is the same. □

Lemma 6.11. $X_i = W_{i+s}$, $Y_i = Z_{i+s}$, $Z_i = Y_{i+s}$, $W_i = X_{i+s}$.

Proof. This is an easy consequence of the equation $[g_\phi] \circ \phi_1 = \phi_1 \circ [g_\phi]$. □

By Lemma 6.8, 6.9, 6.10, we can apply the exact sequence (6.1) to the pair $(\mathcal{M}, \mathcal{M})$. If we forget about the descent data of $[g_\phi]$, then the same computation as in the reducible case shows that

$$\dim_E Hom'_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{M}) = 4pr,$$
$$\dim_E Hom'_{dd,Fil}(\mathcal{M}, \mathcal{M}) = (4p-1)r.$$

Here $'$ means that we do not consider the descent data of $[g_\phi]$. Then by Lemma 6.11 and a similar computation as for equation (6.7), we have

$$\dim_E Hom_{dd,1\otimes Frob}(Fil^1\mathcal{M}, \mathcal{M}) = 4ps,$$
$$\dim_E Hom_{dd,Fil}(\mathcal{M}, \mathcal{M}) = (4p-1)s.$$

Therefore,

$$\dim_E Ext^1_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{M}) = \dim_E Hom_{BrMod^1_{dd,K/F}}(\mathcal{M}, \mathcal{M}) + s = 1 + s. \tag{6.11}$$

6.4. **Proof of Lemma 3.3.** Now we can prove Lemma 3.3 by applying the above computation. We use the notation in section 3.1. Note that we have the following facts.

If $\bar{\rho}_v$ is reducible and nonsplit, $\bar{\rho}_v$ is of type J_v in the sense of Definition 3.1.3 of [20]. Therefore, there exists a Breuil module $\mathcal{M}_{red} \in BrMod^1_{dd,L/F_v}$ (here the coefficient ring is $\mathcal{S} = k_v \otimes \kappa[u]/u^{ep}$) such that the generic fibre of $\mathcal{M}_{red}$ is $\bar{\rho}_v$ and $\mathcal{M}_{red} = \bigoplus_{i:k_v \to \bar{\mathbb{F}}_p} \mathcal{M}_i$ has the following form.

$$\mathcal{M}_i = \kappa[u]/u^{ep}\langle e_i, f_i\rangle$$
$$Fil^1\mathcal{M}_i = \kappa[u]/u^{ep}\langle u^{j_i}e_i, u^{e-j_i}f_i + \lambda_i u^{h_i}e_i\rangle$$
$$\phi_1(u^{j_i}e_i) = (b)_{i+1}e_{i+1}, \quad \phi_1(u^{e-j_i}f_i + \lambda_i u^{h_i}e_i) = (a)_{i+1}f_{i+1}$$
$$[g]e_i = (\psi'_1 \prod_{i\in J_v} \omega_i^{-p}(g))e_i, \quad [g]f_i = (\psi'_2 \prod_{i\notin J_v} \omega_i^{-p})f_i$$

where $\lambda_i \in \kappa$ with $\lambda_i = 0$ if $i+1 \notin J_v$, $j_i = \begin{cases} e & i+1 \in J_v \\ 0 & i+1 \notin J_v \end{cases}$, h_i is an integer between 0 and e, ψ'_1 and ψ'_2 are restrictions of ψ_1 and ψ_2 to $Gal(L/F_v)$. Note that $\mathcal{M}_{red}$ is split if and only if all the $\lambda'_i s$ are 0.

If $\bar{\rho}_v$ is irreducible, from Theorem 4.1.4 of [20], there exists a rank two Breuil module $\mathcal{M}_{irr} \in BrMod^1_{dd,L/F_v}$ (here the coefficient ring is $\mathcal{S} = k'_v \otimes \kappa[u]/u^{ep}$), such that the generic fiber of $\mathcal{M}_{irr}$ is $\bar{\rho}_v$ and $\mathcal{M}_{irr} = \oplus_{i\in S'}\mathcal{M}_i$ has the following form.

$$\mathcal{M}_i = \kappa[u]/u^{ep}\langle e_i, f_i\rangle$$
$$Fil^1\mathcal{M}_i = \kappa[u]/u^{ep}\langle u^{n_i}e_i, u^{m_i}f_i\rangle$$
$$\phi_1(u^{n_i}e_i) = (a)'_{i+1}e_{i+1}, \quad \phi_1(u^{m_i}f_i) = (a)_{i+1}f_{i+1}$$
$$N(e_i) = 0, \quad N(f_i) = 0$$
$$[g_\phi]e_i = f_{i+d_v}, \quad [g_\phi]f_i = e_{i+d_v}$$
$$[g]e_i = ((\prod_{i\in S_v} \omega_i^{a_i} \prod_{i\in J_v} \omega_i^{b_i - p}))(g)e_i, \quad [g]f_i = ((\prod_{i\in S_v} \omega_i^{a_i} \prod_{i\notin J_v} \omega_i^{b_i - p}))(g)f_i.$$

Here $n_i = \delta_{J_v}(i+1)e$ and $m_i = e - n_i$, $(a)_i = \begin{cases} a & \text{if } i = 1 \\ 1 & \text{otherwise} \end{cases}$, $(a)'_i = \begin{cases} a \text{ if } i = d_v + 1 \\ 1 \text{ otherwise} \end{cases}$, $g_\phi \in Gal(L/F_v)$ is a fixed element such that it maps to the nontrivial element in $Gal(F'_v/F_v)$ under the natural surjection $Gal(L/F_v) \to Gal(F'_v/F_v)$, g denotes the elements in $Gal(L/F'_v)$. Notice that $\mathcal{M}_{irr}$ is of type J_v in the sense of Definition 6.5.

Suppose that $\rho_v : G_{F_v} \to GL_2(E)$ is a potentially Barsotti-Tate representation with Galois type τ_v. Since $\tau_v|_{I_L}$ is trivial, ρ_v is Barsotti-Tate when restricted to G_L. Consequently, there exists a p-divisible group Γ over $\mathcal{O}_L$ such that the generic fibre of Γ is $\rho_v|_{G_L}$. Γ also has an action of the Galois group $Gal(L/F_v)$ over the action of $Gal(L/F_v)$ on $Spec(\mathcal{O}_L)$. Let $D(\Gamma/k_v \otimes \kappa)$ be the contravariant Dieudonné module of Γ. Then $D(\Gamma/k_v \otimes \kappa)$ is a free $W(k_v) \otimes \mathcal{O}$-module of rank two with an action of the Dieudonné ring $W(k_v)[\mathbb{F}, \mathbb{V}]$. We may define an action of the Weil group W_v on $D(\Gamma/k_v \otimes \kappa)$ (see for example appendix of [11]). Then

$$WD(\rho_v) \cong Hom_{W(k_v)\otimes\mathcal{O}}(D(\Gamma/k_v \otimes \kappa), W(k_v) \otimes \mathcal{O}) \otimes_{W(k_v)\otimes\mathcal{O}} \bar{\mathbb{Q}}_p.$$

6.4.1. *The reducible case.* Assume that $\bar{\rho}_v$ is reducible and nonsplit. Since $\tau_v = \tau_{J_v} = \tilde{\chi}_{J_v} \oplus \tilde{\chi}_{J_v} \prod_{i\in S_v} \tilde{\omega}_i^{c_i}$, there exist basis elements $\vec{v}$ and $\vec{w}$ of $WD(\rho_v)$ so that for $g \in I_v \subset G_{F_v}$,

$$g(\vec{v}) = \tilde{\chi}_{J_v}(g)\vec{v}, \quad g(\vec{w}) = \tilde{\chi}_{J_v} \prod_{i\in S_v} \tilde{\omega}_i^{c_i}(g)\vec{w}. \tag{6.12}$$

Let $BrMod^1_{dd,L/F_v}$ be the category of Breuil modules with descent data ($\mathcal{S} = k_v \otimes \kappa[u]/u^{ep}$ in this case). Then there is an equivalence between $BrMod^1_{dd,L/F_v}$ and the category of finite flat group schemes over $\mathcal{O}_L$ with an action of $Gal(L/F_v)$. Let $\mathcal{M}_\Gamma \in BrMod^1_{dd,L/F_v}$ be the Breuil module with descent data corresponding to the group scheme Γ. Then there is a canonical isomorphism

$$D(\Gamma/k_v \otimes \kappa) \otimes_{k_v\otimes\kappa, Frob\otimes 1} (k_v \otimes \kappa) \cong \mathcal{M}_\Gamma/u\mathcal{M}_\Gamma$$

under which $\mathbb{F} \otimes Frob$ corresponds to ϕ (in our case, $\phi(x) = \phi_1(u^e \cdot x)$) and $\mathbb{V} \otimes Frob^{-1}$ corresponds to the composition

$$\mathcal{M}_\Gamma/u\mathcal{M}_\Gamma \xrightarrow{\phi_1^{-1}} Fil^1\mathcal{M}_\Gamma/uFil^1\mathcal{M}_\Gamma \to \mathcal{M}_\Gamma/u\mathcal{M}_\Gamma.$$

(This is well defined since ϕ_1^{-1} is a bijection.)

We determine those group schemes with descent data such that the corresponding Breuil modules with descent data have Dieudonné modules with basis satisfying (6.12). Note that if $\mathcal{M}$ is a Breuil module such that the associated Dieudonné module $D(\mathcal{M})$ has a basis satisfying (6.12), then it is easy to see that the descent data of $\mathcal{M}$ are determined by the reductions of $\tilde{\chi}_{J_v}$ and $\tilde{\chi}_{J_v} \prod_{i\in S_v} \tilde{\omega}_i^{c_i}$. More precisely, $\mathcal{M}$ has the following form

$$\mathcal{M}_i = \kappa[u]/u^{ep}\langle e_i, f_i\rangle$$
$$Fil^1\mathcal{M}_i = \kappa[u]/u^{ep}\langle u^{n_i}e_i, u^{m_i}f_i + \lambda_i u^{h_i}e_i\rangle$$
$$\phi_1(u^{n_i}e_i) = (b)_{i+1}e_{i+1}, \quad \phi_1(u^{m_i}f_i + \lambda_i u^{h_i}e_i) = (a)_{i+1}f_{i+1}$$
$$[g]e_i = \prod_{i\in S_v} \omega_i^{a_i} \prod_{i\notin J_v} \omega_i^{b_i-p}(g)e_i, \quad [g]f_i = \prod_{i\in S_v} \omega_i^{a_i} \prod_{i\notin J_v} \omega_i^{b_i-p} \prod_{i\in S_v} \omega_i^{c_i}(g)f_i$$

where m_i and n_i are integers between 0 and e.

Since we know the generic fibre of $\mathcal{M}$, then we can determine $\mathcal{M}$ completely.

Lemma 6.12. *If the Dieudonné module associated to $\mathcal{M}$ satisfies (6.12), then $\mathcal{M}$ is of type J_v.*

Proof. As remarked above, there exists at most one such Breuil module. From the definition, it is easy to see that $D(\mathcal{M}_{red})$ has all the required properties. Thus $\mathcal{M} = \mathcal{M}_{red}$ is of type J_v. □

We then have the following lemma.

Lemma 6.13. *If $\bar{\rho}_v$ is reducible and nonsplit, then*
(1) $\dim_\kappa H^1_f(G_{F_v}, Ad(\bar{\rho}_v)) = 1 + d_v$.
(2) $\dim_\kappa H^1_f(G_{F_v}, Ad^0(\bar{\rho}_v)) = d_v$.

Proof. (1) follows from equation (6.10). (2) follows from the fact that

$$\dim_\kappa Ext^1(\mathcal{M}(m_i, \mu_i, a), \mathcal{M}(m_i, \mu_i, a)) = 1$$

for any rank one Breuil module. (This fact is a special case of Theorem 3.9 of [9].) □

6.4.2. *The irreducible case.* Assume that $\bar{\rho}_v$ is irreducible, then the same argument as in the reducible case shows that there exists a unique Breuil module with descent data $\mathcal{M}$ such that the corresponding Dieudonné module has basis elements $\vec{v}$ and $\vec{w}$ with

$$g(\vec{v}) = (\tilde{\psi}_{J_v}\tilde{\omega}_r \prod_{i=1}^{d_v} \tilde{\omega}_i^{c_i})(g)\vec{v}, \quad g(\vec{w}) = (\tilde{\psi}_{J_v}\tilde{\omega}_r \prod_{i=1}^{d_v} \tilde{\omega}_i^{c_i})^{p^r}(g)\vec{w}.$$

Indeed, $\mathcal{M} = \mathcal{M}_{irr}$ which is the one constructed in Theorem 4.1.4 of [20]. In particular, $\mathcal{M}$ is of type J_v in the sense of Definition 6.5.

Lemma 6.14. *If $\bar{\rho}_v$ is irreducible, then*
(1) $\dim_\kappa H^1_f(G_{F_v}, Ad(\bar{\rho}_v)) = 1 + d_v$.
(2) $\dim_\kappa H^1_f(G_{F_v}, Ad^0(\bar{\rho}_v)) = d_v$.

Proof. The proof is almost the same as the proof of last lemma, except now we use equation (6.11). □

Acknowledgements The author is deeply indebted to Matthew Emerton, who suggested the problem, for many enlightening discussions. He would like to thank Brian Conrad, Fred Diamond, Toby Gee, and Mark Kisin for helpful conversations and correspondences.

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