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$\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules

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ABSTRACT

In this paper, we construct an $\mathcal{O}$-display theory and prove that, under certain conditions on the base ring, the category of nilpotent $\mathcal{O}$-displays and the category of π-divisible formal $\mathcal{O}$-modules are equivalent. Starting with this result, we then construct a Dieudonné $\mathcal{O}$-display theory and prove a similar equivalence between the category of Dieudonné $\mathcal{O}$-displays and the category of π-divisible $\mathcal{O}$-modules. We also show that this equivalence is compatible with duality. These results generalize the corresponding results of Zink and Lau on displays and p-divisible groups.

Contents

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1. Introduction

The theory of displays was introduced by Zink in [23] and then developed by Zink and Lau in a series of papers ([22,9,10] etc.). One of the main results of this theory is that, for any ring R with p nilpotent in it, the category of formal p-divisible groups over R and the category of nilpotent displays over R are equivalent.

Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field with characteristic $(0, p)$ and uniformizer π. The goal of this paper is to generalize the above equivalence to nilpotent $\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules over $\mathcal{O}$-algebras R with π nilpotent in R. For this purpose we combine the idea of Drinfeld in [3] (where Drinfeld established the equivalence between certain Cartier $\mathbb{E}_{\mathcal{O},R}$-modules and formal $\mathcal{O}$-modules over R) and the ideas in [23] and [10]. Hence, we generalize many results needed for establishing the equivalence of nilpotent displays over R and p-divisible formal groups over R and also use the already established equivalence for the $\mathcal{O} = \mathbb{Z}_p$ case. Some parts of this generalized theory are already utilized in [8, Chapter 9]. In a recent paper of Verhoek [18], a more general notion of Cartier A-modules is introduced, where A is the ring of integers of a number field. An equivalence between certain Cartier $\mathbb{E}_A$-modules and certain formal A-modules is also established in [18].

After establishing the equivalence between nilpotent $\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules, we introduce Dieudonné $\mathcal{O}$-displays and extend the equivalence to an equivalence between Dieudonné $\mathcal{O}$-displays and π-divisible $\mathcal{O}$-modules following [22].

In the first part of this introduction, we state the main results and outline the contents of this paper. All the notions related to Witt vectors and formal groups are the standard ones and will be recalled in the second part of this introduction.

1.1. Statement of the main results

Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic $(0, p)$ with uniformizers π. All rings and algebras over a commutative ring are assumed to be commutative.

Let $\mathrm{Nil}_{\mathcal{O}}$ be the category of $\mathcal{O}$-algebras with π nilpotent. Let R be an object of $\mathrm{Nil}_{\mathcal{O}}$. Let $\mathrm{ndisp}_{\mathcal{O}}/R$ be the category of nilpotent $\mathcal{O}$-displays over R (Definitions 2.1 and 2.3). Let (π-divisible formal $\mathcal{O}$-modules$/R$) be the category of π-divisible formal $\mathcal{O}$-modules over R (Definition 1.15). The main result of this paper is the following theorem.

Theorem 1.1. *There exists a functor* $\mathrm{BT}_{\mathcal{O}}$

$$\mathrm{BT}_{\mathcal{O}} : \mathrm{ndisp}_{\mathcal{O}}/R \to (\pi\textit{-divisible formal } \mathcal{O}\textit{-modules}/R),$$

which is an equivalence of categories.

Remark 1.2. This is the main result of the first author's thesis [1]. This paper is mostly based on [1].

Remark 1.3. Assume that R is π-adic, i.e., $R = \varprojlim R/(\pi)^n$. By the discussion at the end of Section 2.2 and taking projective limits, Theorem 1.1 holds for such R as well.

Remark 1.4. The idea of the proof is similar as Drinfeld's idea in [3]. More precisely, let $(\mathcal{O}', \pi')$ be a finite extension of $(\mathcal{O}, \pi)$, we show that if Theorem 1.1 is true for $(\mathcal{O}, \pi)$, then it is true for $(\mathcal{O}', \pi')$. We sketch the strategy in the following.

The functor $\mathrm{BT}_{\mathcal{O}}$ is constructed in Section 2.4. Let $R \in \mathrm{Nil}_{\mathcal{O}'}$. Let $\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}$ be the category of nilpotent $\mathcal{O}$-displays over R with strict $\mathcal{O}'$-action (Definition 2.4). In Section 2.5, we construct a functor $\Gamma(\mathcal{O}, \mathcal{O}') : \mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R \to \mathrm{ndisp}_{\mathcal{O}'}/R$ and show the commutativity of the following diagram

$$\begin{tikzcd}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}"] \arrow[d, "\Gamma(\mathcal{O},\mathcal{O}')"'] & (\pi'\text{-divisible formal } \mathcal{O}'\text{-modules}/R) \\
(\mathrm{ndisp}_{\mathcal{O}'}/R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}'}"'] &
\end{tikzcd} \tag{1.1}$$

By adapting the argument in [10], we show in Section 4 that $\Gamma(\mathcal{O}, \mathcal{O}')$ is an equivalence if and only if it is fully faithful (Proposition 4.10). If $\mathrm{BT}_{\mathcal{O}}$ is an equivalence, then to show that $\mathrm{BT}_{\mathcal{O}'}$ is an equivalence, it suffices to show that $\mathrm{BT}_{\mathcal{O}'}$ is faithful. This faithfulness property is proved in Section 3 using the theory of crystals (Proposition 3.28). Therefore Theorem 1.1 follows from the equivalence of $\mathrm{BT}_{\mathbb{Z}_p}$ [10, Theorem 1.1].

The construction of $\Gamma(\mathcal{O}, \mathcal{O}')$ is divided into two cases (Section 2.5). The first case is that $\mathcal{O} \to \mathcal{O}'$ is unramified with degree f. For this we introduce the category of nilpotent f-$\mathcal{O}$-displays ($f - \mathrm{ndisp}_{\mathcal{O}}$), construct two functors $\Omega_1(\mathcal{O}, \mathcal{O}') : \mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R \to (f - \mathrm{ndisp}_{\mathcal{O}}/R)$, $\Omega_2(\mathcal{O}, \mathcal{O}') : (f - \mathrm{ndisp}_{\mathcal{O}}/R) \to \mathrm{ndisp}_{\mathcal{O}'}/R$, and define $\Gamma_1(\mathcal{O}, \mathcal{O}') = \Omega_2(\mathcal{O}, \mathcal{O}') \circ \Omega_1(\mathcal{O}, \mathcal{O}')$. The second case is that $\mathcal{O} \to \mathcal{O}'$ is totally ramified. For this, we construct $\Gamma(\mathcal{O}, \mathcal{O}')$ directly.

One consequence of Theorem 1.1 is the following result.

Theorem 1.5. *Let p be an odd prime. Let R be a Noetherian complete local $\mathcal{O}$-algebra with perfect residue field of characteristic p. Then the equivalence $\mathrm{BT}_{\mathcal{O}}$ in Theorem 1.1 extends to an equivalence*

$$\mathrm{BT}_{\mathcal{O}} : \mathrm{Ddisp}_{\mathcal{O}}/R \to (\pi\text{-divisible } \mathcal{O}\text{-modules}/R).$$

Furthermore, this equivalence is compatible with duality in the sense that

$$\mathrm{BT}_{\mathcal{O}}(\mathcal{P}^t) \cong G^\vee.$$

Here $\mathrm{Ddisp}_{\mathcal{O}}/R$ is the category of Dieudonné $\mathcal{O}$-displays over R (Definition 5.1), $\mathcal{P}^t$ is the dual of $\mathcal{P}$ (Definition 5.8), $G^\vee$ is the Serre $\mathcal{O}$-dual of G.

As mentioned before, a large part of the paper is a generalization of the $\mathcal{O} = \mathbb{Z}_p$ case. In the body of the paper, we give precise definitions and constructions of the generalized notions. On the other hand, if the argument of a result is the same as the argument in the $\mathcal{O} = \mathbb{Z}_p$ case, we skip the details and only refer to the original references.

In Section 2, we study the category ($f - \mathrm{disp}_{\mathcal{O}}$) of f-$\mathcal{O}$-displays (Definition 2.1) in detail, explain the constructions of the functors in Remark 1.4, and prove the commutativity of the diagram (1.1).

In Section 3, we introduce $\mathcal{O}$-frames and $\mathcal{O}$-windows, which generalize the notion of $\mathcal{O}$-displays. We define the crystals associated with $\mathcal{O}$-displays and the Grothendieck–Messing crystals associated with π-divisible formal $\mathcal{O}$-modules. By the study of the universal extensions of π-divisible formal $\mathcal{O}$-modules, we show that if the π-divisible formal $\mathcal{O}$-module and the $\mathcal{O}$-display are related by the functor $\mathrm{BT}_{\mathcal{O}}$, then the corresponding crystals on $\mathcal{O}$-pd-thickenings are isomorphic (Theorem 3.26). As a consequence of this result, the faithfulness of $\mathrm{BT}_{\mathcal{O}}$ follows (Proposition 3.28).

In Section 4, we apply the ideas of [10] to the functors $\Omega_i(\mathcal{O}, \mathcal{O}')$ and $\Gamma_i(\mathcal{O}, \mathcal{O}')$ we constructed in Section 2.5. We show that the functors are equivalences of categories if they are fully faithful (Proposition 4.10). Combine with the faithfulness of $\mathrm{BT}_{\mathcal{O}}$ (Proposition 3.28), we obtain Theorem 1.1.

In Section 5 we study Dieudonné $\mathcal{O}$-displays and prove Theorem 1.5.

1.2. Background on Witt vectors, formal groups, and Cartier modules

In this section, we recall the basic properties of Witt vectors and formal groups and review Drinfeld's result on Cartier modules. Along the way, we fix the notation we use in this paper. The readers may skip this part to next section and only come back for references.

Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic $(0, p)$ with uniformizer π and residue field $\mathbb{F}_q$.

1.2.1. The functors $W_{\mathcal{O}}(-)$ and $\widehat{W}_{\mathcal{O}}(-)$

Let $W_{\mathcal{O}}(-) : \mathrm{Alg}_{\mathcal{O}} \to \mathrm{Alg}_{\mathcal{O}}$ be the functor of ramified Witt vectors associated with $(\mathcal{O}, \pi)$. See for example [5, Section 5.1] or [7] for more details on this object. The *n-th Witt polynomial* attached to $(\mathcal{O}, \pi)$ is defined by

$$\begin{aligned} \mathrm{w}_n : W_{\mathcal{O}}(R) &\to R \\ \underline{b} = (b_0, b_1, \dots) &\mapsto b_0^{q^n} + \pi b_1^{q^{n-1}} + \dots + \pi^n b_n. \end{aligned}$$

Let ${}^V = {}^{V_\pi} : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ be the *Verschiebung morphism* and ${}^F : W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ the *Frobenius morphism*. Note that ^{V} depends on the choice of π. Define $I_{\mathcal{O},R} = {}^V(W_{\mathcal{O}}(R))$ and $W_{\mathcal{O},n}(R) = W_{\mathcal{O}}(R)/{}^{V^n}(W_{\mathcal{O}}(R)$.

Let $a \in R$, the *Teichmüller lift* of a is $[a] \in W_{\mathcal{O}}(R)$ given by $(a, 0, 0 \dots)$.

Let $\mathcal{O}'$ be a finite extension of $\mathcal{O}$ with a fixed uniformizer π' and residue field k'. Denote by u the natural morphism in [5, Lemma 5.3]

$$u : W_{\mathcal{O}}(-) \to W_{\mathcal{O}'}(-).$$

Recall that we have $u([a]) = [a]$, $u({}^{V_\pi} x) = \frac{\pi}{\pi'} {}^{V_{\pi'} F^{f-1}} u(x)$, and $u({}^{F^f} x) = {}^F u(x)$, where f is the degree of residue extension for $\mathcal{O}'/\mathcal{O}$.

Denote by Δ the unique natural morphism (Cartier morphism) of $\mathcal{O}$-algebras

$$\Delta : W_{\mathcal{O}}(-) \longrightarrow W_{\mathcal{O}}(W_{\mathcal{O}}(-))$$

such that $\mathcal{W}(\Delta(x)) = [{}^{F^n} x]_{n \geq 0}$. Here $\mathcal{W} = (\mathrm{w}_0, \mathrm{w}_1, \dots)$.

Remark 1.6. With the notation as above, let E and E' be the fraction field of $\mathcal{O}$ and $\mathcal{O}'$ respectively. Let E_0/E be the maximal unramified extension of E in E' with ring of integers $\mathcal{O}_{E_0}$. We identify $W_{\mathcal{O}}(k') = \mathcal{O}_{E_0}$. If A is an $\mathcal{O}'$-algebra, we have the following morphism

$$\mathcal{O}_{E_0} = W_{\mathcal{O}}(k') \xrightarrow{\Delta} W_{\mathcal{O}}(\mathcal{O}_{E_0}) \to W_{\mathcal{O}}(A),$$

which makes $W_{\mathcal{O}}(A)$ an $\mathcal{O}_{E_0}$-algebra. The natural morphism $W_{\mathcal{O}}(A) \longrightarrow W_{\mathcal{O}'}(A)$ is then a morphism of $\mathcal{O}_{E_0}$-algebras and induces a natural morphism

$$W_{\mathcal{O}}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}' \longrightarrow W_{\mathcal{O}'}(A),$$

where ${}^{F_E^f} \otimes \mathrm{id}$ on the left hand side corresponds to ${}^{F_{E'}}$ on the right hand side.

If A is a perfect k'-algebra, the reductions modulo π' of the above two algebras coincide with A. Since $W_{\mathcal{O}'}(A)$ has no π'-torsion, we obtain the following isomorphism

$$W_{\mathcal{O}}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}' \xrightarrow{\sim} W_{\mathcal{O}'}(A).$$

Thus, if E_0 is the maximal unramified extension of $\mathbb{Q}_p$ in E with degree $f(E/\mathbb{Q}_p)$, $W = W_{\mathbb{Z}_p}$, for every perfect k-algebra A we have a canonical isomorphism

$$W(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O} \xrightarrow{\sim} W_{\mathcal{O}}(A)$$
$$[a] \otimes 1 \mapsto [a]$$
$${}^{F^{f(E/\mathbb{Q}_p)}} \otimes \mathrm{id} \leftrightarrow {}^{F}.$$

If R is a nilpotent $\mathcal{O}$-algebra, there is a subalgebra $\widehat{W}_{\mathcal{O}}(R)$ of $W_{\mathcal{O}}(R)$ which is stable under ^{F} and ^{V} and defined by

$$\widehat{W}_{\mathcal{O}}(R) = \{(x_0, x_1, \cdots) \in W_{\mathcal{O}}(R) \mid x_i = 0 \text{ for almost all } i\}.$$

Let R be an $\mathcal{O}$-algebra which is a Noetherian local ring with perfect residue field k and is complete with respect to the topology defined by the maximal ideal. In the following, we define an important subring $\widehat{W}_{\mathcal{O}}(R)$ of the ring of Witt vectors $W_{\mathcal{O}}(R)$. The construction follows from [22]. Assume first that R is Artinian. Note that there is a unique ring homomorphism $W_{\mathcal{O}}(k) \to R$, which for any element $a \in k$, maps the Teichmüller representative $[a]$ of a in $W_{\mathcal{O}}(k)$ to the Teichmüller representative of a in R. Let $\mathfrak{m} \subset R$ be the maximal ideal of R. Then we have the following exact sequence

$$0 \to W_{\mathcal{O}}(\mathfrak{m}) \to W_{\mathcal{O}}(R) \xrightarrow{\tau} W_{\mathcal{O}}(k) \to 0.$$

It admits a canonical section $\delta : W_{\mathcal{O}}(k) \xrightarrow{\Delta} W_{\mathcal{O}}(W_{\mathcal{O}}(k)) \to W_{\mathcal{O}}(R)$, which is a ring homomorphism commuting with ^{F}.

Since $\mathfrak{m}$ is nilpotent, we have a subalgebra of $W_{\mathcal{O}}(\mathfrak{m})$:

$$\widehat{W}_{\mathcal{O}}(\mathfrak{m}) = \{(x_0, x_1, \cdots) \in W_{\mathcal{O}}(\mathfrak{m}) \mid x_i = 0 \text{ for almost all } i\}.$$

$\widehat{W}_{\mathcal{O}}(\mathfrak{m})$ is stable under ^{F} and ^{V}. Moreover, $\widehat{W}_{\mathcal{O}}(\mathfrak{m})$ is an ideal of $W_{\mathcal{O}}(R)$. The proof of this fact is exactly the same as the argument in [22, Section 2]. Indeed, by definition, any element in $\widehat{W}_{\mathcal{O}}(\mathfrak{m})$ may be represented as a finite sum $\sum_{i=0}^{N} {}^{V^i}[x_i]$, it suffices to show that $[x]\eta \in \widehat{W}_{\mathcal{O}}(\mathfrak{m})$ for any $x \in \mathfrak{m}$ and for any $\eta \in W_{\mathcal{O}}(R)$. This follows from the formula

$$[x](\eta_0, \eta_1, \cdots, \eta_n, \cdots) = (x\eta_0, x^q\eta_1, \cdots, x^{q^n}\eta_n, \cdots).$$

Definition 1.7. If R is Artinian, we define the subring $\widehat{W}_{\mathcal{O}}(R) \subset W_{\mathcal{O}}(R)$ by

$$\widehat{W}_{\mathcal{O}}(R) = \{\xi \in W_{\mathcal{O}}(R) \mid \xi - \delta\tau(\xi) \in \widehat{W}_{\mathcal{O}}(\mathfrak{m})\}.$$

Again we have an exact sequence

$$0 \to \widehat{W}_{\mathcal{O}}(\mathfrak{m}) \to \widehat{W}_{\mathcal{O}}(R) \xrightarrow{\tau} W_{\mathcal{O}}(k) \to 0$$

with a canonical section δ of τ.

If R is Noetherian, define $\widehat{W}_{\mathcal{O}}(R) = \varprojlim \widehat{W}_{\mathcal{O}}(R/\mathfrak{m}_R^n)$.

We also define $\widehat{I}_{\mathcal{O},R} = {}^V(\widehat{W}_{\mathcal{O}}(R))$.

Lemma 1.8. *(Cf. [22, Lemma 2].) Assume that $p \geq 3$. Then the subring $\widehat{W}_{\mathcal{O}}(R)$ of $W_{\mathcal{O}}(R)$ is stable under ^{F} and ^{V}.*

Proof. Note that δ commutes with ^{F}, the stability under ^{F} is obvious. We only have to check the stability under ^{V}. It suffices to show that

$$\delta({}^V x) - {}^V(\delta x) \in \widehat{W}_{\mathcal{O}}(\mathfrak{m}) \text{ for } x \in W_{\mathcal{O}}(k).$$

By assumption, k is perfect. If we write $x = {}^F y$ and use that $\widehat{W}_{\mathcal{O}}(\mathfrak{m})$ is an ideal in $W_{\mathcal{O}}(R)$, it suffices to show the above claim for $x = 1$. In the ring $W_{\mathcal{O}}(W_{\mathcal{O}}(k))$, let $\mathcal{W} = (\mathrm{w}_0, \mathrm{w}_1, \cdots)$, we have

$$\mathcal{W}(\delta({}^V 1) - {}^V(\delta 1)) = [{}^V 1, 0, \cdots, 0, \cdots] = [\pi, 0, \cdots].$$

Assume that

$$[\pi, 0, \cdots] = \mathcal{W}(u_0, u_1, \cdots) \text{ where } u_i \in W_{\mathcal{O}}(k).$$

Then $\pi = u_0$, and $0 = \mathrm{w}_n(u_0, \cdots, u_n)$ for $n \geq 1$. By induction, we see that $\operatorname{ord}_\pi u_n = q^n - q^{n-1} - \cdots - 1$. The lemma follows. $\square$

1.2.2. $\mathcal{O}$-pd-structures

We recall the definition and basic properties of $\mathcal{O}$-pd-structure following [4, Section 7] and [6, Section B.5.1]. Let R be an $\mathcal{O}$-algebra. Let $\mathfrak{a} \subseteq R$ be an ideal. An *$\mathcal{O}$-pd-structure* on $\mathfrak{a}$ is a map $\gamma : \mathfrak{a} \to \mathfrak{a}$, such that

- $\pi \cdot \gamma(x) = x^q$,
- $\gamma(r \cdot x) = r^q \cdot \gamma(x)$ and
- $\gamma(x + y) = \gamma(x) + \gamma(y) + \sum_{0<i<q}(\binom{q}{i}/\pi) \cdot x^i \cdot y^{q-i}$

hold for all $r \in R$ and $x, y \in \mathfrak{a}$.

Remark 1.9. For any $S \in \mathrm{Alg}_R$, an $\mathcal{O}$-pd-structure on S is defined as above by considering S as the kernel of the morphism $R|S| := R \oplus S \to R$.

Let us denote by γ^n the n-fold iterate of γ. We call γ *nilpotent* if $\mathfrak{a}^{[n]} = 0$ for all $n \gg 0$, where $\mathfrak{a}^{[n]} \subset \mathfrak{a}$ is generated by all products $\prod \gamma^{a_i}(x_i)$ with $x_i \in \mathfrak{a}$ and $\sum q^{a_i} \geq n$. Define $\alpha_0 = \mathrm{id}$ and for each $n \geq 1$, define

$$\alpha_n = \pi^{q^{n-1}+q^{n-2}+\ldots+q+1-n} \cdot \gamma^n : \mathfrak{a} \to \mathfrak{a}.$$

Define the *n-th divided Witt polynomial* by

$$\begin{aligned} \mathrm{w}'_n &: W_{\mathcal{O}}(\mathfrak{a}) \to \mathfrak{a} \\ (x_0, x_1, \ldots, x_n, \ldots) &\mapsto \alpha_n(x_0) + \alpha_{n-1}(x_1) + \ldots + \alpha_1(x_{n-1}) + x_n. \end{aligned}$$

The map w'_n is w_n-linear, i.e., $\mathrm{w}'_n(rx) = \mathrm{w}_n(r)\mathrm{w}'_n(x)$ for all $n \in \mathbb{N}$, $x \in W_{\mathcal{O}}(\mathfrak{a})$ and $r \in W_{\mathcal{O}}(R)$.

The main application of this structure is as follows (cf. [6, Lemma B.5.8]). Define on $\mathfrak{a}^{\mathbb{N}}$ a $W_{\mathcal{O}}(R)$-module structure by setting

$$\xi[a_0, a_1, \ldots] = [\mathrm{w}_0(\xi)a_0, \mathrm{w}_1(\xi)a_1, \ldots]$$

for all $\xi \in W_{\mathcal{O}}(R)$ and $[a_0, a_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$. Then we have an isomorphism of $W_{\mathcal{O}}(R)$-modules

$$\begin{aligned} \log &: W_{\mathcal{O}}(\mathfrak{a}) \to \mathfrak{a}^{\mathbb{N}} \\ \underline{a} = (a_0, a_1, \ldots) &\mapsto [\mathrm{w}'_0(\underline{a}), \mathrm{w}'_1(\underline{a}), \ldots]. \end{aligned}$$

Moreover, if γ is nilpotent, the above isomorphism induces an isomorphism

$$\log : \widehat{W}_{\mathcal{O}}(\mathfrak{a}) \to \mathfrak{a}^{\oplus \mathbb{N}}.$$

We may view $\mathfrak{a}$ as an ideal of $W_{\mathcal{O}}(\mathfrak{a})$ via the map $a \mapsto \tilde{a} = \log^{-1}([a, 0, \ldots])$. Since ^{F} acts on the right hand side by

$${}^F[a_0, a_1, \ldots] = [\pi a_1, \pi a_2, \ldots, \pi a_i, \ldots]$$

for all $[a_0, a_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$, we obtain that, for the ideal $\mathfrak{a} \subset W_{\mathcal{O}}(\mathfrak{a})$, ${}^F\mathfrak{a} = 0$.

Definition 1.10. Let $S \to R$ be a surjection of $\mathcal{O}$-algebras, such that the kernel $\mathfrak{a}$ is equipped with an $\mathcal{O}$-pd-structure. We call $S \to R$ an *$\mathcal{O}$-pd-thickening* if the $\mathcal{O}$-pd-structure over $\mathfrak{a}$ is nilpotent. We call $S \to R$ a *topological $\mathcal{O}$-pd-thickening*, if there is a sequence of ideals $\mathfrak{a}_n \subset \mathfrak{a}$, such that S is complete and separated in the linear topology defined by the $\mathfrak{a}_n$, and each $\mathfrak{a}/\mathfrak{a}_n$ is equipped with a nilpotent $\mathcal{O}$-pd-structure.

Proposition 1.11. *Let S be an $\mathcal{O}$-algebra and $\mathfrak{a} \subset S$ an ideal equipped with an $\mathcal{O}$-pd-structure γ. Then for any nilpotent S-algebra $\mathcal{N}$ the algebra $\mathfrak{a} \otimes_S \mathcal{N}$ inherits a nilpotent $\mathcal{O}$-pd-structure $\widetilde{\gamma}$ from $\mathfrak{a}$ which is uniquely determined by $\widetilde{\gamma}(a \otimes n) = \gamma(a) \otimes n^q$ for $a \in \mathfrak{a}$ and $n \in \mathcal{N}$.*

Proof. We only need to refer to the proof of [14, Chapter III, Lemma (1.8)], where one defines the map $\varphi : S^{(\mathfrak{a} \times \mathcal{N})} \to \mathfrak{a} \otimes \mathcal{N}$ by the formula

$$\varphi(\sum_{i=1}^{l} s_i(a_i, n_i)) = \sum_{i=1}^{l} \gamma(a_i) \otimes (s_i n_i)^q + \sum(\binom{q}{i_1, \ldots, i_l}/\pi) \prod_{j=1}^{l} (s_j a_j \otimes n_j)^{i_j}.$$

Here the last sum runs through all l-tuples $(i_1, \ldots, i_l)$ with $i_j > 0$ and $\sum_{j=1}^{l} i_j = q$. A similar argument, compared to the one there, shows that $\widetilde{\gamma}$ determines an $\mathcal{O}$-pd-structure on $\mathfrak{a} \otimes_S \mathcal{N}$. It is nilpotent since $\mathcal{N}$ is nilpotent.

Note that if $\mathcal{N}$ is flat over S, we obtain an $\mathcal{O}$-pd-structure on $\mathfrak{a}\mathcal{N}$ via the inverse of $\mathfrak{a} \otimes_S \mathcal{N} \to \mathfrak{a}\mathcal{N}$. $\square$

1.2.3. π-divisible formal $\mathcal{O}$-modules

Let R be a commutative unitary ring and Nil_R denote the category of nilpotent R-algebras. We embed the category of R-modules Mod_R into Nil_R by setting $M^2 = 0$ for any $M \in \mathrm{Mod}_R$. In particular, this is the case for the R-module R.

If H is a functor on Nil_R, we denote by t_H its restriction to Mod_R.

Definition 1.12. (Cf. [20, Chapter 2], [23, Definition 80].) A *(finite dimensional) formal group over R* is a functor $F : \mathrm{Nil}_R \to \mathrm{Ab}$ such that

(1) $F(0) = 0$.
(2) F is exact, i.e., if

$$0 \to \mathcal{N}_1 \to \mathcal{N}_2 \to \mathcal{N}_3 \to 0$$

is a sequence in Nil_R, which is exact as a sequence of R-modules, then

$$0 \to F(\mathcal{N}_1) \to F(\mathcal{N}_2) \to F(\mathcal{N}_3) \to 0$$

is an exact sequence of abelian groups.
(3) The functor t_F commutes with infinite direct sums.
(4) $t_F(R)$ is a finitely generated projective R-module. (By [23, 3.1 The functor BT] $t_F(M)$ is in a canonical way an R-module for each $M \in \mathrm{Mod}_R$.)

The module $t_F(R)$ is called the *tangent space* of F. The rank of $t_F(R)$ is called the *dimension* of F. The *morphisms* between two formal groups are the natural transformations between the functors.

Definition 1.13. Let S be a unitary ring and R a unitary S-algebra. A *formal S-module over R* is a formal group over R with an action of S, which induces the natural action on the tangent space, i.e., it coincides with the S-module structure obtained by the R-module structure of the tangent space and restriction of scalars. The *morphisms* between two formal S-modules are the natural transformations between the functors respecting the attached S-actions.

Definition 1.14. (Cf. [20, 5.4 Definition].) Let $R \in \mathrm{Alg}_{\mathcal{O}}$. A morphism $\varphi : G \to H$ of formal $\mathcal{O}$-modules over R of equal dimension is called an *isogeny* if $\operatorname{Ker}\varphi$ is representable (i.e., $\operatorname{Ker}\varphi \simeq \operatorname{Spf} A$ with $A \in \mathrm{Nil}_R$, where $\operatorname{Spf} A : \mathrm{Nil}_R \to \mathrm{Sets}$ is given by $\operatorname{Spf} A(\mathcal{N}) = \operatorname{Hom}_{\mathrm{Alg}_R}(A, R \oplus \mathcal{N})$ for $\mathcal{N} \in \mathrm{Nil}_R$.)

Definition 1.15. (Cf. [20, 5.28 Definition], [6, Definition B.2.1].) A formal $\mathcal{O}$-module G over an $\mathcal{O}$-algebra R is called *π-divisible*, if the multiplication map $\pi : G \to G$ is an isogeny. The category of π-divisible formal $\mathcal{O}$-modules over R is a full subcategory of the category of formal $\mathcal{O}$-modules over R.

Lemma 1.16. *Let $(\mathcal{O}', \pi')$ be a finite extension of $(\mathcal{O}, \pi)$. Then a formal $\mathcal{O}'$-module G is π'-divisible, if and only if $\pi : G \to G$ is an isogeny.*

Proof. This follows easily by [20, 5.10 Satz], which says that the composition of two morphisms is an isogeny if and only if both morphisms are. See [6, Remarque B.2.2] for more discussion. □

1.2.4. Cartier modules and Drinfeld's result

Let $R \in \mathrm{Alg}_{\mathcal{O}}$. Let $\mathbb{E}_{\mathcal{O},R}$ be the Cartier ring defined in [3].

Definition 1.17. (Cf. [3].) A Cartier module M over R and $\mathcal{O}$ (i.e., an $\mathbb{E}_{\mathcal{O},R}$-module) is *reduced*, if the action of V is injective, $M = \varprojlim M/V^k M$, and M/VM is a finite projective R-module. The quotient M/VM is called the *tangent space* of M. Note that in [3], M/VM is required to be free.

Theorem 1.18. *(Cf. [3].) The category of formal $\mathcal{O}$-modules over R is equivalent to the category of reduced $\mathbb{E}_{\mathcal{O},R}$-modules.*

Although Definition 1.17 is slightly different from the one in [3], the proof is an easy combination of the proof in [3] (the *induction* step) and the proof of [20, 4.23 Satz] (the *base* step $\mathcal{O} = \mathbb{Z}_p$). The readers may also find the detail proof in [1, Section 2.4] or [18].

One may give an explicit description of the equivalence in Theorem 1.18. Note that we may consider $\widehat{W}_{\mathcal{O}}(\mathcal{N})$ for each $\mathcal{N} \in \mathrm{Nil}_R$ as an $\mathbb{E}_{\mathcal{O},R}$-module. For $e \in \mathbb{E}_{\mathcal{O},R}$, it can be written in a unique way as

$$e = \sum_{n,m \geq 0} V^n [a_{m,n}] F^m, \tag{1.2}$$

where $a_{m,n} \in R$ and for fixed n the coefficients $a_{m,n}$ are zero for m large enough. Then the module structure is written as a right multiplication and is defined by

$$we = \sum_{n,m \geq 0} {}^{V^m}([a_{n,m}]({}^{F^n}w)),$$

where w is an element of $\widehat{W}_{\mathcal{O}}(\mathcal{N})$. This generalizes [23, Equation (166)]. Clearly, a morphism between $\mathcal{N} \to \mathcal{N}'$ in Nil_R induces a morphism of $\mathbb{E}_{\mathcal{O},R}$-modules $\widehat{W}_{\mathcal{O}}(\mathcal{N}) \to \widehat{W}_{\mathcal{O}}(\mathcal{N}')$.

Lemma 1.19. *(Cf. [21, (2.10) Lemma].) Let R be an $\mathcal{O}$-algebra, $\mathcal{N}$ a nilpotent R-algebra with a nilpotent $\mathcal{O}$-pd-structure and M a reduced $\mathbb{E}_{\mathcal{O},R}$-module. Then we have an isomorphism of $\mathcal{O}$-modules*

$$\widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},R}} M \simeq \mathcal{N} \otimes_R M/VM,$$

given by $\underline{n} \otimes m \mapsto \sum_i \mathrm{w}'_i(\underline{n}) \otimes \overline{F^i m}$ for all $\underline{n} \in \widehat{W}_{\mathcal{O}}(\mathcal{N})$ and $m \in M$, where the w'_i are divided Witt polynomials. The inverse map is given by $n \otimes \overline{m} \mapsto \log^{-1}[n, 0, \ldots] \otimes m$ for all $n \in \mathcal{N}$ and $\overline{m} \in M/VM$, where m is any lift of $\overline{m}$.

Proof. The two arrows are well-defined. One checks that they are inverse of each other via direct computation. □

Lemma 1.20. *(Cf. [20, 4.41 Satz].) Let M be a reduced $\mathbb{E}_{\mathcal{O},R}$-module and $\mathcal{N}$ a nilpotent R-algebra. Then $\mathrm{Tor}_i^{\mathbb{E}_{\mathcal{O},R}}(\widehat{W}_{\mathcal{O}}(\mathcal{N}), M) = 0$ for each $i \geq 1$.*

Proposition 1.21. *For each reduced $\mathbb{E}_{\mathcal{O},R}$-module M, the functor $\widehat{W}_{\mathcal{O}}(-) \otimes_{\mathbb{E}_{\mathcal{O},R}} M$ on Nil_R is a formal $\mathcal{O}$-module. Furthermore, the equivalence functor from the category of reduced $\mathbb{E}_{\mathcal{O},R}$-modules to the category of formal $\mathcal{O}$-modules in Theorem 1.18 is given by this functor.*

Proof. To show that $\widehat{W}_{\mathcal{O}}(-) \otimes_{\mathbb{E}_{\mathcal{O},R}} M$ is a formal $\mathcal{O}$-module, it suffices to show that the tangent space is a finite projective R-module and that it preserves exact sequences. But this follows from Lemmas 1.19 and 1.20.

The second assertion is already confirmed in the $\mathbb{Z}_p$-case (cf. [20, 4.23 Satz]). Hence, as in Drinfeld's proof, it suffices to show that if the assertion is true for some $\mathcal{O}$, for any finite extension $\mathcal{O} \to \mathcal{O}'$ the assertion is true for $\mathcal{O}'$. This is easy to check by writing down Drinfeld's construction explicitly. □

1.2.5. The exponential map

We reformulate Lemma 1.19. Let S be an $\mathcal{O}$-algebra. By Proposition 1.21, we obtain for each formal $\mathcal{O}$-module G over S and each nilpotent S-algebra $\mathcal{N}$ equipped with a nilpotent $\mathcal{O}$-pd-structure an isomorphism

$$\log_G(\mathcal{N}) : G(\mathcal{N}) \to \operatorname{Lie} G \otimes_S \mathcal{N}.$$

Definition 1.22. Let G be a formal $\mathcal{O}$-module over an $\mathcal{O}$-algebra S and $\mathfrak{a} \subseteq S$ be an ideal equipped with an $\mathcal{O}$-pd-structure. We define the *exponential map*

$$\exp_G : \underline{\mathfrak{a} \otimes \operatorname{Lie} G} \to G$$

by

$$\underline{\mathfrak{a} \otimes \operatorname{Lie} G}(\mathcal{N}) = \mathfrak{a} \otimes_S \mathcal{N} \otimes_S \operatorname{Lie} G \overset{\log_G^{-1}(\mathfrak{a} \otimes_S \mathcal{N})}{\longrightarrow} G(\mathfrak{a} \otimes_S \mathcal{N}) \to G(\mathcal{N})$$

for each $\mathcal{N} \in \operatorname{Nil}_S$, where $\log_G$ is defined as above (which makes sense by Proposition 1.11) and the last map is induced by the multiplication morphism $\mathfrak{a} \otimes_S \mathcal{N} \to \mathcal{N}$.

Following [23, Section 3.2. The Universal Extension], one may also define the exponential maps via Cartier modules. For S an $\mathcal{O}$-algebra and L an S-module, define the group $C(L) = \prod_{i \geq 0} V^i L$. We may turn $C(L)$ into an $\mathbb{E}_{\mathcal{O},S}$-module by the equations

$$\begin{aligned} \xi(\sum_{i \geq 0} V^i l_i) &= \sum_{i \geq 0} V^i \mathrm{w}_n(\xi) l_i, \\ V(\sum_{i \geq 0} V^i l_i) &= \sum_{i \geq 0} V^{i+1} l_i, \\ F(\sum_{i \geq 0} V^i l_i) &= \sum_{i \geq 1} V^{i-1} \pi l_i, \end{aligned}$$

for all $\xi \in W_{\mathcal{O}}(S)$ and $l_i \in L$. We may interpret $C(L)$ as the Cartier module of the additive group of L. If $\widehat{L}^+$ denotes the functor from Nil_S to $\operatorname{Mod}_{\mathcal{O}}$ defined by

$$\widehat{L}^+(\mathcal{N}) = (\mathcal{N} \otimes_S L)^+$$

for $\mathcal{N} \in \operatorname{Nil}_S$, then there is a functor isomorphism

$$\mathcal{N} \otimes_S L \simeq \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{\mathbb{E}_{\mathcal{O},S}} C(L) \tag{1.3}$$

given by $n \otimes l \mapsto [n] \otimes V^0 l$ for $n \in \mathcal{N}$ and $l \in L$. The inverse map is given by sending $w \otimes \sum_{i \geq 0} V^i l_i$ to $\sum_{i \geq 0} \mathrm{w}_i(w) \otimes l_i$ for $w \in \widehat{W}_{\mathcal{O}}(\mathcal{N})$ and $l_i \in L$ for all $i \geq 0$ (cf. [21, (2.1) Lemma]). By Proposition 1.21, the following is now clear. (See also [6, Section B.5.3].)

Proposition 1.23. *(Cf. [21, (2.3) Satz, (2.11) Satz].) Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$, M' a reduced $\mathbb{E}_{\mathcal{O},S}$-module and $M = \mathbb{E}_{\mathcal{O},R} \otimes_{\mathbb{E}_{\mathcal{O},S}} M'$. Then there is an exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules*

$$0 \to C(\mathfrak{a} \otimes_S M'/VM') \overset{\exp}{\to} M' \to M \to 0.$$

Here the map $\exp$ *is given by sending $V^i(a \otimes m)$ to $V^i \log^{-1}[a, 0, \ldots]m$. It induces the map*

$$\exp_{G'} : \underline{\mathfrak{a} \otimes_S \operatorname{Lie} G'} \to G',$$

where G' is the formal $\mathcal{O}$-module over S attached to M'.

2. f-$\mathcal{O}$-Displays

In this section we study f-$\mathcal{O}$-displays. Since f-$\mathcal{O}$-displays are generalizations of displays, the materials in the first three subsections are similar to those in [23]. We give details here for completeness.

2.1. Definitions

Definition 2.1. Let R be an $\mathcal{O}$-algebra. An *f-$\mathcal{O}$-display* $\mathcal{P}$ over R is a quadruple (P, Q, F, F_1), where P is a finitely generated projective $W_{\mathcal{O}}(R)$-module, Q is a submodule of P, $F : P \to P$ and $F_1 : Q \to P$ are ${}^{F^f}$-linear maps, such that the following properties are satisfied:

(1) $I_{\mathcal{O},R}P \subset Q$ and there is a decomposition of P as $W_{\mathcal{O}}(R)$-modules $P = L \oplus T$, such that $Q = L \oplus I_{\mathcal{O},R}T$. (We call such a decomposition a *normal decomposition*.)

(2) F_1 is an ${}^{F^f}$-linear epimorphism, i.e., its linearization

$$\begin{aligned} F_1^\sharp : W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} Q &\to P \\ w \otimes q &\mapsto wF_1 q, \end{aligned}$$

where $w \in W_{\mathcal{O}}(R)$ and $q \in Q$, is surjective.

(3) For $x \in P$ and $w \in W_{\mathcal{O}}(R)$, we have

$$F_1({}^V wx) = {}^{F^{f-1}} wFx.$$

The finite projective R-module P/Q is the *tangent space* of $\mathcal{P}$. If $f = 1$, we call $\mathcal{P}$ an *$\mathcal{O}$-display*.

A *morphism* $\alpha : (P, Q, F, F_1) \to (P', Q', F', F_1')$ between two f-$\mathcal{O}$-displays is a morphism of $W_{\mathcal{O}}(R)$-modules $\alpha : P \to P'$, such that $\alpha(Q) \subset Q'$ and α commutes with F and F_1.

Together with these morphisms, the f-$\mathcal{O}$-displays over R form a category, we call it $(f-\operatorname{disp}_{\mathcal{O}}/R)$ or only $(\operatorname{disp}_{\mathcal{O}}/R)$ if $f=1$.

This definition is similar to [23, Definition 1]. Note that

$$F_1({}^V 1x) = Fx$$

for all $x \in P$. Hence F is uniquely determined by F_1. Applying this equation to $y \in Q$, we obtain $Fy = \pi \cdot F_1 y$.

We introduce an operator $V^\sharp$. The following lemma is an easy generalization of the corresponding result in [23].

Lemma 2.2. *(Cf. [23, Lemma 10].) Let R be an $\mathcal{O}$-algebra and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. There exists a unique $W_{\mathcal{O}}(R)$-linear map*

$$V^\sharp : P \to W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P,$$

which satisfies the following equations for all $w \in W_{\mathcal{O}}(R)$, $x \in P$ and $y \in Q$:

$$V^\sharp(wFx) = \pi \cdot w \otimes x,$$
$$V^\sharp(wF_1 y) = w \otimes y.$$

Furthermore, $F^\sharp V^\sharp = \pi \operatorname{id}_P$ and $V^\sharp F^\sharp = \pi \operatorname{id}_{W_{\mathcal{O}}(R)\otimes_{F^f, W_{\mathcal{O}}(R)} P}$.

By $V^{n\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{F^{fn}, W_{\mathcal{O}}(R)} P$ we mean the composite map ${}^{F^{f(n-1)}}V^\sharp \circ \ldots \circ {}^{F^f}V^\sharp \circ V^\sharp$, where ${}^{F^{fi}}V^\sharp$ is the $W_{\mathcal{O}}(R)$-linear map

$$\operatorname{id} \otimes_{F^{fi}, W_{\mathcal{O}}(R)} V^\sharp : W_{\mathcal{O}}(R) \otimes_{F^{fi}, W_{\mathcal{O}}(R)} P \to W_{\mathcal{O}}(R) \otimes_{F^{f(i+1)}, W_{\mathcal{O}}(R)} P.$$

Definition 2.3. Let R be an object in $\operatorname{Nil}_{\mathcal{O}}$ and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. We call $\mathcal{P}$ *nilpotent*, if there is a number N such that the composite map

$$\operatorname{pr} \circ V^{N\sharp} : P \to W_{\mathcal{O}}(R) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P \to W_{\mathcal{O}}(R)/(I_{\mathcal{O},R} + \pi W_{\mathcal{O}}(R)) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P$$

is the zero map.

Denote by $(f-\operatorname{ndisp}_{\mathcal{O}}/R)$ the subcategory of $(f-\operatorname{disp}_{\mathcal{O}}/R)$ consisting of nilpotent objects.

Definition 2.4. Let $\mathcal{O}'$ be a finite extension of $\mathcal{O}$, R an $\mathcal{O}'$-algebra, and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. We call an $\mathcal{O}'$-action of $\mathcal{P}$, i.e., an $\mathcal{O}$-algebra morphism $\iota : \mathcal{O}' \to \operatorname{End}\mathcal{P}$, *strict*, if the induced action $\bar{\iota} : \mathcal{O}' \to \operatorname{End}(P/Q)$ coincides with the $\mathcal{O}'$-module structure given by the R-module structure of P/Q and restriction of scalars.

We denote by $(\mathrm{disp}^{(f)}_{\mathcal{O},\mathcal{O}'}/R)$ (resp. $(\mathrm{ndisp}^{(f)}_{\mathcal{O},\mathcal{O}'}/R)$) the category of f-$\mathcal{O}$-displays (resp. nilpotent f-$\mathcal{O}$-displays) over R equipped with a strict $\mathcal{O}'$-action.

In fact, for this situation, we consider only the case $f = 1$. When $f = 1$, we omit the script f in the notation.

2.2. Base change for f-$\mathcal{O}$-displays

Let $R \to S$ be a morphism of $\mathcal{O}$-algebras.

Definition 2.5. (Cf. [23, Definition 20].) Let $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over R. The *f-$\mathcal{O}$-display obtained by base change* with respect to $R \to S$ is the quadruple $\mathcal{P}_S = (P_S, Q_S, F_S, F_{1,S})$, where

- $P_S := W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P$,
- $Q_S := \mathrm{Ker}(\mathrm{w}_0 \otimes \mathrm{pr} : W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P \to S \otimes_R P/Q)$,
- $F_S := {}^{F^f} \otimes F$, and
- $F_{1,S} : Q_S \to P_S$ is the unique ${}^{F^f}$-linear morphism which satisfies

$$F_{1,S}(w \otimes y) = {}^{F^f} w \otimes F_1 y,$$
$$F_{1,S}({}^{V} w \otimes x) = {}^{F^{f-1}} w \otimes Fx$$

for all $w \in W_{\mathcal{O}}(S)$, $x \in P$ and $y \in Q$.

It is easy to check that $\mathcal{P}_S$ is an f-$\mathcal{O}$-display over S. In particular, if we choose a normal decomposition $P = L \oplus T$, then

$$Q_S \simeq W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} L \oplus I_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(R)} T.$$

Remark 2.6. We remark a very important case of base change, which will be needed for the study of the functor $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, -)$ (cf. [23, Example 23]). Let R be an $\mathcal{O}$-algebra, such that $\pi R = 0$. Let Frob_q denote the Frobenius endomorphism defined by $\mathrm{Frob}_q(r) = r^q$ for all $r \in R$ and $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over R. The Frobenius ^{F} on $W_{\mathcal{O}}(R)$ is given by $W_{\mathcal{O}}(\mathrm{Frob}_q)$. If we set

$$P^{(q)} = W_{\mathcal{O}}(R) \otimes_{{}^{F}, W_{\mathcal{O}}(R)} P,$$
$$Q^{(q)} = I_{\mathcal{O},R} \otimes_{{}^{F}, W_{\mathcal{O}}(R)} P + \mathrm{Im}(W_{\mathcal{O}}(R) \otimes_{{}^{F}, W_{\mathcal{O}}(R)} Q)$$

and define the operators $F^{(q)}$ and $F_1^{(q)}$ in a unique way by

$$F^{(q)}(w \otimes x) = {}^{F^f} w \otimes Fx,$$
$$F_1^{(q)}({}^{V} w \otimes x) = {}^{F^{f-1}} w \otimes Fx,$$
$$F_1^{(q)}(w \otimes y) = {}^{F^f} w \otimes F_1 y$$

for all $w \in W_{\mathcal{O}}(R)$, $x \in P$ and $y \in Q$, we see that the f-$\mathcal{O}$-display obtained by base change with respect to Frob_q is $\mathcal{P}^{(q)} = (P^{(q)}, Q^{(q)}, F^{(q)}, F_1^{(q)})$. It is essential to demand $\pi R = 0$ here, otherwise $Q^{(q)}/I_{\mathcal{O},R}P^{(q)}$ would not necessarily be a direct summand of $P^{(q)}/I_{\mathcal{O},R}P^{(q)}$.

Let us denote the k-fold iterate of this construction by $\mathcal{P}^{(q^k)}$ and consider the map $V^\sharp : P \to W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P$ of Lemma 2.2 and $F^\sharp : W_{\mathcal{O}}(R) \otimes_{F^f, W_{\mathcal{O}}(R)} P \to P$. $V^\sharp$ maps P into $Q^{(q^f)}$ and $F^\sharp$ maps $Q^{(q^f)}$ into $I_{\mathcal{O},R}P$. Both maps commute with the pairs $(F, F^{(q^f)})$ and $(F_1, F_1^{(q^f)})$ respectively, so $V^\sharp$ induces the so called Frobenius morphism of $\mathcal{P}$, which is a morphism of f-$\mathcal{O}$-displays

$$\mathrm{Fr}_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}^{(q^f)}, \tag{2.1}$$

and $F^\sharp$ induces the so called Verschiebung

$$\mathrm{Ver}_{\mathcal{P}} : \mathcal{P}^{(q^f)} \to \mathcal{P}.$$

By Lemma 2.2, we obtain two analogous relations

$$\mathrm{Fr}_{\mathcal{P}} \mathrm{Ver}_{\mathcal{P}} = \pi \cdot \mathrm{id}_{\mathcal{P}^{(q^f)}} \text{ and } \mathrm{Ver}_{\mathcal{P}} \mathrm{Fr}_{\mathcal{P}} = \pi \cdot \mathrm{id}_{\mathcal{P}}.$$

Let R be a topological $\mathcal{O}$-algebra, where the linear topology is given by the ideals $R = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \ldots \supset \mathfrak{a}_n \ldots$, such that $\mathfrak{a}_i \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$. Suppose further that π is nilpotent in $R/\mathfrak{a}_1$ (and hence in all $R/\mathfrak{a}_i$) and that R is complete and separated with respect to this filtration.

Definition 2.7. With R as above, an f-$\mathcal{O}$-display over R is called *nilpotent*, if the f-$\mathcal{O}$-display obtained by base change to $R/\mathfrak{a}_1$ is nilpotent in the sense of Definition 2.3.

Let $\mathcal{P}$ be a nilpotent f-$\mathcal{O}$-display over R. We denote by $\mathcal{P}_i$ the f-$\mathcal{O}$-display over $R/\mathfrak{a}_i$ obtained by base change. Then $\mathcal{P}_i$ is a nilpotent f-$\mathcal{O}$-display in the sense of Definition 2.3. There are obvious transition isomorphisms

$$\phi_i : (\mathcal{P}_{i+1})_{R/\mathfrak{a}_i} \to \mathcal{P}_i.$$

Conversely, assume we are given for each index i a nilpotent f-$\mathcal{O}$-display $\mathcal{P}_i$ over $\mathcal{O}$-algebra $R/\mathfrak{a}_i$ and transition isomorphisms ϕ_i as above. It is easily seen that the system $(\mathcal{P}_i, \phi_i)$ is obtained from a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over R. The category of systems of nilpotent f-$\mathcal{O}$-displays $(\mathcal{P}_i, \phi_i)$ and the category of nilpotent f-$\mathcal{O}$-displays over R are equivalent by the above association. This equivalence fits well to [14, Chapter II, Lemma (4.16)].

2.3. Descent data for f-$\mathcal{O}$-displays

We introduce descent theory for f-$\mathcal{O}$-displays.

Lemma 2.8. *(Cf. [23, 1.3. Descent].) Let $R \to S$ be a faithfully flat $\mathcal{O}$-algebra morphism. Then we have the exact sequence*

$$R \to S \underset{q_2}{\overset{q_1}{\rightrightarrows}} S \otimes_R S \overset{q_{12}}{\underset{q_{13}}{\overset{\xrightarrow{q_{23}}}{\rightrightarrows}}} S \otimes_R S \otimes_R S,$$

where q_i is the map, which sends an element of S to the i-th factor of $S \otimes_R S$ and q_{ij} is given by sending the first component of $S \otimes_R S$ to the i-th component of $S \otimes_R S \otimes_R S$ and the second one to the j-th component of it.

Definition 2.9. With $\mathcal{O}$, $R \to S$, q_i and q_{ij} as above. Let $\mathcal{P}$ be an f-$\mathcal{O}$-display over S. Denote the f-$\mathcal{O}$-display over $S \otimes_R S$ obtained by base change via q_i by $q_i^\star \mathcal{P}$ and similarly for f-$\mathcal{O}$-displays over $S \otimes_R S \otimes_R S$ and q_{ij}. A descend datum for $\mathcal{P}$ relative to $R \to S$ is an isomorphism of f-$\mathcal{O}$-displays $\alpha : q_1^\star \mathcal{P} \to q_2^\star \mathcal{P}$, such that the cocycle condition holds, i.e., the diagram

$$\begin{tikzcd}
q_{12}^\star q_1^\star \mathcal{P} \arrow[r, "q_{12}^\star \alpha"] \arrow[d, equal] & q_{12}^\star q_2^\star \mathcal{P} \arrow[d, equal] \\
q_{13}^\star q_1^\star \mathcal{P} \arrow[d, "q_{13}^\star \alpha"] & q_{23}^\star q_1^\star \mathcal{P} \arrow[d, "q_{23}^\star \alpha"] \\
q_{13}^\star q_2^\star \mathcal{P} \arrow[r, equal] & q_{23}^\star q_2^\star \mathcal{P}
\end{tikzcd}$$

is commutative.

It is obvious that we obtain for any f-$\mathcal{O}$-display $\mathcal{P}$ over R a canonical descent datum $\alpha_{\mathcal{P}}$ for the base change $\mathcal{P}_S$ over S relative to $R \to S$.

Theorem 2.10. *(Cf. [23, Theorem 37].) With the terminology as in Definition 2.9, the functor $\mathcal{P} \mapsto (\mathcal{P}, \alpha_{\mathcal{P}})$ from the category of f-$\mathcal{O}$-displays over R to the category of f-$\mathcal{O}$-displays over S equipped with a descent datum relative to $R \to S$ is an equivalence of categories. We also obtain an equivalence, when we restrict to nilpotent f-$\mathcal{O}$-display structures.*

The following result is important in Section 4.

Proposition 2.11. *Let $R \to S$ be a faithfully flat morphism of $\mathcal{O}'$-algebras, and f, f' two natural numbers ≥ 1. Let G_A be a functor between the category of (nilpotent)*

f-$\mathcal{O}$-displays over A and the category of (nilpotent) f'-$\mathcal{O}'$-displays over A for $A = R, S, S \otimes_R S, S \otimes_R S \otimes_R S$. Assume that these functors are compatible with the base change functors induced by q_i, q_{ij} (with the obvious notation) and $R \to S$, that $G_{S\otimes_R S}$ is fully faithful and $G_{S\otimes_R S\otimes_R S}$ is faithful. Let $\mathcal{P}'$ be a (nilpotent) f'-$\mathcal{O}'$-display over R, such that the base change ${\mathcal{P}'}_S$ lies in the image of G_S. Then $\mathcal{P}'$ lies in the image of G_R. The same assertion is true, when the domain of G_A is the category of (nilpotent) f-$\mathcal{O}$-displays over A equipped with a strict $\mathcal{O}'$-action for each A as above.

Proof. Let $\mathcal{P}$ be a (nilpotent) f-$\mathcal{O}$-display over S, such that $G_S(\mathcal{P}) = {\mathcal{P}'}_S$. It suffices to construct for $\mathcal{P}$ a descent datum relative to $R \to S$, so we would obtain by Theorem 2.10 a (nilpotent) f-$\mathcal{O}$-display over R, which has the image $\mathcal{P}'$. We have the obvious descent datum for ${\mathcal{P}'}_S$. One may lift the isomorphism $\alpha' : q_1^\star {\mathcal{P}'}_S \cong q_2^\star {\mathcal{P}'}_S$ to $\alpha : q_1^\star \mathcal{P} \cong q_2^\star \mathcal{P}$, since $G_{S\otimes_R S}$ is fully faithful. Now we may establish the cocycle diagram for α and the commutativity of the diagram because of the faithfulness of $G_{S\otimes_R S\otimes_R S}$ and the compatibility of the G's with the base change functors.

The last assertion follows from the same argument as above by attaching a strict $\mathcal{O}'$-action to the objects of the categories in Theorem 2.10. □

2.4. *The formal $\mathcal{O}$-module* $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, -)$

For a given f-$\mathcal{O}$-display $\mathcal{P} = (P, Q, F, F_1)$ we consider the following $W_{\mathcal{O}}(R)$-modules, which can be considered as $\mathcal{O}$-modules by restriction of scalars via $\mathcal{O} \to W_{\mathcal{O}}(R)$:

$$\widehat{P}_{\mathcal{N}} = \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \tag{2.2}$$

$$\widehat{Q}_{\mathcal{N}} = \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} L \oplus \widehat{I}_{\mathcal{O},\mathcal{N}} \otimes_{W_{\mathcal{O}}(R)} T. \tag{2.3}$$

Here $P = L \oplus T$ is a normal decomposition and $\mathcal{N} \in \mathrm{Nil}_R$. Let S be the unitary R-algebra $R|\mathcal{N}| = R \oplus \mathcal{N}$ with an addition in the obvious way and a multiplication given by

$$(r_1, n_1)(r_2, n_2) = (r_1 r_2, r_1 n_2 + r_2 n_1 + n_1 n_2) \tag{2.4}$$

for all $n_i \in \mathcal{N}$ and $r_i \in R$. If we denote by $\mathcal{P}_S = (P_S, Q_S, F_S, F_{1,S})$ the f-$\mathcal{O}$-display over S obtained from $\mathcal{P}$ via base change relative to $R \to S$, we can consider $\widehat{P}_{\mathcal{N}}$ as a submodule of P_S and obtain $\widehat{Q}_{\mathcal{N}} = \widehat{P}_{\mathcal{N}} \cap Q_S$. By restricting $F_S : P_S \to P_S$ and $F_{1,S} : Q_S \to P_S$, we obtain operators

$$F : \widehat{P}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}},$$
$$F_1 : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}.$$

Now we are able to associate to an f-$\mathcal{O}$-display $\mathcal{P}$ a finite dimensional formal $\mathcal{O}$-module $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, -)$. In the case that $f = 1$, we will just refer to $\mathrm{BT}_{\mathcal{O}}(\mathcal{P}, -)$. The following theorem is a modified version of [23, Theorem 81].

Theorem 2.12. *Let $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over R. Then the functor from* Nil_R *(the category of nilpotent R-algebras) to the category of $\mathcal{O}$-modules, which associates to any $\mathcal{N} \in \mathrm{Nil}_R$ the cokernel of the morphism of abelian groups*

$$F_1 - \mathrm{id} : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}} \tag{2.5}$$

is a finite dimensional formal $\mathcal{O}$-module, when considered as a functor to abelian groups equipped with a natural $\mathcal{O}$-action. Denote this functor by $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, -)$. Then we have an exact sequence of $\mathcal{O}$-modules

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{F_1 - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \longrightarrow 0. \tag{2.6}$$

First note the following lemma.

Lemma 2.13. *(Cf. [23, Lemma 38].) Let $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over R and $\mathfrak{a} \subseteq R$ an ideal equipped with a nilpotent $\mathcal{O}$-pd-structure. Then there is a unique extension of F_1 to*

$$F_1 : W_{\mathcal{O}}(\mathfrak{a})P + Q \to P,$$

such that $F_1 \mathfrak{a} P = 0$.

Proof. If we choose a normal decomposition $P = L \oplus T$, then

$$W_{\mathcal{O}}(\mathfrak{a})P + Q = \mathfrak{a}T \oplus L \oplus I_{\mathcal{O},R}T.$$

Define $F_1 : W_{\mathcal{O}}(\mathfrak{a})P + Q \to P$ by setting $F_1(\mathfrak{a}T) = 0$. Then $F_1 \mathfrak{a} L = 0$ since ${}^{F}\mathfrak{a} = 0$. (See Section 1.2.2.) $\square$

Proof of Theorem 2.12. First, we show that (2.5) is injective. Since any nilpotent $\mathcal{N}$ admits a filtration

$$0 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \ldots \subset \mathcal{N}_r = \mathcal{N}$$

with $\mathcal{N}_i^2 \subset \mathcal{N}_{i-1}$, it suffices to prove the injectivity for $\mathcal{N}$ with $\mathcal{N}^2 = 0$. In this case, $\mathcal{N}$ has a trivial $\mathcal{O}$-pd-structure $\gamma = 0$. Extend $F_1 : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}$ to a map

$$F_1 : \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \to \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \tag{2.7}$$

by applying Lemma 2.13 to $F_1 : Q_S \to P_S$ first (with $S = R \oplus \mathcal{N}$) and then restricting to $\widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P$. By our assumption on $\mathcal{N}$, we get an isomorphism

$$\widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \to \bigoplus_{i \geq 0} \mathcal{N} \otimes_{\mathrm{w}_i, W_{\mathcal{O}}(R)} P. \tag{2.8}$$

Define the operators K_i for $i \geq 0$ by

$$\begin{aligned} K_i : \mathcal{N} \otimes_{\mathrm{w}_{f+i}, W_{\mathcal{O}}(R)} P &\longrightarrow \mathcal{N} \otimes_{\mathrm{w}_i, W_{\mathcal{O}}(R)} P \\ a \otimes x &\longmapsto \pi^{f-1} a \otimes F x. \end{aligned}$$

It is easy to check that F_1 on the right side of (2.8) is given by

$$F_1[u_0, u_1, \ldots] = [K_1 u_f, K_2 u_{f+1}, \ldots].$$

In particular, F_1 is pointwise nilpotent. Therefore, the morphism

$$F_1 - \mathrm{id} : \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \to \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P, \tag{2.9}$$

is an isomorphism. Here F_1 is the map (2.7). The injectivity of (2.5) follows.

Define $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{N})$ by the exact sequence

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{F_1 - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \longrightarrow 0.$$

There is an obvious $\mathcal{O}$-module structure on $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{N})$. For an R-algebra morphism $\eta : \mathcal{N} \to \mathcal{M}$ in Nil_R, we obtain an $\mathcal{O}$-module morphism $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \eta) : \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \to \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{M})$ by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{Q}_{\mathcal{N}} \arrow[r, "F_1 - \mathrm{id}"] \arrow[d, "\eta'"'] & \widehat{P}_{\mathcal{N}} \arrow[r] \arrow[d, "\eta''"] & \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \arrow[r] \arrow[d, "\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \eta)"] & 0 \\
0 \arrow[r] & \widehat{Q}_{\mathcal{M}} \arrow[r, "F_1 - \mathrm{id}"] & \widehat{P}_{\mathcal{M}} \arrow[r] & \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{M}) \arrow[r] & 0,
\end{tikzcd}$$

where η'' is the induced morphism $\overline{\eta} \otimes \mathrm{id} : \widehat{P}_{\mathcal{N}} \to \widehat{P}_{\mathcal{M}}$ and η' is the restriction of η'' to $\widehat{Q}_{\mathcal{N}}$. It is easily seen that the image of η' is contained in $\widehat{Q}_{\mathcal{M}}$. This shows that $\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, -)$ is a functor.

We need to verify that the conditions of Definition 1.12 hold. The first two conditions are clear because the functors $\mathcal{N} \mapsto \widehat{P}_{\mathcal{N}}$ and $\mathcal{N} \mapsto \widehat{Q}_{\mathcal{N}}$ are exact. For the remaining conditions we need to study $t_{\mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, -)}$. Because we only consider Mod_R in Nil_R, we may assume that $\mathcal{N}^2 = 0$. Thus $\mathcal{N}$ has the trivial $\mathcal{O}$-pd-structure. Define a morphism

$$\exp_{\mathcal{P}} : \mathcal{N} \otimes_R P/Q \longrightarrow \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}, \mathcal{N})$$

by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{Q}_{\mathcal{N}} & \xrightarrow{\mathrm{id}} & \widehat{P}_{\mathcal{N}} & \longrightarrow & \mathcal{N} \otimes_R P/Q & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle F_1 - \mathrm{id}} & & \downarrow{\scriptstyle \exp_{\mathcal{P}}} & & \\ 0 & \longrightarrow & \widehat{Q}_{\mathcal{N}} & \xrightarrow{F_1 - \mathrm{id}} & \widehat{P}_{\mathcal{N}} & \longrightarrow & \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, \mathcal{N}) & \longrightarrow & 0. \end{array}$$

One can easily deduce from this diagram that $\exp_{\mathcal{P}}$ is an isomorphism. We see furthermore that $t_{\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P},-)}$ is isomorphic to $M \mapsto M \otimes_R P/Q$ via this exponential map. Then it is easy to see that the last two conditions of Definition 1.12 are satisfied. We conclude that $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is a formal group (with $\mathcal{O}$-action). Since $W_{\mathcal{O}}(R) \to R$ is $\mathcal{O}$-linear, the two $\mathcal{O}$-actions on the tangent space coincide. Hence it is a formal $\mathcal{O}$-module. $\square$

Let $\alpha : R \to S$ be an $\mathcal{O}$-algebra morphism and $\mathcal{P}$ an f-$\mathcal{O}$-display over R. We get an f-$\mathcal{O}$-display $\alpha_\star \mathcal{P}$ over S by base change and obtain a formal $\mathcal{O}$-module $\mathrm{BT}_{\mathcal{O}}^{(f)}(\alpha_\star \mathcal{P}, -)$ over S. On the other hand, we obtain a formal $\mathcal{O}$-module $\alpha_\star \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ over S by restriction. The following corollary says that the functor $\mathcal{P} \to \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ commutes with base change.

Corollary 2.14. *(Cf. [23, Corollary 86].) With the conditions as above we get an isomorphism of formal $\mathcal{O}$-modules over S*

$$\alpha_\star \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -) \cong \mathrm{BT}_{\mathcal{O}}^{(f)}(\alpha_\star \mathcal{P}, -).$$

Proof. The isomorphism is induced from the isomorphism

$$\widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \cong \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(S)} W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} P = \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(S)} \alpha_\star P,$$

for $\mathcal{N} \in \mathrm{Nil}_S$. $\square$

We cite two propositions of [23], from which we deduce that $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is a π-divisible formal $\mathcal{O}$-module for nilpotent f-$\mathcal{O}$-displays $\mathcal{P}$.

Proposition 2.15. *(Cf. [23, Proposition 87].) Let R be an $\mathcal{O}$-algebra, such that $\pi R = 0$, and $\mathcal{P}$ a nilpotent f-$\mathcal{O}$-display over R. Furthermore, let $\mathrm{Fr}_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}^{(q^f)}$ be the Frobenius endomorphism (see (2.1)) and $G = \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ resp. $G^{(q^f)} = \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}^{(q^f)}, -)$ be the formal $\mathcal{O}$-module attached to $\mathcal{P}$ resp. $\mathcal{P}^{(q^f)}$. Because $\mathrm{BT}_{\mathcal{O}}^{(f)}$ commutes with base change, we obtain a morphism of formal $\mathcal{O}$-modules*

$$\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathrm{Fr}_{\mathcal{P}}) : G \to G^{(q^f)},$$

which is the Frobenius morphism of the formal $\mathcal{O}$-module G (with respect to $x \mapsto x^q$) iterated f times Fr_G^f. (This Frobenius is the obvious generalization of [20, Kapitel V].)

Proposition 2.16. *(Cf. [23, Proposition 88].) With the setting as in Proposition 2.15, there is a number N and a morphism of nilpotent f-$\mathcal{O}$-displays*

$$\gamma : \mathcal{P} \to \mathcal{P}^{(q^{fN})},$$

such that the diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\pi"] \arrow[d, "\mathrm{Fr}_{\mathcal{P}}^{N}"'] & \mathcal{P} \arrow[dl, "\gamma"] \\ \mathcal{P}^{(q^{fN})} & \end{tikzcd}$$

is commutative.

Corollary 2.17. *(Cf. [23, Proposition 89].) Let $R \in \mathrm{Nil}_{\mathcal{O}}$ and $\mathcal{P}$ be a nilpotent f-$\mathcal{O}$-display over R. Then $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is a π-divisible formal $\mathcal{O}$-module (cf. Definition 1.15).*

Proof. First assume that $\pi R = 0$. In this case, we may apply $\mathrm{BT}_{\mathcal{O}}^{(f)}$ to the diagram of Proposition 2.16 and obtain that some iteration of the Frobenius on $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ factors through π and some other morphism. By [20, 5.18 Lemma] and [20, 5.10 Satz], we obtain that π is an isogeny. Hence, $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is a π-divisible formal $\mathcal{O}$-module over R.

If π is nilpotent in R, then a formal $\mathcal{O}$-module is π-divisible, if and only if its reduction modulo π is π-divisible (cf. [20, 5.12 Korollar]). Hence, $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is π-divisible. □

The following proposition explains how we may describe the reduced Cartier module of a formal $\mathcal{O}$-module associated with an f-$\mathcal{O}$-display over an $\mathcal{O}$-algebra R.

Proposition 2.18. *(Cf. [23, Proposition 90].) Let $\mathcal{P} = (P, Q, F, F_1)$ be an f-$\mathcal{O}$-display over an $\mathcal{O}$-algebra R. The reduced $\mathbb{E}_{\mathcal{O},R}$-module associated with the formal $\mathcal{O}$-module $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}, -)$ is given by*

$$M(\mathcal{P}) = M_{\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P},-)} = \mathbb{E}_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P/(F \otimes x - V^{f-1} \otimes Fx, V^f \otimes F_1 y - 1 \otimes y)_{x\in P, y\in Q}.$$

The proof is the same as the proof of [23, Proposition 90]. Here we use equation (2.6), which corresponds to [23, Equation (147)]. The only difference is that now F_1 is ${}^{F^f}$-linear and the morphism V^{-1} in [23] is ^{F}-linear.

2.5. Construction of $\Gamma(\mathcal{O}, \mathcal{O}')$

Let $\mathcal{O}'$ be a finite extension of $\mathcal{O}$. In this section, we construct a functor $\Gamma(\mathcal{O}, \mathcal{O}')$ which makes the following diagram commutes

$$
\begin{tikzcd}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}"] \arrow[d, "\Gamma(\mathcal{O},\mathcal{O}')"'] & (\pi'\text{-divisible formal } \mathcal{O}'\text{-modules}/R) \\
(\mathrm{ndisp}_{\mathcal{O}'}/R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}'}"'] &
\end{tikzcd}
$$

In order to do this, it suffices to treat the cases when $\mathcal{O}'/\mathcal{O}$ is unramified and $\mathcal{O}'/\mathcal{O}$ is totally ramified. The constructions in these two cases are different and we explain them separately. The above commutative diagram is obtained by combining Propositions 2.21, 2.23, and 2.29.

2.5.1. Unramified case

Assume that $\mathcal{O} \to \mathcal{O}'$ is unramified of degree f. In the following, we introduce the functors $\Omega_i(\mathcal{O},\mathcal{O}')$ $(i = 1,2)$ and $\Gamma_1(\mathcal{O},\mathcal{O}')$.

Lemma 2.19. *Let $\mathcal{O} \to \mathcal{O}'$ be unramified of degree f, R an $\mathcal{O}'$-algebra, and $\mathcal{P} = (P,Q,F,F_1)$ an $\mathcal{O}$-display over R equipped with a strict $\mathcal{O}'$-action ι. Then we may decompose P and Q canonically as $P = \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} P_i$, $Q = \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} Q_i$, where P_i and $Q_i = P_i \cap Q$ are $W_{\mathcal{O}}(R)$-modules, $P_i = Q_i$ for all $i \neq 0$, $F(P_i), F_1(Q_i) \subseteq P_{i+1}$ for all i (where we consider i modulo f) and*

$$\mu_{i,j} : W_{\mathcal{O}}(R) \otimes_{F^i, W_{\mathcal{O}}(R)} P_j \to P_{i+j},$$

given by $w \otimes p_j \mapsto wF_1^i p_j$ is an isomorphism for all i,j with $0 \le i \le (f-1)$, $1 \le j \le (f-1)$, $i + j \le f$.

Proof. Let σ denote the relative Frobenius of the extension $\mathcal{O} \to \mathcal{O}'$. The $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) = W_{\mathcal{O}}(R)^f$-modules P and Q decompose as

$$P = \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} P_i, \; Q = \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} Q_i,$$

where

$$P_i = \{\, x \in P \,|\, (a \otimes 1)x = (1 \otimes \sigma^i(a))x \text{ for all } a \in \mathcal{O}' \,\}$$

and $Q_i = Q \cap P_i$. Because the $\mathcal{O}'$-action is strict, it is easy to check that the modules P_i and Q_i satisfy the conditions in the lemma. □

Definition 2.20. With the setting as in Lemma 2.19, we define a functor

$$\Omega_1(\mathcal{O},\mathcal{O}') : (\mathrm{disp}_{\mathcal{O},\mathcal{O}'}/R) \to (f - \mathrm{disp}_{\mathcal{O}}/R)$$

by sending (P, Q, F, F_1) equipped with a strict $\mathcal{O}'$-action to $(P_0, Q_0, F_1^{f-1}F, F_1^f)$ and restricting a morphism between two $(\mathcal{O}, \mathcal{O}')$-displays respecting the attached $\mathcal{O}'$-actions to the zeroth component. Furthermore, if $R \in \mathrm{Nil}_{\mathcal{O}'}$, we obtain by restriction the functor

$$\Omega_1(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (f - \mathrm{ndisp}_{\mathcal{O}}/R).$$

It is easy to check that the functors commute with base change.

Proposition 2.21. *Let $\mathcal{O} \to \mathcal{O}'$ be unramified of degree f and R an $\mathcal{O}'$-algebra. Then the following diagram is commutative:*

$$\begin{tikzcd}
(\mathrm{disp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}"] \arrow[d, "\Omega_1(\mathcal{O},\mathcal{O}')"'] & (\text{formal } \mathcal{O}'\text{-modules}/R) \\
(f - \mathrm{disp}_{\mathcal{O}}/R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}}^{(f)}"'] &
\end{tikzcd}$$

If π' is nilpotent in R, then the restriction of the above diagram

$$\begin{tikzcd}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}"] \arrow[d, "\Omega_1(\mathcal{O},\mathcal{O}')"'] & (\pi'\text{-divisible formal } \mathcal{O}'\text{-modules}/R) \\
(f - \mathrm{ndisp}_{\mathcal{O}}/R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}}^{(f)}"'] &
\end{tikzcd}$$

is commutative.

Proof. Let $\mathcal{P}$ be a (nilpotent) $\mathcal{O}$-display over R and $\mathcal{P}_0$ its image via $\Omega_1(\mathcal{O}, \mathcal{O}')$. It suffices to construct an isomorphism

$$\mathrm{BT}_{\mathcal{O}}(\mathcal{P}, -) \to \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -).$$

For a nilpotent R-algebra $\mathcal{N}$, consider the sequence

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{F_1 - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow \mathrm{BT}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \longrightarrow 0$$

and the one defining $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -)$. Using the $\mathbb{Z}/f\mathbb{Z}$-grading of Q and P, we obtain from the above sequence

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \widehat{Q}_{i,\mathcal{N}} \xrightarrow{F_1 - \mathrm{id}} \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \widehat{P}_{i,\mathcal{N}} \longrightarrow \mathrm{BT}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \longrightarrow 0,$$

where $\widehat{P}_{i,\mathcal{N}}$ and $\widehat{Q}_{i,\mathcal{N}}$ have the obvious meaning. Note that $\widehat{P}_{i,\mathcal{N}} = \widehat{Q}_{i,\mathcal{N}}$ for all $i \neq 0$.

There is a map θ from $\bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} \widehat{P}_{i,\mathcal{N}} = \widehat{P}_{\mathcal{N}}$ to $\widehat{P}_{0,\mathcal{N}}$ defined by $\theta(x_0, x_1, \ldots, x_{f-1}) = \sum_{j=1}^{f} F_1^{f-j} x_j$ (with index taken modulo f) and we claim that the image of $\theta \circ (F_1 - \mathrm{id})$ is contained in the image of $F_1^f - \mathrm{id}$, which would establish a map $\mathrm{BT}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \to \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}_0, \mathcal{N})$.

Indeed, an element $(x_0, \ldots, x_{f-1})$ of $\widehat{P}_{\mathcal{N}} = \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} \widehat{P}_{i,\mathcal{N}}$ is contained in $(F_1 - \mathrm{id})(\widehat{Q}_{\mathcal{N}})$ if and only if there is a $(q_0, \ldots, q_{f-1}) \in \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} \widehat{Q}_{i,\mathcal{N}} = \widehat{Q}_{\mathcal{N}}$, such that

$$x_i = F_1 q_{i-1} - q_i \tag{2.10}$$

for all i. For such an element $(x_0, \ldots, x_{f-1})$, we have

$$q_i = F_1^i q_0 - \sum_{j=1}^{i} F_1^{i-j} x_j \tag{2.11}$$

for all $i = 0, \ldots, f$. So we get

$$F_1^f q_0 - q_0 = \sum_{j=1}^{f} F_1^{f-j} x_j, \tag{2.12}$$

from which we can deduce $\theta(F_1 - \mathrm{id})(\widehat{Q}_{\mathcal{N}}) \subseteq (F_1^f - \mathrm{id})(\widehat{Q}_{0,\mathcal{N}})$. The morphism θ induces a well defined morphism

$$\overline{\theta} : \mathrm{BT}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \to \mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}_0, \mathcal{N}).$$

It is obvious that $\overline{\theta}$ is a morphism respecting the $\mathcal{O}'$-module structure and that $\overline{\theta} = \overline{\theta}(\mathcal{N})$ is functorial in $\mathcal{N}$. The surjectivity of $\overline{\theta}$ follows from the fact $\overline{\theta}(\overline{(x_0, 0, \ldots, 0)}) = \overline{x_0}$. We check the injectivity. Indeed, assume that $(x_0, \ldots, x_{f-1}) \in \widehat{P}_{\mathcal{N}}$ such that $\overline{\theta}(\overline{(x_0, \ldots, x_{f-1})}) = 0$, then equation (2.12) holds for some $q_0 \in \widehat{Q}_{0,\mathcal{N}}$. For $i = 1, \ldots, f-1$, define q_i via equation (2.11). Then $(F_1 - \mathrm{id})(q_0, \ldots, q_{f-1}) = (x_0, \ldots, x_{f-1})$ and $\overline{(x_0, \ldots, x_{f-1})} = 0$. The proposition follows. $\square$

In order to obtain a functor from $(\mathcal{O}, \mathcal{O}')$-displays over R to $\mathcal{O}'$-displays over R, it suffices to construct a functor from f-$\mathcal{O}$-displays over R to $\mathcal{O}'$-displays over R.

Definition 2.22. With $\mathcal{O} \to \mathcal{O}'$ unramified of degree f and R an $\mathcal{O}'$-algebra, we define a functor

$$\Omega_2(\mathcal{O}, \mathcal{O}') : (f - \mathrm{disp}_{\mathcal{O}} / R) \to (\mathrm{disp}_{\mathcal{O}'} / R)$$

by sending $\mathcal{P}_0 = (P_0, Q_0, F_0, F_{1,0})$ with a normal decomposition $L_0 \oplus T_0 = P_0$ to $\mathcal{P}' = (P', Q', F', F_1')$, where the elements of the quadruple are given by

$$
\begin{aligned}
P' &= W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_0, \\
Q' &= I_{\mathcal{O}',R} \otimes_{W_{\mathcal{O}}(R)} T_0 \oplus W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} L_0, \\
F' &= {}^{F'} \otimes_{W_{\mathcal{O}}(R)} F_0, \\
F_1'(w \otimes z) &= {}^{F'} w \otimes_{W_{\mathcal{O}}(R)} F_{1,0}(z), \\
F_1'({}^{V'} w \otimes x) &= w \otimes_{W_{\mathcal{O}}(R)} F_0 x,
\end{aligned}
$$

for all $w \in W_{\mathcal{O}'}(R)$, $x \in P_0$ and $z \in Q_0$. Here u is the natural morphism $u : W_{\mathcal{O}}(R) \to W_{\mathcal{O}'}(R)$, the operators related to $W_{\mathcal{O}'}(R)$ are marked with a dash.

For $R \in \mathrm{Nil}_{\mathcal{O}'}$ this defines a functor

$$\Omega_2(\mathcal{O}, \mathcal{O}') : (f - \mathrm{ndisp}_{\mathcal{O}} /R) \to (\mathrm{ndisp}_{\mathcal{O}'} /R).$$

We define

$$\Gamma_1(\mathcal{O}, \mathcal{O}') : (\mathrm{disp}_{\mathcal{O},\mathcal{O}'} /R) \to (\mathrm{disp}_{\mathcal{O}'} /R)$$

to be the composite of $\Omega_2(\mathcal{O}, \mathcal{O}')$ and $\Omega_1(\mathcal{O}, \mathcal{O}')$ and analogously for the nilpotent case.

It is easily checked that the definition of Q' is independent of the normal decomposition of P_0 and that the functors commute with base change.

Proposition 2.23. *Let $\mathcal{O} \to \mathcal{O}'$ be unramified of degree f and R an $\mathcal{O}'$-algebra. Then the following diagram is commutative:*

$$
\begin{tikzcd}
(f - \mathrm{disp}_{\mathcal{O}} /R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}^{(f)}"] \arrow[d, "\Omega_2(\mathcal{O},\mathcal{O}')"'] & (\text{ formal } \mathcal{O}'\text{-modules}/R) \\
(\mathrm{disp}_{\mathcal{O}'} /R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}'}"'] &
\end{tikzcd}
$$

If π' is nilpotent in R, then the restriction of the above diagram

$$
\begin{tikzcd}
(f - \mathrm{ndisp}_{\mathcal{O}} /R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}^{(f)}"] \arrow[d, "\Omega_2(\mathcal{O},\mathcal{O}')"'] & (\pi'\text{-divisible formal } \mathcal{O}'\text{-modules}/R) \\
(\mathrm{ndisp}_{\mathcal{O}'} /R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}'}"'] &
\end{tikzcd}
$$

is commutative.

Proof. Let $\mathcal{P}_0 = (P_0, Q_0, F_0, F_{1,0})$ be a (nilpotent) f-$\mathcal{O}$-display over R and $\mathcal{P}' = (P', Q', F', F_1')$ its image via $\Omega_2(\mathcal{O}, \mathcal{O}')$. We need to show that $\mathrm{BT}_{\mathcal{O}}^{(f)}(\mathcal{P}_0, -)$ and

$\mathrm{BT}_{\mathcal{O}'}(\mathcal{P}', -)$ are isomorphic in the category of (π'-divisible) formal $\mathcal{O}'$-modules over R. For $\mathcal{N} \in \mathrm{Nil}_R$ the equations

$$\begin{aligned} \widehat{P'}_{\mathcal{N}} &= \widehat{W}_{\mathcal{O}'}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P_0 \\ \widehat{Q'}_{\mathcal{N}} &= \widehat{I}_{\mathcal{O}',\mathcal{N}} \otimes_{W_{\mathcal{O}}(R)} T_0 \oplus \widehat{W}_{\mathcal{O}'}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} L_0 \end{aligned}$$

hold (for a normal decomposition $L_0 \oplus T_0$ of P_0) and we define a map

$$\mu = u_{\mathcal{N}} \otimes \mathrm{id} : \widehat{P}_{0,\mathcal{N}} \to \widehat{P'}_{\mathcal{N}},$$

where $u_{\mathcal{N}}$ is the natural map $\widehat{W}_{\mathcal{O}}(\mathcal{N}) \to \widehat{W}_{\mathcal{O}'}(\mathcal{N})$. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{Q}_{0,\mathcal{N}} \arrow[r, "F_{1,0}-\mathrm{id}"] \arrow[d, "\mu|_{\widehat{Q}_{0,\mathcal{N}}}"'] & \widehat{P}_{0,\mathcal{N}} \arrow[r] \arrow[d, "\mu"] & \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}_0, \mathcal{N}) \arrow[r] \arrow[d, "\overline{\mu}"] & 0 \\
0 \arrow[r] & \widehat{Q'}_{\mathcal{N}} \arrow[r, "F'_1-\mathrm{id}"] & \widehat{P'}_{\mathcal{N}} \arrow[r] & \mathrm{BT}_{\mathcal{O}'}(\mathcal{P}', \mathcal{N}) \arrow[r] & 0,
\end{tikzcd}$$

where $\overline{\mu}$ is the induced map. In order to show that $\overline{\mu}$ is in fact an isomorphism, we may reduce to the case that $\mathcal{N}^2 = 0$. Consider the exact sequence

$$0 \longrightarrow Q' \longrightarrow P' = W_{\mathcal{O}'}(R) \otimes_{W_{\mathcal{O}}(R)} P_0 \xrightarrow{\ \omega\ } R \otimes_R P_0/Q_0 = P_0/Q_0 \longrightarrow 0,$$

where $\omega = \mathrm{w}'_0 \otimes \mathrm{pr}$, we see that P'/Q' is isomorphic to P_0/Q_0 as R-modules. It is easily seen that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{Q}_{0,\mathcal{N}} \arrow[r, "\mathrm{id}"] \arrow[d, equal] \arrow[dd, bend right, "\mu|_{\widehat{Q}_{0,\mathcal{N}}}"'] & \widehat{P}_{0,\mathcal{N}} \arrow[r] \arrow[d, "F_{1,0}-\mathrm{id}"'] \arrow[dd, bend left, "\mu"] & \mathcal{N} \otimes_R P_0/Q_0 \arrow[r] \arrow[d, "\exp_{\mathcal{O},\mathcal{P}_0}"'] \arrow[dd, bend left=60, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \widehat{Q}_{0,\mathcal{N}} \arrow[r, "F_{1,0}-\mathrm{id}"'] \arrow[dd, bend right, "\mu|_{\widehat{Q}_{0,\mathcal{N}}}"'] & \widehat{P}_{0,\mathcal{N}} \arrow[r] \arrow[dd, bend left, "\mu"] & \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}_0, \mathcal{N}) \arrow[r] \arrow[dd, bend right=60, "\overline{\mu}"'] & 0 \\
0 \arrow[r] & \widehat{Q'}_{\mathcal{N}} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \widehat{P'}_{\mathcal{N}} \arrow[r] \arrow[d, "F'_1-\mathrm{id}"'] & \mathcal{N} \otimes_R P'/Q' \arrow[r] \arrow[d, "\exp_{\mathcal{O}',\mathcal{P}'}"] & 0 \\
0 \arrow[r] & \widehat{Q'}_{\mathcal{N}} \arrow[r, "F'_1-\mathrm{id}"'] & \widehat{P'}_{\mathcal{N}} \arrow[r] & \mathrm{BT}_{\mathcal{O}'}(\mathcal{P}', \mathcal{N}) \arrow[r] & 0
\end{tikzcd}$$

is commutative, where the upper two rows and the lower two rows are as in the diagram at the end of the proof of Theorem 2.12 for the construction of $\exp_{\mathcal{O},\mathcal{P}_0}$ and $\exp_{\mathcal{O}',\mathcal{P}'}$, respectively. Since both exp maps are isomorphisms, $\overline{\mu} : \mathrm{BT}^{(f)}_{\mathcal{O}}(\mathcal{P}_0, \mathcal{N}) \to \mathrm{BT}_{\mathcal{O}'}(\mathcal{P}', \mathcal{N})$ is an isomorphism. The proposition follows easily. $\square$

2.5.2. Totally ramified case

We now construct $\Gamma_2(\mathcal{O},\mathcal{O}')$ in the case $\mathcal{O} \to \mathcal{O}'$ is totally ramified. Let R be an $\mathcal{O}'$-algebra. First, we prove some lemmas.

Lemma 2.24. *Let $\omega \in \pi\mathcal{O}$. Then there are units $\epsilon, \delta \in W_{\mathcal{O}}(\mathcal{O})$ such that*

$$\pi - [\pi] = {}^V\epsilon, \quad \omega - [\omega^q] = \delta\omega.$$

Proof. The claim on ϵ follows from

$$\pi \mathrm{w}_0(\epsilon) = \mathrm{w}_1(\pi - [\pi]) = \pi - \pi^q.$$

Applying ^{F} to the first identity of the lemma, we obtain

$$\pi - [\pi^q] = \pi\epsilon.$$

In particular, $[\pi^q]/\pi \in W_{\mathcal{O}}(\mathcal{O})$ and hence $[\pi^{qm}]/\pi^m \in W_{\mathcal{O}}(\mathcal{O})$. Therefore, $[\omega^q]/\omega \in W_{\mathcal{O}}(\mathcal{O})$. Applying w_0 to $1-[\omega^q]/\omega$, we see that $1-[\omega^q]/\omega$ is a unit. The lemma follows. $\square$

Lemma 2.25. *Let $(S,\mathfrak{m}) \hookrightarrow (\overline{S},\overline{\mathfrak{m}})$ be an embedding of local rings which send $\mathfrak{m}$ into $\overline{\mathfrak{m}}$ and let P be a finite S-module. If $\overline{P} = \overline{S} \otimes_S P$ is free over $\overline{S}$, then P is free over S.*

Proof. Since $P/\mathfrak{m}P$ is free over the field $S/\mathfrak{m}$, we may take a basis $\overline{x_1}, \dots, \overline{x_d}$ of $P/\mathfrak{m}P$ and consider lifts $x_1, \dots, x_d \in P$, which lift the corresponding $\overline{x_i}$. Because

$$\overline{S}/\overline{\mathfrak{m}} \otimes_{S/\mathfrak{m}} P/\mathfrak{m}P = \overline{P}/\overline{\mathfrak{m}}\overline{P},$$

the elements $1 \otimes \overline{x_i} \in \overline{S}/\overline{\mathfrak{m}} \otimes_{S/\mathfrak{m}} P/\mathfrak{m}P$ form a basis of $\overline{P}/\overline{\mathfrak{m}}\overline{P}$. By Nakayama Lemma, if we consider now the elements $1 \otimes x_i \in \overline{S} \otimes_S P = \overline{P}$, we obtain a basis of $\overline{P}$. Define

$$\begin{aligned} \beta : S^d &\to P \\ e_i &\mapsto x_i \end{aligned}$$

we obtain the commutative diagram of S-modules

$$\begin{tikzcd} S^d \arrow[r, "\beta"] \arrow[d] & P \arrow[d] \\ \overline{S}^d \arrow[r, "\alpha"] & \overline{P}. \end{tikzcd}$$

The injectivity of β follows, since α is an isomorphism of $\overline{S}$-modules. The surjectivity follows by Nakayama Lemma again. Hence β is an isomorphism and P is free over S. $\square$

Proposition 2.26. *Let $R \in \mathrm{Nil}_{\mathcal{O}'}$. Let P be a finite projective $W_{\mathcal{O}}(R)$-module equipped with an $\mathcal{O}$-algebra morphism $\mathcal{O}' \to \mathrm{End}_{W_{\mathcal{O}}(R)} P$. Then P is a finite and projective $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$-module.*

Proof. First we consider the case, where $R = k$ is a perfect field of characteristic p, which extends the residue fields of $\mathcal{O}$ and $\mathcal{O}'$. Then $W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$ is isomorphic to $W_{\mathcal{O}'}(k)$, hence a PID. Since P is finite and torsion free over $W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$, it must be free.

Now let $R = k'$ be an arbitrary field extending the residue fields of $\mathcal{O}$ and $\mathcal{O}'$. We consider the algebraic closure k of k' and the result follows from Lemma 2.25 if we take $S = W_{\mathcal{O}}(k') \otimes_{\mathcal{O}} \mathcal{O}'$ and $\overline{S} = W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$.

Next we assume that $(R, \mathfrak{m})$ is local with residue field k. The module $W_{\mathcal{O}}(k) \otimes_{W_{\mathcal{O}}(R)} P$ is free over $W_{\mathcal{O}}(k) \otimes_{\mathcal{O}} \mathcal{O}'$, so there is a basis of the form $1 \otimes y_1, \ldots, 1 \otimes y_d$ with $y_i \in P$. We claim that the y_i form a basis of the $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-module P. Let us consider the morphism of $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-modules

$$\begin{aligned} \gamma : (W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}')^d &\to P \\ e_i &\mapsto y_i. \end{aligned}$$

Clearly the cokernel B of γ is finitely generated as a $W_{\mathcal{O}}(R)$-module and $W_{\mathcal{O}}(k) \otimes_{W_{\mathcal{O}}(R)} B$ is zero. Since R is local, we obtain that $W_{\mathcal{O}}(R)$ is local with the maximal ideal $\overline{M} = W_{\mathcal{O}}(\mathfrak{m}) + I_{\mathcal{O},R}$. By the above we get $\overline{M}B = B$ and so $B = 0$ by Nakayama. Hence, γ is surjective. Since P is finite and projective as a $W_{\mathcal{O}}(R)$-module and $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$ is finitely generated over $W_{\mathcal{O}}(R)$, the kernel of γ is also finitely generated over $W_{\mathcal{O}}(R)$. By tensoring with $W_{\mathcal{O}}(k)$ we obtain the zero module, hence the kernel of γ is zero by Nakayama again and P is free over $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$.

Finally let R be a general $\mathcal{O}'$-algebra with π' nilpotent in R. The module P is projective over $W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}'$, if and only if $P_n := W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P$ is projective over $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$ for each $n \geq 1$, where $W_{\mathcal{O},n}(R) = W_{\mathcal{O}}(R)/{}^{V^n}W_{\mathcal{O}}(R)$.

We first show that P_n is finitely presented over $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$. For any collection $x_1, \ldots, x_k$ of generators of P_n over $W_{\mathcal{O},n}(R)$, the kernel of the $W_{\mathcal{O},n}(R)$-linear surjection

$$\begin{aligned} W_{\mathcal{O},n}(R)^k &\to P_n \\ e_i &\mapsto x_i \end{aligned}$$

is finitely generated. For a fixed choice of generators $y_1, \ldots, y_d$ of P_n over $W_{\mathcal{O},n}(R)$, we consider the $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-linear surjection

$$\begin{aligned} \delta : (W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}')^d &\to P_n \\ e_i &\mapsto y_i. \end{aligned}$$

Since $(W_{\mathcal{O}}(R) \otimes_{\mathcal{O}} \mathcal{O}')^d$ is finite free over $W_{\mathcal{O}}(R)$, the $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$-module $\mathrm{Ker}\, \delta$ is finitely generated over $W_{\mathcal{O},n}(R)$, hence also over $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$, which establishes the fact that P_n is finitely presented over $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$. To finish the proof, it suffices to show that for each maximal ideal of $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$, the localization of P_n at this ideal is free over the localized ring. It is not too hard to verify that the maximal ideals of $A := W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}'$ are of the form

$$\overline{M} = \pi'^{\,0}(W_{\mathcal{O},n}(\mathfrak{m}) + I_{\mathcal{O},n,R}) + \pi' W_{\mathcal{O},n}(R) + \ldots + \pi'^{\,e-1} W_{\mathcal{O},n}(R),$$

where $\mathfrak{m}$ runs through the maximal ideals of R and $I_{\mathcal{O},n,R} \subseteq W_{\mathcal{O},n}(R)$ is the image of Verschiebung. We claim that

$$A_{\overline{M}} \simeq W_{\mathcal{O},n}(R_{\mathfrak{m}}) \otimes_{\mathcal{O}} \mathcal{O}'. \tag{2.13}$$

Indeed, first one sees that every element of the image of $A \backslash \overline{M}$ via the obvious morphism $W_{\mathcal{O},n}(R) \otimes_{\mathcal{O}} \mathcal{O}' \to W_{\mathcal{O},n}(R_{\mathfrak{m}}) \otimes_{\mathcal{O}} \mathcal{O}'$ is a unit. Now let B be any A-algebra such that $A \backslash \overline{M} \subset B^{\times}$. By considering $W_{\mathcal{O},n}(R)$ as a subring of A in the canonical way, there is a unique morphism of $W_{\mathcal{O},n}(R)$-algebras $g : W_{\mathcal{O},n}(R_{\mathfrak{m}}) \to B$, since $W_{\mathcal{O},n}(R_{\mathfrak{m}})$ is the localization of $W_{\mathcal{O},n}(R)$ at $W_{\mathcal{O},n}(\mathfrak{m}) + I_{\mathcal{O},n,R}$. By considering the value z of $\pi' \in A$ in B, we get a unique morphism of A-algebras $W_{\mathcal{O},n}(R_{\mathfrak{m}}) \otimes_{\mathcal{O}} \mathcal{O}' \to B$ given by g and $\pi' \mapsto z$. By the universal property of localizations the isomorphism (2.13) is established. The general case follows since $(W_{\mathcal{O},n}(R_{\mathfrak{m}}) \otimes_{\mathcal{O}} \mathcal{O}') \otimes_A P_n$ is free over $W_{\mathcal{O},n}(R_{\mathfrak{m}}) \otimes_{\mathcal{O}} \mathcal{O}'$. $\square$

The $W_{\mathcal{O}}(R)$ module $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ is free. It has a basis

$$1 \otimes 1, (\pi')^m \otimes 1 - 1 \otimes [(\pi')^m], \text{ for } m = 1, \cdots, e-1.$$

Assume now that $R \in \mathrm{Nil}_{\mathcal{O}'}$. With the above basis, we can write $J = \mathrm{Ker}(\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) \to R)$ as

$$J = 1 \otimes I_{\mathcal{O},R} \oplus (\pi' \otimes 1 - 1 \otimes [\pi'])(\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)).$$

Let $\mathcal{P} = (P, Q, F, F_1)$ be an object in $\mathrm{disp}_{\mathcal{O},\mathcal{O}'}/R$. An $\mathcal{O}'$-equivariant normal decomposition of $\mathcal{P}$ consists of two finitely generated projective direct summands L, T of the $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$-module P such that

$$P = L \oplus T, \quad Q = L \oplus JT.$$

The existence of an $\mathcal{O}'$-equivariant normal decomposition is shown as for the classical displays. Indeed, consider the exact sequence of R-modules

$$0 \to Q/JP \to P/JP \to P/Q \to 0.$$

We choose finitely generated projective $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$-modules L and T, such that $L/JL \cong Q/JP$ and $T/JT \cong P/Q$. Then we obtain factorizations

$$L \to Q \to Q/JP, \quad T \to P \to P/Q.$$

By Nakayama Lemma, it is easy to see that $P \cong L \oplus T$ and the existence follows.

Define

$$\begin{aligned} P' &= W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P, \\ Q' &= \mathrm{Ker}(W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P \to P/Q : w \otimes x \mapsto \mathrm{w}_0 \, \mathrm{pr}(x)). \end{aligned} \tag{2.14}$$

If $P = L \oplus T$ is an $\mathcal{O}'$-equivariant normal decomposition of $\mathcal{P}$, we set

$$L' = W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} L, \quad T' = W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} T.$$

Then we obtain an induced decomposition

$$P' = L' \oplus T', \quad Q' = L' \oplus I_{\mathcal{O}',R} T'.$$

Proposition 2.27. *With the notation as above, there are uniquely determined* ${}^{F'}$*-linear maps*

$$F' : P' \to P', \quad F'_1 : Q' \to P',$$

such that the quadruple (P', Q', F', F'_1) *becomes an* $\mathcal{O}'$*-display over* R *and such that the following diagram is commutative*

$$\begin{array}{ccc} Q & \xrightarrow{F_1} & P \\ \downarrow & & \downarrow \\ Q' & \xrightarrow[F'_1]{} & P'. \end{array}$$

Proof. Applying Lemma 2.24 to $\mathcal{O}'$, we obtain units $\epsilon, \delta \in W_{\mathcal{O}'}(R)$ such that

$${}^{V'}\epsilon = \pi' - [\pi'], \; \pi'\epsilon = \pi' - [(\pi')^q], \; (\pi')^e - [(\pi')^{eq}] = \delta\pi. \tag{2.15}$$

The commutativity of the diagram forces us to define F' and F'_1 as

$$F'(w \otimes x) = {}^{F'}w \cdot \epsilon^{-1} \otimes F_1((\pi' - [\pi'])x), \tag{2.16}$$

$$F'_1({}^{V'}w \otimes x) = \epsilon^{-1} w \otimes F_1((\pi' - [\pi'])x), \tag{2.17}$$

$$F'_1(w \otimes z) = {}^{F'}w \otimes F_1(z), \tag{2.18}$$

for all $w \in W_{\mathcal{O}'}(R)$, $x \in P$ and $z \in Q$. Here we have used the morphism

$$\begin{aligned} \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) &\to W_{\mathcal{O}'}(R) \\ a \otimes w &\mapsto au(w), \end{aligned}$$

where $a \in \mathcal{O}'$ and $w \in W_{\mathcal{O}}(R)$. Indeed, equation (2.18) follows from the commutativity easily. Equation (2.16) is determined by (2.18), since for all $w \in W_{\mathcal{O}'}(R)$ and $x \in P$,

$$\begin{aligned} \epsilon^{-1\,F'}w \otimes F_1((\pi' - [\pi'])x) &\overset{(2.18)}{=} \epsilon^{-1}F_1'(w \otimes (\pi' - [\pi'])x) \\ &= \epsilon^{-1}F_1'(w^{V'}\epsilon \otimes x) \\ &= \epsilon^{-1}F_1'((^{V'}(^{F'}w\epsilon) \otimes 1)(1 \otimes x)) \\ &= \epsilon^{-1}((^{F'}w\epsilon) \otimes 1)F'(1 \otimes x) = F'(w \otimes x) \end{aligned}$$

must hold. Finally, equation (2.17) is determined by equation (2.16), since $wF'(x) = F_1'(^{V'}wx)$ must hold for all $w \in W_{\mathcal{O}'}(R)$ and $x \in P'$.

We need to check that, with this definition, $\mathcal{P}' = (P', Q', F', F_1')$ is an $\mathcal{O}'$-display over R and the diagram is commutative. For the first assertion, we only need to check that F_1' is an $^{F'}$-linear epimorphism. But this follows from the commutativity of the diagram. Now we prove this commutativity.

We have to show that for $y \in Q = L \oplus JT$

$$F_1'(1 \otimes y) = 1 \otimes F_1(y). \tag{2.19}$$

This is clear if $y \in L$. Consider now the case $y = {^V}\rho t$, where $\rho \in W_{\mathcal{O}}(R)$ and $t \in T$. Denote by $\tilde{\rho}$ the image of ρ under the map $u : W_{\mathcal{O}}(R) \to W_{\mathcal{O}'}(R)$. Then

$$\begin{aligned} F_1'(1 \otimes {^V}\rho t) = F_1'(u(^V\rho) \otimes t) = F_1'(\frac{\pi}{\pi'}{^{V'}}\tilde{\rho} \otimes t) = \frac{\pi}{\pi'}\tilde{\rho}F'(1 \otimes t) \\ = \tilde{\rho}\delta^{-1}\epsilon^{F'}(\frac{(\pi')^e - [\pi']^e}{\pi' - [\pi']})F'(1 \otimes t) \\ = \tilde{\rho}\delta^{-1}\epsilon F'(1 \otimes \frac{(\pi')^e - [\pi']^e}{\pi' - [\pi']}t) \\ = \delta^{-1}\tilde{\rho} \otimes F_1(((\pi')^e - [\pi']^e)t). \end{aligned} \tag{2.20}$$

Let $a \in \mathcal{O}'$ and $\zeta \in W_{\mathcal{O}}(R)$, such that

$$(\pi')^e = a\pi, \quad \pi - [\pi] = {^V}\zeta.$$

With this notation, we may write

$$(\pi')^e - [\pi']^e = a\pi - [a][\pi] = a\pi - [a]\pi + [a]^V\zeta.$$

Thus

$$F_1(((\pi')^e - [\pi']^e)t) = F_1(\pi(a-[a])t) + F_1([a]^V\zeta t) = (a-[a^q])F(t) + [a^q]\zeta F(t),$$

and

$$F_1'(1 \otimes {}^V\rho t) = \delta^{-1}(a - [a^q] + [a^q]\tilde{\zeta}) \otimes \rho F(t).$$

But in $W_{\mathcal{O}'}(R)$ we have $\pi - [\pi^q] = \pi\tilde{\zeta}$. From this we obtain

$$(\pi')^e - [\pi']^{eq} = a\pi - [a\pi]^q = (a - [a^q] + [a^q]\tilde{\zeta})\pi$$

and hence

$$\delta^{-1}(a - [a^q] + [a^q]\tilde{\zeta}) = 1.$$

Equation (2.19) is true for $y = {}^V\rho t$.

It remains to prove the commutativity of the diagram at the elements of Q with the form $(\pi' \otimes 1 - 1 \otimes [\pi'])t$, where $t \in T$. This means we have to show the equality

$$F_1'(1 \otimes (\pi' \otimes 1 - 1 \otimes [\pi'])t) = 1 \otimes F_1((\pi' \otimes 1 - 1 \otimes [\pi'])t).$$

This follows from

$$\begin{aligned} F_1'(1 \otimes (\pi' \otimes 1 - 1 \otimes [\pi'])t) &= F_1'({}^{V'}\epsilon \otimes t) \\ &= \epsilon F'(1 \otimes t) = \epsilon\epsilon^{-1} \otimes F_1((\pi' \otimes 1 - 1 \otimes [\pi'])t). \end{aligned}$$

The proposition follows. $\square$

Finally, we define the functor $\Gamma_2(\mathcal{O}, \mathcal{O}')$. The following definition is well defined by the previous results.

Definition 2.28. With $\mathcal{O} \to \mathcal{O}'$ totally ramified and $R \in \mathrm{Nil}_{\mathcal{O}'}$, we define a functor

$$\Gamma_2(\mathcal{O}, \mathcal{O}') : (\mathrm{disp}_{\mathcal{O},\mathcal{O}'}/R) \to (\mathrm{disp}_{\mathcal{O}'}/R)$$

by sending the $\mathcal{O}$-display $\mathcal{P}$ equipped with a strict $\mathcal{O}'$-action to the $\mathcal{O}'$-display $\mathcal{P}'$, which is defined by

$$\begin{aligned} P' &= W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P, \\ Q' &= \mathrm{Ker}(W_{\mathcal{O}'}(R) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P \to P/Q : w \otimes x \mapsto \mathrm{w}_0\, \mathrm{pr}(x)), \\ F'(w \otimes x) &= {}^{F'}w \cdot \epsilon^{-1} \otimes F_1((\pi' - [\pi'])x), \\ F_1'({}^{V'}w \otimes x) &= \epsilon^{-1} w \otimes F_1((\pi' - [\pi'])x), \qquad (2.21) \\ F_1'(w \otimes z) &= {}^{F'}w \otimes F_1(z), \qquad (2.22) \end{aligned}$$

for all $w \in W_{\mathcal{O}'}(R)$, $x \in P$ and $z \in Q$. Here we have used the morphism

$$\begin{aligned} \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) &\to W_{\mathcal{O}'}(R) \\ a \otimes w &\mapsto au(w), \end{aligned}$$

where $a \in \mathcal{O}'$ and $w \in W_{\mathcal{O}}(R)$, and $\epsilon \in W_{\mathcal{O}'}(R)$ is given by ${}^{V'}\epsilon = \pi' - [\pi']$. Also, P is considered as an $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$-module. The functor

$$\Gamma_2(\mathcal{O}, \mathcal{O}') : (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \to (\mathrm{ndisp}_{\mathcal{O}'}/R)$$

is defined by restriction.

One can easily verify that the functors commute with base change.

Proposition 2.29. *Let $\mathcal{O} \to \mathcal{O}'$ be totally ramified and R an $\mathcal{O}'$-algebra with π' nilpotent in R. Then the following diagram is commutative:*

$$\begin{tikzcd} (\mathrm{disp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}"] \arrow[d, "\Gamma_2(\mathcal{O},\mathcal{O}')"'] & (\text{formal } \mathcal{O}'\text{-modules}/R) \\ (\mathrm{disp}_{\mathcal{O}'}/R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}'}"'] & \end{tikzcd}$$

Also the restriction of the above diagram

$$\begin{tikzcd} (\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\mathrm{BT}_{\mathcal{O}}"] \arrow[d, "\Gamma_2(\mathcal{O},\mathcal{O}')"'] & (\pi'\text{-divisible formal } \mathcal{O}'\text{-modules}/R) \\ (\mathrm{ndisp}_{\mathcal{O}'}/R) \arrow[ur, "\mathrm{BT}_{\mathcal{O}'}"'] & \end{tikzcd}$$

is commutative.

Proof. Let $\mathcal{P} = (P, Q, F, F_1)$ be a (nilpotent) $\mathcal{O}$-display over R with a strict $\mathcal{O}'$-action and $\mathcal{P}' = (P', Q', F', F_1')$ be its image via $\Gamma_2(\mathcal{O}, \mathcal{O}')$. We need to show that $\mathrm{BT}_{\mathcal{O}}(\mathcal{P}, -)$ and $\mathrm{BT}_{\mathcal{O}'}(\mathcal{P}', -)$ are isomorphic in the category of (π'-divisible) formal $\mathcal{O}'$-modules over R. For a nilpotent R-algebra $\mathcal{N}$ we have $\widehat{\mathcal{P}'}_{\mathcal{N}} \simeq \widehat{W}_{\mathcal{O}'}(\mathcal{N}) \otimes_{\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)} P$ and we may define

$$\mu = u_{\mathcal{N}} \otimes \mathrm{id} : \widehat{P}_{\mathcal{N}} = \widehat{W}_{\mathcal{O}}(\mathcal{N}) \otimes_{W_{\mathcal{O}}(R)} P \to \widehat{\mathcal{P}'}_{\mathcal{N}}.$$

It is easy to check that the diagram

$$\begin{tikzcd} \widehat{Q}_{\mathcal{N}} \arrow[r, "F_1 - \mathrm{id}"] \arrow[d, "\mu|_{\widehat{Q}_{\mathcal{N}}}"'] & \widehat{P}_{\mathcal{N}} \arrow[d, "\mu"] \\ \widehat{Q'}_{\mathcal{N}} \arrow[r, "F_1' - \mathrm{id}"] & \widehat{P'}_{\mathcal{N}} \end{tikzcd}$$

is commutative (cf. Proposition 2.27). It induces an $\mathcal{O}'$-module morphism $\overline{\mu} : \mathrm{BT}_{\mathcal{O}}(\mathcal{P}, \mathcal{N}) \to \mathrm{BT}_{\mathcal{O}'}(\mathcal{P}', \mathcal{N})$. To show that $\overline{\mu}$ is an isomorphism of $\mathcal{O}'$-modules, we can reduce to the case $\mathcal{N}^2 = 0$ and proceed in a similar manner as in the unramified case for $\mathrm{BT}_{\mathcal{O}}(\mathcal{P}_0, \mathcal{N})$ and $\mathrm{BT}_{\mathcal{O}'}(\mathcal{P}', \mathcal{N})$. The proposition follows. □

3. Crystals

In this section, we attach to each nilpotent $\mathcal{O}$-display $\mathcal{P}$ over R a crystal $\mathcal{D}_{\mathcal{P}}$ on $\mathcal{O}$-pd-thickenings and attach to each π-divisible formal $\mathcal{O}$-module G a crystal $\mathbb{D}_G$ on $\mathcal{O}$-pd-thickenings. If $G = \mathrm{BT}_{\mathcal{O}}(\mathcal{P}, -)$, following the ideas in [23], we show that $\mathcal{D}_{\mathcal{P}} \cong \mathbb{D}_G$. As a consequence, we deduce the faithfulness of $\mathrm{BT}_{\mathcal{O}}$.

3.1. $\mathcal{O}$-frames and $\mathcal{O}$-windows

Definition 3.1. (Cf. [11, Definition 2.1].) An *$\mathcal{O}$-frame* is a quintuple $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$, where S is an $\mathcal{O}$-algebra, $I \subseteq S$ an ideal, $R = S/I$ together with an $\mathcal{O}$-algebra morphism $\sigma : S \to S$ and a σ-linear morphism of S-modules $\sigma_1 : I \to S$, which satisfy the following conditions:

(1) $I + \pi S \subseteq \mathrm{Rad}(S)$,
(2) $\sigma(a) \equiv a^q \pmod{\pi S}$ for all $a \in S$, and
(3) $\sigma_1(I)$ generates S as an S-module.

Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ and $\mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be two $\mathcal{O}$-frames. A morphism of $\mathcal{O}$-frames $\alpha : \mathcal{F} \to \mathcal{F}'$ is an $\mathcal{O}$-algebra morphism $\alpha : S \to S'$, such that $\alpha(I) \subseteq I'$, $\sigma'\alpha = \alpha\sigma$ and $\sigma_1'\alpha = \alpha\sigma_1$. In the sense of Lau [11, Definition 2.6], these are strict morphisms.

A special example is the so called Witt $\mathcal{O}$-frame

$$\mathcal{W}_{\mathcal{O},R} = (W_{\mathcal{O}}(R), I_{\mathcal{O},R}, W_{\mathcal{O}}(R)/I_{\mathcal{O},R} = R, {}^F, {}^{V^{-1}}).$$

Let $\rho : A \to B$ be a ring homomorphism. For any A-module M, define the B-module $M^{(\rho)}$ by $B \otimes_{\rho,A} M$. For any B-module N and ρ-linear map $g : M \to N$, define the B-linear map $g^\sharp : M^{(\rho)} \to N$ by $b \otimes m \mapsto bg(m)$.

Lemma 3.2. *(Cf. [11, Lemma 2.2].) Let $\mathcal{F}$ be an $\mathcal{O}$-frame. Then there is a unique $\theta \in S$, such that $\sigma(a) = \theta\sigma_1(a)$ for all $a \in I$.*

Proof. The third condition of Definition 3.1 says that the linearization $\sigma_1^\sharp : I^{(\sigma)} \to S$ is surjective. If $b \in I^{(\sigma)}$ satisfies $\sigma_1^\sharp(b) = 1$, then necessarily $\theta = \sigma^\sharp(b)$. For $a \in I$ we obtain $\sigma(a) = \sigma_1^\sharp(b)\sigma(a) = \sigma_1^\sharp(ba) = \sigma^\sharp(b)\sigma_1(a)$, which confirms the assertion. □

Definition 3.3. (Cf. [11, Definition 2.3].) Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ be an $\mathcal{O}$-frame. An *$\mathcal{O}$-window* over $\mathcal{F}$, or an *$\mathcal{F}$-window*, is a quadruple $\mathcal{P} = (P, Q, F, F_1)$, where P is a finitely generated projective S-module, $Q \subset P$ is a submodule, $F : P \to P$ and $F_1 : Q \to P$ are σ-linear maps of S-modules, such that the following conditions hold:

(1) There is a decomposition $P = T \oplus L$ with $Q = IT \oplus L$.
(2) $F_1(ax) = \sigma_1(a)F(x)$ for $a \in I$ and $x \in P$.
(3) $F_1(Q)$ generates P as an S-module.

If $\mathcal{P} = (P, Q, F, F_1)$ is an $\mathcal{F}$-window, define a morphism of S-modules $V^\sharp : P \to S \otimes_{\sigma,S} P$ by $V^\sharp(F_1 y) = 1 \otimes y$ for $y \in Q$ and $V^\sharp(Fx) = \theta \otimes x$ for $x \in P$. Here $\theta \in S$ is the element in Lemma 3.2. Let $(V^N)^\sharp$ be the composition of the following maps

$$P \xrightarrow{V^\sharp} S \otimes_{\sigma,S} P \xrightarrow{\mathrm{id} \otimes V^\sharp} S \otimes_{\sigma,S} (S \otimes_{\sigma,S} P) \to \cdots \to S \otimes_{\sigma^N,S} P.$$

We say that $\mathcal{P}$ is *nilpotent* if $(V^N)^\sharp \equiv 0 \pmod{I + \pi S}$ for some $N \in \mathbb{Z}_{>0}$.

Remark 3.4. The operator F is determined by F_1. Indeed, assume that $\sigma_1^\sharp(b) = 1$ with $b \in I^{(\sigma)}$. Then $F(x) = F_1^\sharp(bx)$ for $x \in P$. In particular, $F(x) = \theta F_1(x)$ if $x \in Q$.

Remark 3.5. It is easy to see that the $\mathcal{O}$-windows over $\mathcal{W}_{\mathcal{O},R}$ are precisely the $\mathcal{O}$-displays over R.

The $\mathcal{O}$-windows have similar properties as windows. We collect some of them in the following and refer to [11] for proofs and more details.

By writing down the structure equation explicitly, we have the following lemma.

Lemma 3.6. *(Cf. [11, Lemma 2.5].) Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ be an $\mathcal{O}$-frame, $P = L \oplus T$ a finitely generated projective S-module and $Q = L \oplus IT$, where L, T are S-submodules of P. Then the set of $\mathcal{O}$-window structures (P, Q, F, F_1) over $\mathcal{F}$ corresponds bijectively to the set of σ-linear isomorphisms $\Psi : L \oplus T \to P$ given by $\Psi(l + t) = F_1(l) + F(t)$ for $l \in L$ and $t \in T$. Conversely, if we start with a Ψ, we obtain an $\mathcal{O}$-window over $\mathcal{F}$ by $F(l + t) = \theta\Psi(l) + \Psi(t)$ and $F_1(l + at) = \Psi(l) + \sigma_1(a)\Psi(t)$ for $l \in L$, $t \in T$ and $a \in I$.*

We call the triple (L, T, Ψ) a *normal decomposition of* (P, Q, F, F_1).

Definition 3.7. Let $\mathcal{P}$ (respectively $\mathcal{P}'$) be an $\mathcal{F}$-window (respectively $\mathcal{F}'$-window). Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a homomorphism of $\mathcal{O}$-frames. A homomorphism of $\mathcal{O}$-windows $g : \mathcal{P} \to \mathcal{P}'$ over α, also called an α-homomorphism, is an S-linear map $g : P \to P'$ with $g(Q) \subset Q'$, such that $F'g = gF$ and $F_1'g = gF_1$. A homomorphism of $\mathcal{F}$-windows is an $\mathrm{id}_{\mathcal{F}}$-homomorphism in the previous sense.

Definition 3.8. With α as above, we associate an $\mathcal{O}$-window $\alpha_\star\mathcal{P} =: \mathcal{P}' = (P', Q', F', F_1')$ over $\mathcal{F}'$ to an $\mathcal{O}$-window $\mathcal{P}$ over $\mathcal{F}$ in the following way.

$$\begin{aligned}
P' &= S' \otimes_S P \\
Q' &= S' \otimes_S L \oplus I' \otimes_S T \\
F' &= \sigma' \otimes F \\
F_1'(s' \otimes q) &= \sigma'(s') \otimes F_1 y \\
F_1'(i' \otimes p) &= \sigma_1'(i') \otimes Fx.
\end{aligned}$$

Here $P = L \oplus T$ is a normal decomposition and $s' \in S'$, $i' \in I'$, $y \in Q$ and $x \in P$. There is an obvious map $\mathrm{Hom}_{\mathcal{F}'}(\alpha_\star\mathcal{P}, \widetilde{\mathcal{P}}) \to \mathrm{Hom}_\alpha(\mathcal{P}, \widetilde{\mathcal{P}})$ for all $\mathcal{O}$-windows $\widetilde{\mathcal{P}}$ over $\mathcal{F}'$ given by composing maps, which is in fact an isomorphism (cf. [11, Lemma 2.9]). This property determines $\alpha_\star\mathcal{P}$ uniquely.

Definition 3.9. Let $\mathcal{F}$ and $\mathcal{F}'$ be two $\mathcal{O}$-frames and $\alpha : \mathcal{F} \to \mathcal{F}'$ a morphism between them. We say that α is *crystalline* if it induces an equivalence of categories between $\mathcal{O}$-windows over $\mathcal{F}$ and $\mathcal{O}$-windows over $\mathcal{F}'$. We say that α is *nilcrystalline* if it induces an equivalence of categories between the nilpotent $\mathcal{O}$-windows.

Theorem 3.10. *(Cf. [11, Theorem 3.2, Theorem 10.3].) Let $\alpha : \mathcal{F} = (S, I, R, \sigma, \sigma_1) \to \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be a morphism of $\mathcal{O}$-frames, such that it induces $R \cong R'$ and a surjection $S \to S'$ with kernel $\mathfrak{b} \subset I$. If there is a finite sequence $\mathfrak{b} = \mathfrak{b}_0 \supseteq \ldots \supseteq \mathfrak{b}_n = 0$ with $\sigma(\mathfrak{b}_i) \subseteq \mathfrak{b}_{i+1}$ and $\sigma_1(\mathfrak{b}_i) \subseteq \mathfrak{b}_i$ such that σ_1 is elementwise nilpotent on $\mathfrak{b}_i/\mathfrak{b}_{i+1}$ and finitely generated projective S'-modules lift to projective S-modules, then α is crystalline.*

Let $J = (I, \pi)$. If $J^n\mathfrak{b} = 0$ for some $n \in \mathbb{Z}_{>0}$, then α is nilcrystalline.

Definition 3.11. Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ be an $\mathcal{O}$-frame. The *Hodge filtration of an $\mathcal{O}$-window* $\mathcal{P}$ over $\mathcal{F}$ is the R-submodule $Q/IP \subseteq P/IP$.

Lemma 3.12. *(Cf. [11, Lemma 4.2].) Let $\alpha : \mathcal{F} = (S, I, R, \sigma, \sigma_1) \to \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be a morphism between two $\mathcal{O}$-frames, such that $S \cong S'$. (Then $R \to R'$ is surjective and $I \subseteq I'$.) The $\mathcal{O}$-windows $\mathcal{P}$ over $\mathcal{F}$ are equivalent to pairs $(\mathcal{P}', V)$ consisting of an $\mathcal{O}$-window $\mathcal{P}'$ over $\mathcal{F}'$ together with a lift of its Hodge filtration to a direct summand $V \subseteq P/IP$. The same is true for nilpotent objects.*

3.2. The crystals associated with $\mathcal{O}$-displays

Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$. Let $\tilde{I} = I_{\mathcal{O},S} + W_{\mathcal{O}}(\mathfrak{a})$. We extend ${}^{V^{-1}} : I_{\mathcal{O},S} \to W_{\mathcal{O}}(S)$ to

$$\sigma_1'' : \tilde{I} \to W_{\mathcal{O}}(S)$$

by setting $\sigma_1''(\mathfrak{a}) = 0$. Here we consider $\mathfrak{a}$ as an ideal of $W_{\mathcal{O}}(\mathfrak{a})$ via the log map (Section 1.2.2). Then it is easy to check that $\mathcal{W}_{\mathcal{O},S/R} = (W_{\mathcal{O}}(S), \tilde{I}, R, \sigma = {}^F, \sigma_1'')$ is an $\mathcal{O}$-frame. Consider the obvious morphisms

$$\mathcal{W}_{\mathcal{O},S} \overset{\alpha_1}{\to} \mathcal{W}_{\mathcal{O},S/R} = (W_{\mathcal{O}}(S), \tilde{I}, R, \sigma, \sigma_1'') \overset{\alpha_2}{\to} \mathcal{W}_{\mathcal{O},R}. \tag{3.1}$$

The morphism α_2 satisfies the conditions in Theorem 3.10 and hence is nilcrystalline.

For an $\mathcal{O}$-display $\mathcal{P}$ over an $\mathcal{O}$-algebra $R \in \mathrm{Nil}_{\mathcal{O}}$, we define a functor $\mathcal{K}_{\mathcal{P}}$ on $\mathcal{O}$-pd-thickenings as follows. For an $\mathcal{O}$-pd-thickening $S \to R$, we define

$$\mathcal{K}_{\mathcal{P}}(S \to R) = \widetilde{P},$$

where $(\widetilde{P}, \widehat{Q}, F, F_1)$ is the uniquely determined window over $\mathcal{W}_{\mathcal{O},S/R}$. We also denote it by $\mathcal{K}_{\mathcal{P}}(S)$ if the setting is clear.

Definition 3.13. The functor $\mathcal{K}_{\mathcal{P}}$ is called the *Witt crystal attached to* $\mathcal{P}$. We define the *Dieudonné crystal* by

$$\mathcal{D}_{\mathcal{P}}(S) = \mathcal{K}_{\mathcal{P}}(S)/I_{\mathcal{O},S}\mathcal{K}_{\mathcal{P}}(S).$$

We define for any topological $\mathcal{O}$-pd-thickening $(S, \mathfrak{a}_n) \to R$ the crystals by

$$\mathcal{K}_{\mathcal{P}}(S) = \varprojlim_n \mathcal{K}_{\mathcal{P}}(S/\mathfrak{a}_n)$$
$$\mathcal{D}_{\mathcal{P}}(S) = \varprojlim_n \mathcal{D}_{\mathcal{P}}(S/\mathfrak{a}_n).$$

Both crystals are compatible with base change in the following sense. If we consider a morphism of $\mathcal{O}$-pd-thickenings

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

we obtain

$$\mathcal{K}_{\mathcal{P}_{R'}}(S') \simeq W_{\mathcal{O}}(S') \otimes_{W_{\mathcal{O}}(S)} \mathcal{K}_{\mathcal{P}}(S),$$
$$\mathcal{D}_{\mathcal{P}_{R'}}(S') \simeq S' \otimes_S \mathcal{D}_{\mathcal{P}}(S).$$

These isomorphisms also hold for topological $\mathcal{O}$-pd-thickenings.

Now let us consider the canonical morphism

$$\mathrm{w}_0 : W_{\mathcal{O}}(R) \to R.$$

The kernel $I_{\mathcal{O},R}$ may be equipped with an $\mathcal{O}$-pd-structure defined by

$$\gamma({}^{V}w) = \pi^{q-2}\,{}^{V}(w^q) \tag{3.2}$$

for all $w \in W_{\mathcal{O}}(R)$. The morphism $\mathrm{w}_0 : W_{\mathcal{O}}(R) \to R$ is a topological $\mathcal{O}$-pd-thickening, since $\mathrm{w}_0 : W_{\mathcal{O},n}(R) \to R$ are $\mathcal{O}$-pd-thickenings with an $\mathcal{O}$-pd-structure given by γ.

If $S \to R$ is an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$, then $\mathfrak{a}$ considered as an ideal of $W_{\mathcal{O}}(S)$ has the same $\mathcal{O}$-pd-structure as considered as an ideal of S. The kernel of the composite map $W_{\mathcal{O}}(S) \to S \to R$ is $I_{\mathcal{O},S} \oplus \mathfrak{a}$, where on both summands we have $\mathcal{O}$-pd-structures. So this follows for the whole kernel ([2, 3.12. Proposition]). Hence, $W_{\mathcal{O}}(S) \to R$ is a topological $\mathcal{O}$-pd-thickening by considering $W_{\mathcal{O},n}(S) \to R$ for each n.

Theorem 3.14. *(Cf. [23, Proposition 53, Corollary 56].) Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$ and $\mathcal{P} = (P, Q, F, F_1)$ be a nilpotent $\mathcal{O}$-display over R. Then*

$$\mathcal{K}_{\mathcal{P}}(S) = \mathcal{D}_{\mathcal{P}}(W_{\mathcal{O}}(S)).$$

If $W_{\mathcal{O}}(R) \to S$ is a morphism of (topological) $\mathcal{O}$-pd-thickenings over R, then

$$\mathcal{K}_{\mathcal{P}}(S) \simeq W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(R)} \mathcal{K}_{\mathcal{P}}(R)$$
$$\mathcal{D}_{\mathcal{P}}(S) \simeq S \otimes_{W_{\mathcal{O}}(R)} \mathcal{K}_{\mathcal{P}}(R),$$

where $W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(S)$ is given by $W_{\mathcal{O}}(R) \overset{\Delta}{\to} W_{\mathcal{O}}(W_{\mathcal{O}}(R)) \to W_{\mathcal{O}}(S)$.

The proof here is analogous to the proof of [23, Proposition 53]. Note that in [23], the result is proved using $\mathcal{P}$-triples. Yet it is easy to see that the category of $\mathcal{P}$-triples with respect to $S \to R$ is equivalent to the category of $\mathcal{O}$-windows over the frame $\mathcal{W}_{\mathcal{O},S/R}$, thus the argument in [23] carries over. The last assertion of the theorem follows by considering the trivial $\mathcal{O}$-pd-thickening $R \to R$ and then making a base change with respect to $W_{\mathcal{O}}(R) \to S$. The most important situations, in which we will use this fact, are for $S = W_{\mathcal{O},n}(R)$ (e.g., Proposition 3.28).

3.3. Universal extensions of π-divisible formal $\mathcal{O}$-modules

In this subsection, we prove the existence of the universal extension of a π-divisible formal $\mathcal{O}$-module. The argument here is taken from [19], where the case for p-divisible formal groups is treated. Let R be a unitary ring. We consider functors from the category Alg_R to the category of abelian groups Ab. Let M be an R-module. We define a functor $\underline{M}$

$$\underline{M}(T) = M \otimes_R T,$$

where $T \in \mathrm{Alg}_R$ and with the additive group structure on the right hand side.

Let $G = \operatorname{Spec} A$ be a finite locally free group scheme. Let $G^* = \operatorname{Spec} A^*$ be its Cartier dual. Let $J \subset A^*$ be the ideal of neutral element. Set

$$\omega_{G^*} = J/J^2.$$

The following result is well-known. See for example [14, Chap 4, Section 1].

Proposition 3.15. *There is a canonical isomorphism*

$$\operatorname{Hom}_{\mathrm{Ab}}(G, \underline{M}) \cong \operatorname{Hom}_{\mathrm{Mod}_R}(\omega_{G^*}, M).$$

As in [19], for a functor $X : \mathrm{Alg}_R \to \mathrm{Ab}$, we consider its completion $\widehat{X} : \mathrm{Aug}_R \to \mathrm{Ab}$ on the category of augmented nilpotent algebras, which is defined by

$$\widehat{X}(A) = \operatorname{Ker}(X(A) \to X(R)).$$

In the following, we assume that G is a local, finite locally free group scheme, i.e., the augmentation ideal of G is nilpotent. Then $\operatorname{Hom}(\widehat{G}, \widehat{\underline{M}}) = \operatorname{Hom}(G, \underline{M})$. We also assume that $R \in \mathrm{Nil}_{\mathcal{O}}$.

From now on we work with functors on Aug_R. We write $\underline{M}$ and mean the completion. The functors $F : \mathrm{Aug}_R \to \mathrm{Ab}$, with $F(R) = 0$ form an abelian category. Exact sequences and extensions are meant in this category unless otherwise stated.

Proposition 3.16. *Let G be a π-divisible formal $\mathcal{O}$-module over $R \in \mathrm{Nil}_{\mathcal{O}}$. There exist a locally free and finite R module U and an $\mathcal{O}$-extension of formal $\mathcal{O}$-modules over R*

$$0 \to \underline{U} \to L \to G \to 0, \tag{3.3}$$

such that for any extension

$$0 \to \underline{M} \to E \to G \to 0, \tag{3.4}$$

where M is an R-module, there is a unique morphism of R-modules $U \to M$ (inducing $\underline{U} \to \underline{M}$) which sits in a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \underline{U} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{M} \arrow[r] & E \arrow[r] & G \arrow[r] & 0.
\end{tikzcd}$$

In particular, the map

$$\operatorname{Hom}_{\mathrm{Mod}_R}(U, M) \to \operatorname{Ext}^1_{\mathcal{O}}(G, \underline{M})$$

induced from the connecting homomorphism is an isomorphism.

The extension (3.3) is called the *universal extension* of G. It commutes with base change.

Proof. Without the strict $\mathcal{O}$-action, the result is well-known. Fix a natural number n such that $\pi^n R = 0$. Let $G_n = G(\pi^n)$. Let $I = \mathrm{Ker}(\mathcal{O} \otimes R \to R)$.

In the category of formal groups (i.e., without $\mathcal{O}$-action), there exists a universal extension of G given by

$$0 \to \underline{\omega_{G_n^*}} \to L^u \to G \to 0.$$

Note that G is an $\mathcal{O}$-module, thus $I.\,\mathrm{Lie}\, L^u \subset \omega_{G_n^*}$. The push-out of this sequence via $\omega_{G_n^*} \to \omega_{G_n^*}^{\mathcal{O}} := \omega_{G_n^*}/I.\,\mathrm{Lie}\, L^u$ induces the following extension

$$0 \to \underline{\omega_{G_n^*}^{\mathcal{O}}} \to L \to G \to 0. \tag{3.5}$$

We check that this extension satisfies the properties in the statement.

First, $\mathrm{Lie}\, L \cong \mathrm{Lie}\, L^u/I.\,\mathrm{Lie}\, L^u$. The induced $\mathcal{O}$-action on L is strict and L is an $\mathcal{O}$-module.

Secondly, for any extension (3.4) of $\mathcal{O}$-modules, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega_{G_n^*}} \arrow[r] \arrow[d] & L^u \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{M} \arrow[r] & E \arrow[r] & G \arrow[r] & 0.
\end{tikzcd}$$

The vertical arrows obviously factor through the sequence (3.5).

Note that the universal extension is compatible with base change because it is compatible with base change in the $\mathbb{Z}_p$-case.

Finally, there exists an isomorphism $\mathrm{Ext}^1(G, \underline{M}) \cong \mathrm{Hom}(\omega_{G_n^*}, M)$ induced from the connection homomorphism in the category of formal groups (cf. Proposition 3.15). The last statement follows easily. $\square$

Remark 3.17. In the sense of Messing / Fargues [6, Definition B.3.2], π-divisible formal $\mathcal{O}$-modules are fppf-sheaves in $\mathcal{O}$-modules with additional structures. This notion is closely related to the notion of π-divisible formal $\mathcal{O}$-modules in this paper. As in [1], one may also prove Proposition 3.16 by a close study of the relations and by [6, Proposition B.3.3, Remarque B.3.6].

Let H be a π-divisible formal $\mathcal{O}$-module over S. Let $\mathfrak{a} \subset S$ be an ideal of S equipped with an $\mathcal{O}$-pd-structure. Let V be a finitely generated locally free S-module. Assume we are given an extension

$$0 \to \underline{V} \to E \to H \to 0.$$

Let $\mathcal{N} \in \mathrm{Nil}_S$. By Definition 1.22, we have an exact sequence

$$t_E \otimes_S \mathfrak{a} \otimes_S \mathcal{N} \to E(\mathcal{N}) \to E(\mathcal{N}/\mathfrak{a}\mathcal{N}) \to 0,$$

where the first arrow is the composition of $\log_G^{-1}(\mathfrak{a} \otimes_S \mathcal{N}) : t_E \otimes_S \mathfrak{a} \otimes_S \mathcal{N} \to E(\mathfrak{a} \otimes_S \mathcal{N})$ and $E(\mathfrak{a} \otimes_S \mathcal{N}) \to E(\mathcal{N})$. The first arrow is injective if $\mathcal{N}$ is a flat S-module, in which case $\mathfrak{a} \otimes_S \mathcal{N} \cong \mathfrak{a}\mathcal{N}$. Note that $\mathrm{Lie}\,\underline{V} \xrightarrow{\exp} \underline{V}$ is an isomorphism. Thus we obtain an exact sequence

$$(V \oplus (\mathfrak{a} \otimes_S t_E)/(\mathfrak{a} \otimes_S V)) \otimes_S \mathcal{N} \to E(\mathcal{N}) \to H(\mathcal{N}/\mathfrak{a}\mathcal{N}) \to 0. \tag{3.6}$$

This sequence is left exact if $\mathcal{N}$ is a flat S-module.

Let $S \to R$ be an $\mathcal{O}$-pd-extension in $\mathrm{Nil}_{\mathcal{O}}$ with kernel $\mathfrak{a}$. We start with a π-divisible formal $\mathcal{O}$-module G over S. Let

$$0 \to \underline{U} \to L \to G \to 0$$

be the universal extension of G, where U is a finite locally free S-module. Denote by G_0 the base change of G to R. The base change of $\underline{U}$ is the functor attached to the R-module $U_0 = U \otimes_S R$. Let $\bar{\rho} : G_0 \to H_0$ be a morphism of π-divisible formal $\mathcal{O}$-modules over R.

Because the universal extension commutes with base change, we obtain a uniquely determined diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{U}_0 \arrow[r] \arrow[d, "\bar{\tau}"] & L_0 \arrow[r] \arrow[d, "\bar{\mu}"] & G_0 \arrow[r] \arrow[d, "\bar{\rho}"] & 0 \\ 0 \arrow[r] & \underline{V}_0 \arrow[r] & E_0 \arrow[r] & H_0 \arrow[r] & 0. \end{tikzcd}$$

Here $\bar{\tau}$ is induced by an R-module morphism $\tau_0 : U_0 \to V_0$, which we denote by the same letter.

Theorem 3.18. *There is a unique morphism of formal $\mathcal{O}$-modules $\mu : L \to E$ which lifts $\bar{\mu}$ and which has the following property:*

Let $\tau : U \to V$ be an arbitrary S-module homomorphism which lifts τ_0 and consider the following diagram

$$\begin{tikzcd} \underline{U} \arrow[r] \arrow[d, "\tau"] & L \arrow[d, "\mu"] \\ \underline{V} \arrow[r] & E. \end{tikzcd}$$

It needs not to be commutative but the difference of the two maps $\underline{U} \to E$ factors as

$$\underline{U} \to \underline{\mathfrak{a} \otimes_S t_E} \xrightarrow{\exp} E.$$

Here the first arrow is induced by an S-module homomorphism $U \to \mathfrak{a} \otimes_S t_E$, the morphism exp *is defined in Section 1.2.5.*

Proof. We begin with the construction of μ. Fix a natural number n with $\pi^n S = 0$. For $\mathcal{N} \in \mathrm{Nil}_S$, we consider the canonical map

$$G(\mathcal{N}) \to G(\mathcal{N}/\mathfrak{a}) = G_0(\mathcal{N}/\mathfrak{a}) \xrightarrow{\bar{\mu}} H_0(\mathcal{N}/\mathfrak{a}). \tag{3.7}$$

We first construct a morphism $t : G \to E$ such that the following diagram is commutative.

$$\begin{tikzcd} G(\pi^n)(\mathcal{N}) \arrow[r] & G(\mathcal{N}) \arrow[r, "\pi^n"] \arrow[d, "t"] & G(\mathcal{N}) \arrow[d] & \\ ((V \oplus \mathfrak{a} \otimes_S t_E)/(\mathfrak{a} \otimes_S V)) \otimes_S \mathcal{N} \arrow[r] & E(\mathcal{N}) \arrow[r] & H_0(\mathcal{N}/\mathfrak{a}\mathcal{N}) \arrow[r] & 0. \end{tikzcd} \tag{3.8}$$

The first arrow of the upper row is injective. Let $\xi \in G(\mathcal{N})$. Let η be the image of ξ in $H_0(\mathcal{N}/\mathfrak{a}\mathcal{N})$ under the map (3.7). Let $\tilde{\eta} \in E(\mathcal{N})$ be a lift of η. We define

$$t(\xi) = \pi^n \tilde{\eta}.$$

It is clear that this is well defined and the diagram is commutative.

Let M denote the S-module $(V \oplus \mathfrak{a} \otimes_S t_E)/(\mathfrak{a} \otimes_S V)$. If $\mathcal{N}$ is a flat S-module, then the first arrow of the lower row of diagram (3.8) is also injective. In this case we obtain a map

$$G(\pi^n)(\mathcal{N}) \to \underline{M}(\mathcal{N}).$$

Inserting for $\mathcal{N}$ the flat augmentation ideal of $G(\pi^n)$ and taking the image of id $\in G(\pi^n)(\mathcal{N})$ we obtain a morphism of functors

$$G(\pi^n) \to \underline{M}.$$

It is a morphism of group functors. Indeed, let $G(\pi^n) = \operatorname{Spec} A$. Since $A \otimes_S A$ is flat, the map

$$G(\pi^n)(A \otimes_S A) \to \underline{M}(A \otimes_S A)$$

is a group homomorphism. Thus $G(\pi^n) \to \underline{M}$ is a morphism of group functors. Moreover, it fits into a commutative diagram

$$\begin{tikzcd} G(\pi^n) \arrow[r] \arrow[d] & G \arrow[d] \\ \underline{M} \arrow[r] & E. \end{tikzcd}$$

By the property of the universal extensions, the morphism factors as

$$G(\pi^n) \to \underline{U} \to \underline{M}.$$

Thus we obtain another commutative diagram

$$\begin{tikzcd} G(\pi^n) \arrow[r] \arrow[d] & G \arrow[d] \\ \underline{U} \arrow[r] & E. \end{tikzcd}$$

By the construction of the universal extension the following diagram is a cofiber product in the abelian category of abelian sheaves.

$$\begin{tikzcd} G(\pi^n) \arrow[r] \arrow[d] & G \arrow[d] \\ \underline{U} \arrow[r] & L. \end{tikzcd}$$

The last two diagram thus provide the desired map $\mu : L \to E$. It enjoys the required properties because $\underline{U} \to \underline{M}$ is induced by an S-module homomorphism.

Finally, we show the uniqueness of μ. Let μ be an arbitrary lifting of $\bar{\mu}$ with the required properties. Note that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & G(\pi^n) \arrow[r] \arrow[d] & G \arrow[r, "\pi^n"] \arrow[d, "\lambda"] & G \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & \underline{U} \arrow[r] & L \arrow[r] & G \arrow[r] & 0. \end{tikzcd}$$

First, we show that the map $\mu \circ \lambda$ coincides with the map t in (3.8). The assumption that μ lifts $\bar{\mu}$ gives us for each $\mathcal{N} \in \mathrm{Nil}_S$ a commutative diagram

$$\begin{tikzcd}
G(\mathcal{N}) \arrow[r, "\pi^n"] \arrow[d, "\lambda"] & G(\mathcal{N}) \arrow[d] \\
L(\mathcal{N}) \arrow[r] \arrow[d, "\mu"] & G(\mathcal{N}/\mathfrak{a}\mathcal{N}) \arrow[d, "\bar{\rho}"] \\
E(\mathcal{N}) \arrow[r] & H(\mathcal{N}/\mathfrak{a}\mathcal{N}).
\end{tikzcd} \tag{3.9}$$

Let $\xi \in G(\mathcal{N})$. Let $\bar{\xi} \in H(\mathcal{N}/\mathfrak{a}\mathcal{N})$ be its image under the two arrows on the right hand side. Let $\eta \in E(\mathcal{N})$ be a preimage of $\bar{\xi}$. Then to show $\mu \circ \lambda = t$, it is equivalent to showing that

$$\mu \circ \lambda(\xi) = \pi^n \eta. \tag{3.10}$$

As sheaves, $\mu \circ \lambda = t : G \to E$ is a local property. We may assume that $\xi = \pi^n \xi_1$ for $\xi_1 \in G(\mathcal{N})$. Thus $\mu \circ \lambda(\xi) = \pi^n(\mu \circ \lambda(\xi_1))$. From this, equation (3.10) is obvious. In particular, we see that $\mu \circ \lambda$ is uniquely determined. Consider the diagram

$$\begin{tikzcd}
G(\pi^n) \arrow[r, "\pi^n"] \arrow[d] & G \arrow[d, "\lambda"] \\
\underline{U} \arrow[r] \arrow[d, "\check{\mu}"] & L \arrow[d, "\mu"] \\
E \arrow[r, "\mathrm{id}"] & E.
\end{tikzcd} \tag{3.11}$$

Here $\check{\mu}$ denotes the restriction of μ. Because the upper square is a push-out in the category of sheaves, it follows that μ is uniquely determined by $\check{\mu}$ and $\mu \circ \lambda$. It remains to show that $\check{\mu}$ is uniquely determined.

Let $\mathcal{N} \in \mathrm{Nil}_S$ be flat as an S-module. Then we obtain a morphism from (3.11)

$$G(\pi^n)(\mathcal{N}) \to \underline{U}(\mathcal{N}) \xrightarrow{\check{\mu}} \underline{M}(\mathcal{N}).$$

Since the composition of the above two arrows is uniquely determined and $\check{\mu}$ is induced from a map of S-modules $U \to M$, $\check{\mu}$ is uniquely determined by Proposition 3.15. The theorem follows. $\square$

3.4. Extensions of Cartier modules and Grothendieck–Messing crystals

In this subsection we follow the idea of [23] to prove the faithfulness of the functor $\mathrm{BT}_{\mathcal{O}}$.

Definition 3.19. Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$ and G a (π-divisible) formal $\mathcal{O}$-module over R with Cartier module M, which we consider as an $\mathbb{E}_{\mathcal{O},S}$-module. Then an *extension* (L, ι, N, κ, M) *of* M *by the* S*-module* L is an exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules

$$0 \to C(L) \xrightarrow{\iota} N \xrightarrow{\kappa} M \to 0,$$

with N a reduced $\mathbb{E}_{\mathcal{O},S}$-module and $\mathfrak{a}N \subset V^0 L$. Here $C(L)$ is defined in Section 1.2.5. For simplicity, we just write (with abuse of notation) (L, N, M) instead of (L, ι, N, κ, M).

Let G, G' be two formal $\mathcal{O}$-modules over R, M, M' their Cartier modules and $\beta : M \to M'$ a morphism over R. Furthermore, let (L, N, M) and (L', N', M') be extensions of M and M' respectively. Then a morphism of extensions $(L, N, M) \to (L', N', M')$ consists of a morphism of S-modules $\varphi : L \to L'$, a morphism of $\mathbb{E}_{\mathcal{O},S}$-modules $u : N \to N'$, and the $\mathbb{E}_{\mathcal{O},R}$-linear morphism β, such that the diagram of $\mathbb{E}_{\mathcal{O},S}$-modules

$$\begin{tikzcd}
0 \arrow[r] & C(L) \arrow[r] \arrow[d, "C(\varphi)"'] & N \arrow[r] \arrow[d, "u"] & M \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & C(L') \arrow[r] & N' \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

is commutative, where $C(\varphi)$ is given by sending $V^i l$ to $V^i \varphi(l)$ for each $i \geq 0$ and $l \in L$.

Definition 3.20. With the above notation, we define the category $\mathrm{Ext}_{1,S\to R}$ by the objects (L, N, M), such that M is the Cartier module of a π-divisible formal $\mathcal{O}$-module over R. The morphisms are those previously described.

We want to show that the universal objects exist in the category $\mathrm{Ext}_{1,S\to R}$. For this purpose, we introduce another category of extensions, which is similar to the one explained in [15, 5.19].

Let $S \to R$ be an $\mathcal{O}$-pd-thickening. We consider sextuples $(\mathbb{W}, \iota, E, \rho, \widetilde{G}, G)$, where $\widetilde{G}$ is a π-divisible formal $\mathcal{O}$-module over S, G its base change to R, E a formal $\mathcal{O}$-module over S, and $\mathbb{W}$ a vector group attached to a finite projective S-module, such that $\iota : \mathbb{W} \to E$ and $\rho : E \to \widetilde{G}$ induce an $\mathcal{O}$-extension of $\widetilde{G}$

$$0 \to \mathbb{W} \to E \to \widetilde{G} \to 0.$$

A morphism $(\mathbb{W}, \iota, E, \rho, \widetilde{G}, G) \to (\mathbb{W}', \iota', E', \rho', \widetilde{G'}, G')$ is a tuple (v, β), where $v : E \to E'$ is a morphism of formal $\mathcal{O}$-modules over S and β a morphism of formal $\mathcal{O}$-modules $G \to G'$ over R, which gives rise to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{W}_R \arrow[r, "\iota_R"'] \arrow[d, "v_0"'] & E_R \arrow[r] \arrow[d, "v_R"'] & G \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & \mathbb{W}'_R \arrow[r, "\iota'_R"'] & E'_R \arrow[r] & G' \arrow[r] & 0, \end{tikzcd}$$

where v_0 is required to be a morphism of vector groups. Furthermore, we require that for each lift of v_0 to a morphism of vector groups $\widetilde{v}_0 : \mathbb{W} \to \mathbb{W}'$ the map

$$\iota' \circ \widetilde{v}_0 - v \circ \iota : \mathbb{W} \to E'$$

factors through

$$\mathbb{W} \xrightarrow{\xi} \underline{\mathfrak{a} \otimes_S \operatorname{Lie} E'} \xrightarrow{\exp_{E'}} E',$$

where ξ is a morphism of vector groups.

Definition 3.21. We define the category $\operatorname{Ext}_{2,S\to R}$ by the above objects and morphisms.

By the same argument in the Remark after [23, Lemma 91], we have the following result.

Proposition 3.22. *Let $S \to R$ be an $\mathcal{O}$-pd-thickening with nilpotent kernel $\mathfrak{a}$. There is an equivalence between $\operatorname{Ext}_{1,S\to R}$ and $\operatorname{Ext}_{2,S\to R}$, such that*

$$\begin{tikzcd} \operatorname{Ext}_{1,S\to R} \arrow[rr] \arrow[d] & & \operatorname{Ext}_{2,S\to R} \arrow[dll] \\ (\pi\text{-divisible formal } \mathcal{O}\text{-modules}/R) & & \end{tikzcd}$$

is commutative.

By Proposition 3.22 and Theorem 3.18, the following result is clear.

Theorem 3.23. *(Cf. [23, Theorem 92].) If $S \to R$ is an $\mathcal{O}$-pd-thickening with nilpotent kernel and G a π-divisible formal $\mathcal{O}$-module over R, then there is a universal extension $(L^{\text{univ}}, N^{\text{univ}}, M_G) \in \operatorname{Ext}_{1,S\to R}$. Here the universality means that, for any π-divisible formal $\mathcal{O}$-module G' over R, any morphism of $\mathbb{E}_{\mathcal{O},R}$-modules $\beta : M_G \to M_{G'}$ and any extension $(L, N, M_{G'}) \in \operatorname{Ext}_{1,S\to R}$, there is a unique morphism*

$$(\varphi, u, \beta) : (L^{\text{univ}}, N^{\text{univ}}, M_G) \to (L, N, M_{G'}).$$

Definition 3.24. With the notation as in the theorem, we define the *Grothendieck–Messing crystal* attached to G on $\mathcal{O}$-pd-thickenings by

$$\mathbb{D}_G(S) = \operatorname{Lie} N^{\mathrm{univ}}.$$

Lemma 3.25. *(Cf. [23, Lemma 93].) Let $S \to R$ be an $\mathcal{O}$-pd-thickening and $\mathcal{P} = (P, Q, F, F_1)$ a nilpotent $\mathcal{O}$-display over R. Let $(\widetilde{P}, \widehat{Q}, F, F_1)$ be the unique window over $\mathcal{W}_{\mathcal{O},S/R}$ which lifts $\mathcal{P}$. The exact sequence of $\mathbb{E}_{\mathcal{O},S}$-modules*

$$0 \to C(\widehat{Q}/I_{\mathcal{O},S}\widetilde{P}) \to \mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}/U \to M(\mathcal{P}) \to 0 \tag{3.12}$$

lies in $\operatorname{Ext}_{1,S\to R}$, where $M(\mathcal{P})$ is given by Proposition 2.18, the second arrow maps $y \in \widehat{Q}$ to $V \otimes F_1 y - 1 \otimes y$, the third arrow is given by the canonical map $\widetilde{P} \to P$, and U is the $\mathbb{E}_{\mathcal{O},S}$-submodule of $\mathbb{E}_{\mathcal{O},S} \otimes_{W_{\mathcal{O}}(S)} \widetilde{P}$ generated by $(F \otimes x - 1 \otimes Fx)_{x\in\widetilde{P}}$.

Moreover, the sequence (3.12) is the universal extension of $\mathrm{BT}(\mathcal{P}, -)$.

Theorem 3.26. *(Cf. [23, Theorem 94].) Let $R \in \mathrm{Nil}_{\mathcal{O}}$. For a nilpotent $\mathcal{O}$-display $\mathcal{P}$ over R and the associated π-divisible formal $\mathcal{O}$-module $G = \mathrm{BT}_{\mathcal{O}}(\mathcal{P}, -)$, there is a canonical isomorphism of crystals on the category of $\mathcal{O}$-pd-thickenings $S \to R$:*

$$\mathcal{D}_{\mathcal{P}} \simeq \mathbb{D}_G.$$

Proof. By Lemma 3.25, we obtain $\mathcal{D}_{\mathcal{P}}(S) = \widetilde{P}/I_{\mathcal{O},S}\widetilde{P} = \mathbb{D}_G(S)$. $\square$

Remark 3.27. (Cf. [23, Corollary 95].) Let $S \to R$ be an $\mathcal{O}$-pd-thickening or a surjection with nilpotent kernel, then the following diagram of categories is Cartesian.

$$\begin{array}{ccc} (\mathrm{ndisp}_{\mathcal{O}}/S) & \longrightarrow & (\pi\text{-divisible formal } \mathcal{O}\text{-modules}/S) \\ \downarrow & & \downarrow \\ (\mathrm{ndisp}_{\mathcal{O}}/R) & \longrightarrow & (\pi\text{-divisible formal } \mathcal{O}\text{-modules}/R). \end{array}$$

In particular, $\mathrm{BT}_{\mathcal{O}}$ is an equivalence for S if and only if it is an equivalence for R.

Now we can prove the faithfulness of the functor $\mathrm{BT}_{\mathcal{O}}$.

Proposition 3.28. *(Cf. [23, Proposition 98].) Let $R \in \mathrm{Nil}_{\mathcal{O}}$. Then $\mathrm{BT}_{\mathcal{O}}$ is faithful.*

Proof. Let $\mathcal{P}$ and $\mathcal{P}'$ be two nilpotent $\mathcal{O}$-displays over R and $\alpha : \mathcal{P} \to \mathcal{P}'$ a morphism between them. If we denote by G and G' the associated π-divisible formal $\mathcal{O}$-modules, then α induces a morphism $a : G \to G'$ and hence a morphism $b : M_G \to M_{G'}$. For each $n \geq 1$, we obtain with $S = W_{\mathcal{O},n}(R)$ and Lemma 3.25 that there is a unique morphism of the above described universal extensions lying over b. Since α induces such a morphism of extensions as well, it must be induced by it. By Theorem 3.14 and Theorem 3.26 we obtain $\mathbb{D}_G(W_{\mathcal{O},n}(R)) = W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P$ and $\mathbb{D}_{G'}(W_{\mathcal{O},n}(R)) = W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P'$ for each $n \geq 1$. If we now apply a to the functor $\mathbb{D}$, we obtain for each $n \geq 1$ a morphism

$$W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P = \mathbb{D}_G(W_{\mathcal{O},n}(R)) \to \mathbb{D}_{G'}(W_{\mathcal{O},n}(R)) = W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P',$$

which is given by $1 \otimes \alpha$. Since we clearly obtain by these morphisms a morphism of the inverse systems $(W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P)_n$ and $(W_{\mathcal{O},n}(R) \otimes_{W_{\mathcal{O}}(R)} P')_n$, we get α back by passing to the projective limit. Hence, the faithfulness follows. □

Remark 3.29. The above argument does not say much about the functor $\mathrm{BT}_{\mathcal{O}}^{(f)}$ or the relations between the categories $(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R)$, $(f-\mathrm{ndisp}_{\mathcal{O}}/R)$, and $(\mathrm{ndisp}_{\mathcal{O}'}/R)$ when $\mathcal{O} \to \mathcal{O}'$ is unramified of degree f.

In Section 3.5, by deformation theory and adapting the ideas of [23] and [10], we show the faithfulness of the functor $\mathrm{BT}_{\mathcal{O}}^{(f)}$ and prove Theorem 1.1 for those $R \in \mathrm{Nil}_{\mathcal{O}'}$, which are complete local rings with perfect residue field and nilpotent nilradical (Theorem 3.30). Then in Section 4, we apply the ideas of [10] to the functors $\Omega_i(\mathcal{O},\mathcal{O}')$ and $\Gamma_i(\mathcal{O},\mathcal{O}')$ and complete the proof of Theorem 1.1 in the way sketched in Remark 1.4. The category $(f-\mathrm{ndisp}_{\mathcal{O}}/R)$ has useful applications. See for example [16].

3.5. More on the functors Ω_i and Γ_i

The main result of this section is the following one.

Theorem 3.30. *Let R be a complete local ring with maximal ideal $\mathfrak{m}$, perfect residue field, nilpotent nilradical, and p nilpotent in R. Then the following assertions hold.*

- *Let $\mathcal{O} \to \mathcal{O}'$ be an unramified extension and R equipped with an additional $\mathcal{O}'$-algebra structure. If $\mathrm{BT}_{\mathcal{O}}$ is an equivalence over R, then $\Omega_1(\mathcal{O},\mathcal{O}')$ is an equivalence of categories. Hence $\mathrm{BT}_{\mathcal{O}}^{(f)}$ is an equivalence.*
- *Let $\mathcal{O} \to \mathcal{O}'$ be an unramified extension and R equipped with an additional $\mathcal{O}'$-algebra structure. If $\mathrm{BT}_{\mathcal{O}}^{(f)}$ is an equivalence over R, then $\Omega_2(\mathcal{O},\mathcal{O}')$ is an equivalence of categories. Hence $\mathrm{BT}_{\mathcal{O}'}$ is an equivalence.*
- *Let $\mathcal{O} \to \mathcal{O}'$ be a totally ramified extension and R equipped with an additional $\mathcal{O}'$-algebra structure. If $\mathrm{BT}_{\mathcal{O}}$ is an equivalence over R, then $\Gamma_2(\mathcal{O},\mathcal{O}')$ is an equivalence of categories. Hence $\mathrm{BT}_{\mathcal{O}'}$ is an equivalence.*

In particular, since $\mathrm{BT}_{\mathbb{Z}_p}$ is an equivalence over R by [10, Proposition 4.4], the functors $\Omega_1(\mathcal{O},\mathcal{O}')$, $\Omega_2(\mathcal{O},\mathcal{O}')$, $\Gamma_2(\mathcal{O},\mathcal{O}')$ are equivalences over R.

The main tool of the proof is deformation theory. We sketch the main ideas following the proofs of [23, Theorem 103] and [10, Proposition 4.1]. First, we prove the following base case.

Proposition 3.31. *Let $R = l$ be a perfect field of characteristic p extending the residue field of $\mathcal{O}'$. Then Theorem 3.30 holds.*

Proof. Consider the case that $\mathcal{O}'$ is unramified over $\mathcal{O}$ and l extends the residue field of $\mathcal{O}'$. We first show the essential surjectivity of $\Omega_1(\mathcal{O}, \mathcal{O}')$. Let $\mathcal{P}_0 = (P_0, Q_0, F_0, F_{10})$ be a nilpotent f-$\mathcal{O}$-display over l. We define for each $i = 1, \dots, f-1$

$$P_i = W_{\mathcal{O}}(l) \otimes_{F^{i-f}, W_{\mathcal{O}}(l)} P_0$$

and consider

$$P = \bigoplus_{i=0}^{f-1} P_i, \quad Q = Q_0 \oplus \bigoplus_{i=1}^{f-1} P_i.$$

The operators F and F_1 are given by

$$F(x_0, 1 \otimes x_1, \dots, 1 \otimes x_{f-1}) = (x_{f-1}, 1 \otimes F_0 x_0, 1 \otimes x_1, \dots, 1 \otimes x_{f-2})$$
$$F_1(y_0, 1 \otimes x_1 \dots, 1 \otimes x_{f-1}) = (x_{f-1}, 1 \otimes F_{10} y_0, 1 \otimes x_1, \dots, 1 \otimes x_{f-2})$$

with $x_i \in P_0$ and $y_0 \in Q_0$. Then $\mathcal{P} = (P, Q, F, F_1)$ is a nilpotent $\mathcal{O}$-display over l. By letting the $\mathcal{O}'$-action of P_0 act on the second factors of the tensor products of the P_i we obtain a strict $\mathcal{O}'$-action of $\mathcal{P}$. It is clear that $\mathcal{P}$ is mapped via $\Omega_1(\mathcal{O}, \mathcal{O}')$ to $\mathcal{P}_0$. The fully faithfulness is easy.

Now consider $\Omega_2(\mathcal{O}, \mathcal{O}')$. Since $u_l : W_{\mathcal{O}}(l) \to W_{\mathcal{O}'}(l)$ is an isomorphism, it is easily seen that $\Omega_2(\mathcal{O}, \mathcal{O}')$ is essentially surjective. Since $\mathrm{BT}_{\mathcal{O}}^{(f)}$ is an equivalence, we need for the fully faithfulness only to show that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{P}_0, \mathcal{P}_{\star 0}) \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{P}', \mathcal{P}'_{\star})$$

is surjective, where $\mathcal{P}_0, \mathcal{P}_{\star 0}$ are nilpotent f-$\mathcal{O}$-displays over l and $\mathcal{P}', \mathcal{P}'_{\star}$ are the respective associated $\mathcal{O}'$-displays over l. This is again fairly obvious.

Let $\mathcal{O}'$ be totally ramified over $\mathcal{O}$ and let l extend the residue field of $\mathcal{O}'$ and $\mathcal{O}$. We consider $\Gamma_2(\mathcal{O}, \mathcal{O}')$ and assume that $\mathrm{BT}_{\mathcal{O}}$ is an equivalence. Let $\mathcal{P} = (P, Q, F, F_1)$ be an $\mathcal{O}$-display over l equipped with a strict $\mathcal{O}'$-action and $\mathcal{P}' = (P', Q', F', F_1')$ its image via $\Gamma_2(\mathcal{O}, \mathcal{O}')$. Note that we have an isomorphism of rings $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l) \simeq W_{\mathcal{O}'}(l)$. Thus the module P' is P interpreted as $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(l)$-module, from which the essential surjectivity follows easily. It is also easy to verify that $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is fully faithful for l, which shows that it is an equivalence. $\square$

Proposition 3.32. *Let $\mathcal{O} \to \mathcal{O}'$ be an unramified extension of rings of integers of non-Archimedean local fields of characteristic $(0, p)$ of degree f and R an $\mathcal{O}'$-algebra with nilpotent nilradical and π' nilpotent in R. Then $\mathrm{BT}_{\mathcal{O}}^{(f)}$ is faithful.*

Proof. Let k' be the residue field of $\mathcal{O}'$.

If $R = l$ is a perfect field extending k', the fully faithfulness of $\mathrm{BT}_{\mathcal{O}}^{(f)}$ follows from Proposition 3.31.

Now let k be any field extending k' and l the algebraic closure of k. If $\mathcal{P}, \mathcal{P}_\star$ are two nilpotent f-$\mathcal{O}$-displays over k, $\mathcal{P}_l, \mathcal{P}_{\star,l}$ the corresponding nilpotent f-$\mathcal{O}$-displays over l obtained by base change and $X, X_\star, X_l, X_{\star,l}$ the corresponding π'-divisible formal $\mathcal{O}'$-modules, then the faithfulness of the $\mathrm{BT}_{\mathcal{O},k}^{(f)}$ functor follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O},k}(\mathcal{P}, \mathcal{P}_\star) \arrow[r, "\mathrm{BT}_{\mathcal{O},k}^{(f)}"] \arrow[d, hook] & \mathrm{Hom}_{\mathcal{O}',k}(X, X_\star) \arrow[d] \\
\mathrm{Hom}_{\mathcal{O},l}(\mathcal{P}_l, \mathcal{P}_{\star,l}) \arrow[r, hook, "\mathrm{BT}_{\mathcal{O},l}^{(f)}"] & \mathrm{Hom}_{\mathcal{O}',l}(X_l, X'_{\star,l}),
\end{tikzcd}$$

where the indices of the Hom-sets should indicate over which $\mathcal{O}'$-algebra we consider them.

Now let R be a reduced $\mathcal{O}'$-algebra with $\pi' R = 0$ and $\mathcal{P}, \mathcal{P}_\star$ two nilpotent f-$\mathcal{O}$-displays over R. We may embed R into a product $\prod_{i \in I} K_i$ of fields, each extending k'. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O},R}(\mathcal{P}, \mathcal{P}_\star) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathcal{O}',R}(X, X_\star) \arrow[d] \\
\prod_{i \in I} \mathrm{Hom}_{\mathcal{O},K_i}(\mathcal{P}_{K_i}, \mathcal{P}_{\star,K_i}) \arrow[r, hook] & \prod_{i \in I} \mathrm{Hom}_{\mathcal{O}',K_i}(X_{K_i}, X_{\star,K_i})
\end{tikzcd}$$

the faithfulness follows for this case.

Now assume that R is an $\mathcal{O}'$-algebra with π nilpotent in R and with nilpotent nil-radical $\mathfrak{a}$. Let $R_1 = R/\mathfrak{a}$ and $\mathcal{P}, \mathcal{P}_\star$ be nilpotent f-$\mathcal{O}$-displays over R. We obtain the injectivity of $\mathrm{Hom}_{\mathcal{O},R}(\mathcal{P}, \mathcal{P}_\star) \to \mathrm{Hom}_{\mathcal{O},R_1}(\mathcal{P}_{R_1}, \mathcal{P}_{\star,R_1})$ by Lemma 3.12. (Here we should define f-$\mathcal{O}$-windows and their Hodge filtrations. But this is straightforward.) With the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O},R}(\mathcal{P}, \mathcal{P}_\star) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathcal{O}',R}(X, X_\star) \arrow[d] \\
\mathrm{Hom}_{\mathcal{O},R_1}(\mathcal{P}_{R_1}, \mathcal{P}_{\star,R_1}) \arrow[r, hook] & \mathrm{Hom}_{\mathcal{O}',R_1}(X_{R_1}, X_{\star,R_1})
\end{tikzcd}$$

the result follows. $\square$

Proposition 3.33. *Let $\mathcal{O} \to \mathcal{O}'$ be an unramified / totally ramified extension, $S \to R$ a surjection of $\mathcal{O}'$-algebras with π' nilpotent in S and nilpotent kernel. Let $\widehat{\mathcal{P}}$ be a nilpotent*

f-$\mathcal{O}$-display over S (for $\Omega_1(\mathcal{O},\mathcal{O}')$) resp. a nilpotent $\mathcal{O}'$-display over S (for $\Omega_2(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$), such that $\widehat{\mathcal{P}}_R$ lies in the image of $\Omega_1(\mathcal{O},\mathcal{O}')_R$ resp. $\Omega_2(\mathcal{O},\mathcal{O}')_R$ resp. $\Gamma_1(\mathcal{O},\mathcal{O}')_R$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')_R$. Then $\widehat{\mathcal{P}}$ lies in the image of the respective functor over S. In particular, if one of the functors $\Omega_1(\mathcal{O},\mathcal{O}')$, $\Omega_2(\mathcal{O},\mathcal{O}')$, $\Gamma_1(\mathcal{O},\mathcal{O}')$ or $\Gamma_2(\mathcal{O},\mathcal{O}')$ is essentially surjective over R, then it is also essentially surjective over S.

Proof. The assertions for $\Gamma_1(\mathcal{O},\mathcal{O}')$ follows from the assertions for $\Omega_i(\mathcal{O},\mathcal{O}')$, so we only consider $\Omega_i(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$.

Let $\mathfrak{a}$ be the kernel of $S \to R$ and $\mathfrak{a}^n = 0$ for an integer $n \geq 0$. By considering the sequence $S/\mathfrak{a}^i$ for $i = 0, \dots, n$ and the $\mathcal{O}'$-algebra surjections $S/\mathfrak{a}^i \to S/\mathfrak{a}^{i-1}$, we may assume that $\mathfrak{a}^2 = 0$. So we may assume that $S \to R$ is an $\mathcal{O}$-pd-thickening. Consider the morphisms of $\mathcal{O}$-frames (see (3.1))

$$W_{\mathcal{O},S} \overset{\alpha_1}{\to} (W_{\mathcal{O}}(S), \tilde{I}, R, \sigma, \sigma_1) \overset{\alpha_2}{\to} W_{\mathcal{O},R}.$$

Applying Theorem 3.10 and Lemma 3.12, we get that the category of nilpotent (f-)$\mathcal{O}$-displays over S is equivalent to the category of nilpotent (f-)$\mathcal{O}$-displays over R equipped with a lift of the Hodge filtration. The same is true for $\mathcal{O}'$. Additionally, the equivalence assertions over $\mathcal{O}$ continue to hold, if we add a strict $\mathcal{O}'$-action to each object and consider only those morphisms respecting the $\mathcal{O}'$-actions. Hence, we obtain commutative diagrams

$$\begin{tikzcd}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/S) \arrow[r, "\Omega_{1,S}"] \arrow[d, equal] & (f-\mathrm{ndisp}_{\mathcal{O}}/S) \arrow[d, equal] \\
(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\alpha_1"] \arrow[d] & (f-\mathrm{ndisp}^{\dagger}_{\mathcal{O}}/R) \arrow[d] \\
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\Omega_{1,R}"] & (f-\mathrm{ndisp}_{\mathcal{O}}/R),
\end{tikzcd}
\qquad
\begin{tikzcd}
(f-\mathrm{ndisp}_{\mathcal{O}}/S) \arrow[r, "\Omega_{2,S}"] \arrow[d, equal] & (\mathrm{ndisp}_{\mathcal{O}'}/S) \arrow[d, equal] \\
(f-\mathrm{ndisp}^{\dagger}_{\mathcal{O}}/R) \arrow[r, "\alpha_2"] \arrow[d] & (\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R) \arrow[d] \\
(f-\mathrm{ndisp}_{\mathcal{O}}/R) \arrow[r, "\Omega_{2,R}"] & (\mathrm{ndisp}_{\mathcal{O}'}/R)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/S) \arrow[r, "\Gamma_{2,S}"] \arrow[d, equal] & (\mathrm{ndisp}_{\mathcal{O}'}/S) \arrow[d, equal] \\
(\mathrm{ndisp}^{\dagger}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\alpha'"] \arrow[d] & (\mathrm{ndisp}^{\dagger}_{\mathcal{O}'}/R) \arrow[d] \\
(\mathrm{ndisp}_{\mathcal{O},\mathcal{O}'}/R) \arrow[r, "\Gamma_{2,R}"] & (\mathrm{ndisp}_{\mathcal{O}'}/R),
\end{tikzcd}$$

where the dagger at each category in the middle of each diagram should indicate the further structure (i.e., the lift of the Hodge filtration) and the horizontal functors are $\Omega_1(\mathcal{O},\mathcal{O}'),\Omega_2(\mathcal{O},\mathcal{O}')$ and $\Gamma_2(\mathcal{O},\mathcal{O}')$ (over S and R) or the functors induced by them (for the α-arrows in the middle of each diagram).

To prove the essential surjectivity of the three functors in the first rows of the diagrams, it suffices to prove the essential surjectivity of the three functors in the middle rows of the diagrams. This is equivalent to showing that lifts of Hodge filtrations on the right hand sides of the diagrams are from lifts of Hodge filtrations on the left hand sides of the diagrams, which follows from the construction. The proposition follows. □

Proof of Theorem 3.30. By Proposition 3.32, we only have to show the essential surjectivity of the corresponding functors in each case. By Proposition 3.33, we may assume that R is reduced. By Proposition 3.31 this is immediate when R is a perfect field of characteristic p extending the residue field of $\mathcal{O}'$. For general reduced complete local $\mathcal{O}'$-algebra R with perfect residue field and p nilpotent in R, applying Proposition 3.33, the equivalences are established for $R/\mathfrak{m}^n$ for each n.

We explain the proof for $\Omega_1(\mathcal{O},\mathcal{O}')$ and the other two cases are similar. Since $\Omega_1(\mathcal{O},\mathcal{O}')$ is compatible with base change, we may take a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over R, make a base change to $R/\mathfrak{m}^n$ for each $n \geq 1$, and obtain a nilpotent f-$\mathcal{O}$-display $\mathcal{P}_{R/\mathfrak{m}^n}$. These nilpotent displays correspond to nilpotent $\mathcal{O}$-displays over $R/\mathfrak{m}^n$ with strict $\mathcal{O}'$-actions and they form an inverse system.

By building the projective limit we obtain an $\mathcal{O}$-display over R with a strict $\mathcal{O}'$-action, say $\mathcal{P}_\star$, which is mapped to $\mathcal{P}$ via $\Omega_1(\mathcal{O},\mathcal{O}')$, when the functor is considered as a functor from general display structures, i.e., not necessarily nilpotent ones.

To show the essential surjectivity of $\Omega_1(\mathcal{O},\mathcal{O}')$, when restricted to nilpotent display structures, it remains to show that $\mathcal{P}_\star$ is nilpotent. Since R is reduced and may be embedded into a product of algebraic closed fields of characteristic p, we may assume that R is an algebraically closed field of characteristic p which extends the residue field of $\mathcal{O}'$. Consider the commutative diagram of $W_{\mathcal{O}}(R)$-modules

$$\begin{tikzcd}
P_\star \arrow[r] \arrow[d, "V_\star^{fN\sharp}"'] & P = P_{\star,0} \arrow[d, "V^{N\sharp}"] \\
W_{\mathcal{O}}(R) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P_\star \arrow[r] \arrow[d] & W_{\mathcal{O}}(R) \otimes_{F^{fN}, W_{\mathcal{O}}(R)} P_{\star,0} \arrow[d] \\
R \otimes_{\mathrm{w}_{fN}, W_{\mathcal{O}}(R)} P_\star \arrow[r, two heads] & R \otimes_{\mathrm{w}_{fN}, W_{\mathcal{O}}(R)} P_{\star,0},
\end{tikzcd}$$

where N is chosen large enough such that the right vertical composite map is zero. The nilpotence of $\mathcal{P}_\star$ follows, since $P_{\star,i} = F_1^i(Q_{\star,0})$ for each $i = 1,\ldots,f-1$ with the usual grading and so the composite map

$$P_\star \overset{V^{f(N+1)\sharp}}{\longrightarrow} W_{\mathcal{O}}(R) \otimes_{F^{f(N+1)}, W_{\mathcal{O}}(R)} P_\star \to R \otimes_{\mathrm{w}_{f(N+1)}, W_{\mathcal{O}}(R)} P_\star$$

is zero. $\square$

4. The stack of truncated f-$\mathcal{O}$-displays

In this section we assume that the reader is familiar with the basic terminology of stacks, as it can be found in [13]. We take the ideas of [10], but instead of applying them to the functors $\mathrm{BT}_{\mathcal{O}}$ or $\mathrm{BT}_{\mathcal{O}}^{(f)}$, we apply them to the functors $\Omega_i(\mathcal{O}, \mathcal{O}')$ and $\Gamma_i(\mathcal{O}, \mathcal{O}')$, where $\mathcal{O} \to \mathcal{O}'$ is an unramified / totally ramified extension of rings of integers of non-Archimedean local fields of characteristic $(0, p)$. The primary ideas are essentially taken from [10], but with the definition of a truncated f-$\mathcal{O}$-display inspired by [12, Chapter 3]. The main result is Proposition 4.10, which completes the proof of Theorem 1.1.

4.1. Truncated f-$\mathcal{O}$-displays

Let R be a π-adic $\mathcal{O}$-algebra. Let n be a positive integer. Denote by $W_{\mathcal{O},n}(R)$ the ring of truncated ramified Witt vectors of length n, by $I_{\mathcal{O},R,n}$ the kernel of w_0. We have an $\mathcal{O}$-algebra morphism ${}^{F_n} : W_{\mathcal{O},n+1}(R) \to W_{\mathcal{O},n}(R)$ induced by the Frobenius on $W_{\mathcal{O}}(R)$ and an ^{F_n}-linear bijective map ${}^{V_n^{-1}} : I_{\mathcal{O},R,n+1} \to W_{\mathcal{O},n}(R)$ induced by the inverse of the Verschiebung of $W_{\mathcal{O}}(R)$.

If $\pi R = 0$, the Frobenius induces an $\mathcal{O}$-algebra endomorphism ^{F_n} of $W_{\mathcal{O},n}(R)$ and the ideal $I_{\mathcal{O},R,n+1}$ of $W_{\mathcal{O},n+1}(R)$ is a $W_{\mathcal{O},n}(R)$-module.

Definition 4.1. Let $f \geq 1$, $\mathcal{O}'$ an unramified extension of $\mathcal{O}$ with degree f and residue field k, R a k-algebra. An *f-$\mathcal{O}$-pre-display over R* is a sextuple $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$, where P and Q are $W_{\mathcal{O}}(R)$-modules with morphisms

$$I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P,$$

and $F : P \to P$ and $F_1 : Q \to P$ are ${}^{F^f}$-linear maps, such that $\iota\varepsilon : I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \to P$ and $\varepsilon(1 \otimes \iota) : I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} Q \to Q$ are the multiplication morphisms and $F_1\varepsilon = {}^{F^{f-1}V^{-1}} \otimes F$.

If P and Q are $W_{\mathcal{O},n}(R)$-modules, we call $\mathcal{P}$ an f-$\mathcal{O}$-pre-display of level n.

A morphism between two f-$\mathcal{O}$-pre-displays $\mathcal{P}$, $\mathcal{P}'$ consists of a tuple of morphisms (α_0, α_1), such that

$$\begin{tikzcd}
I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P \arrow[r, "\varepsilon"] \arrow[d, "1\otimes\alpha_1"'] & Q \arrow[r, "\iota"] \arrow[d, "\alpha_0"'] & P \arrow[d, "\alpha_1"'] \\
I_{\mathcal{O},R} \otimes_{W_{\mathcal{O}}(R)} P' \arrow[r, "\varepsilon'"] & Q' \arrow[r, "\iota'"] & P'
\end{tikzcd}$$

commutes, $\alpha_1 \circ F_1 = F_1' \circ \alpha_0$, and $\alpha_1 \circ F = F' \circ \alpha_1$. It is easily seen that we obtain an abelian category, named $(f - \text{pre-disp}_{\mathcal{O}}/R)$, which contains $(f - \text{disp}_{\mathcal{O}}/R)$ as a full subcategory. We denote the abelian subcategory of f-$\mathcal{O}$-pre-displays of level n by $(f - \text{pre-disp}_{\mathcal{O},n}/R)$.

Definition 4.2. A *truncated pair of level n over R* is a quadruple $\mathcal{B} = (P, Q, \iota, \varepsilon)$, where P and Q are $W_{\mathcal{O},n}(R)$-modules with module morphisms

$$I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P$$

such that

- $\iota\varepsilon : I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} P \to P$ and $\varepsilon(1 \otimes \iota) : I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} Q \to Q$ are the multiplication maps, i.e., they coincide with

$$I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} P \to I_{\mathcal{O},n,R} \otimes_{W_{\mathcal{O},n}(R)} P \xrightarrow{\text{mult}} P$$

  and

$$I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} Q \to I_{\mathcal{O},n,R} \otimes_{W_{\mathcal{O},n}(R)} Q \xrightarrow{\text{mult}} Q,$$

  respectively, where $I_{\mathcal{O},n+1,R} \to I_{\mathcal{O},n,R}$ is the restriction map and mult the multiplication map,
- P is projective and of finite type over $W_{\mathcal{O},n}(R)$,
- $\text{Coker}(\iota)$ is projective over R, and
- There exists an exact sequence

$$0 \to J_{R,n+1} \otimes_R \text{Coker}(\iota) \xrightarrow{\overline{\varepsilon}} Q \xrightarrow{\iota} P \to \text{Coker}(\iota) \to 0,$$

  where $J_{R,n+1}$ is defined as the kernel of the restriction map $W_{\mathcal{O},n+1}(R) \to W_{\mathcal{O},n}(R)$ and $\overline{\varepsilon}$ is induced by ε.

A *normal decomposition* for a truncated pair is a pair of projective $W_{\mathcal{O},n}(R)$-modules (L, T) with $L \subseteq Q$ and $T \subseteq P$, such that

$$L \oplus T \xrightarrow{\iota + \text{id}} P \text{ and } L \oplus (I_{\mathcal{O},R,n+1} \otimes_{W_{\mathcal{O},n}(R)} T) \xrightarrow{\text{id} + \varepsilon} Q$$

are bijective. By the obvious generalization of [12, Lemma 3.3], every ramified truncated pair admits a normal decomposition.

Definition 4.3. A *truncated f-$\mathcal{O}$-display of level n over R* is an f-$\mathcal{O}$-pre-display $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$ of level n over R, such that $(P, Q, \iota, \varepsilon)$ is a truncated pair of level n and the image of F_1 generates P as a $W_{\mathcal{O},n}(R)$-module.

The *rank* of $\mathcal{P}$ is defined to be the rank of P over $W_{\mathcal{O},n}(R)$. We denote the category of truncated f-$\mathcal{O}$-displays of level n over R by $(f - \operatorname{disp}_{\mathcal{O},n}/R)$. This is a full subcategory of the category of f-$\mathcal{O}$-pre-displays of level n over R.

If we are given a truncated pair $(P, Q, \iota, \varepsilon)$ with normal decomposition (L, T), then we have a bijection between the set of pairs (F, F_1) such that $(P, Q, \iota, \varepsilon, F, F_1)$ is a truncated f-$\mathcal{O}$-display and the set of ${}^{F_n^f}$-linear isomorphisms $\Psi : L \oplus T \to P$, such that $\Psi|_L = F_1|_L$ and $\Psi|_T = F|_T$. If L and T are free $W_{\mathcal{O},n}(R)$-modules, then Ψ is described by an invertible matrix with coefficients in $W_{\mathcal{O},n}(R)$. The proof of the bijection is an obvious variation of [23, Lemma 9] and the case, when L and T are free, is a variation of the discussion after that Lemma. We call (L, T, Ψ) a *normal decomposition of* $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$.

Furthermore, we need to remark that morphisms (α_0, α_1) between two truncated f-$\mathcal{O}$-displays of level n, say $\mathcal{P}, \mathcal{P}'$, may be described in a reduced way. If we are given a normal decomposition (L, T) of $\mathcal{P}$, it suffices to know $(\alpha_0|_L, \alpha_1|_T)$, since we obtain by the definition of a morphism that $\alpha_1|_{\iota L} = \iota' \circ \alpha_0|_L$ and $\alpha_0|_{\varepsilon(I_{\mathcal{O},n+1,R} \otimes_{W_{\mathcal{O},n}(R)} T)} = \varepsilon'(1 \otimes \alpha_1|_T)$.

All assertions from Lemma 3.5 to Lemma 3.17 in [12] are true in their obvious generalization, and the proofs are essentially the same. We state two of the results in the following, since we need them in next section.

Fix integers $h \geq 0, f \geq 1$ and the ring $\mathcal{O}$ and denote by $f - \operatorname{Disp}_{\mathcal{O},n} \to \operatorname{Spec} k$ the fibered category of truncated f-$\mathcal{O}$-displays of level n and rank h. Hence, $f - \operatorname{Disp}_{\mathcal{O},n}(\operatorname{Spec} R)$ is the groupoid of truncated f-$\mathcal{O}$-displays of level n and rank h over R. There is an obvious morphism $\tau_{\mathcal{O},n} : f - \operatorname{Disp}_{\mathcal{O},n+1} \to f - \operatorname{Disp}_{\mathcal{O},n}$ induced by the truncation functors.

Lemma 4.4. *(Cf. [12, Proposition 3.15].) The fibered category $f - \operatorname{Disp}_{\mathcal{O},n}$ is a smooth Artin algebraic stack with affine diagonal. The truncation morphism $f - \operatorname{Disp}_{\mathcal{O},n+1} \to f - \operatorname{Disp}_{\mathcal{O},n}$ is smooth and surjective.*

Proof. By the generalization of [12, Proposition 3.14], we know that $f - \operatorname{Disp}_{\mathcal{O},n}$ is an fpqc stack. To show the affineness of the diagonal, we have to show that for truncated f-$\mathcal{O}$-displays $\mathcal{P}_1$ and $\mathcal{P}_2$ of level n and rank h over a k-algebra R, the sheaf $\underline{\operatorname{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$ is represented by an affine scheme. By passing to an open cover of $\operatorname{Spec} R$, we may assume that $\mathcal{P}_1$ and $\mathcal{P}_2$ have normal decompositions with free modules. The homomorphisms of the underlying truncated pairs are clearly represented by an affine scheme. Commuting with F and F_1 is a closed condition and a homomorphism of truncated pairs is an isomorphism if and only if it induces isomorphisms on $\operatorname{Coker}(\iota)$ and $\operatorname{Coker}(\varepsilon)$, which is equivalent to saying that two determinants are invertible. Hence, $\underline{\operatorname{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$ is represented by an affine scheme.

For each integer d with $0 \leq d \leq h$, let $f - \operatorname{Disp}_{\mathcal{O},n,d}$ be the substack of $f - \operatorname{Disp}_{\mathcal{O},n}$ where $\operatorname{Coker}(\iota)$ has rank d. We define the functor $X_{\mathcal{O},n,d}$ from the category of affine k-schemes to (Sets) by defining $X_{\mathcal{O},n,d}(\operatorname{Spec} R)$ as the set of invertible $W_{\mathcal{O},n}(R)$-matrices of rank h. Hence, $X_{\mathcal{O},n,d}$ is an affine open subscheme of the affine space of dimension nh^2

over k. We now define the morphism $\pi_{\mathcal{O},n,d} : X_{\mathcal{O},n,d} \to f - \mathrm{Disp}_{\mathcal{O},n,d}$ in the way that $\pi_{\mathcal{O},n,d}(M)$ is the truncated f-$\mathcal{O}$-display given by the normal representation (L, T, Ψ), where $L = W_{\mathcal{O},n}(R)^{h-d}$, $T = W_{\mathcal{O},n}(R)^d$, and M is the matrix representation of Ψ. We define the sheaf of groups $G_{\mathcal{O},n,d}$ by associating to each k-algebra R the group of invertible matrices $\binom{AB}{CD}$ with $A \in \mathrm{Aut}(L)$, $B \in \mathrm{Hom}(T, L)$, $C \in \mathrm{Hom}(L, I_{\mathcal{O},R,n+1} \otimes_{W_{\mathcal{O},n}(R)} T)$ and $D \in \mathrm{Aut}(T)$, where L and T are as above. $G_{\mathcal{O},n,d}$ is an affine open subscheme of the affine space of dimension nh^2 over k and $\pi_{\mathcal{O},n,d}$ is a $G_{\mathcal{O},n,d}$-torsor. So we see that $f - \mathrm{Disp}_{\mathcal{O},n,d}$ and $f - \mathrm{Disp}_{\mathcal{O},n}$ are smooth algebraic stacks over k. The truncation morphism $\tau_{\mathcal{O},n}$ is smooth and surjective because it commutes with the obvious projection $X_{\mathcal{O},n+1,d} \to X_{\mathcal{O},n,d}$, which is smooth and surjective. $\square$

For a truncated f-$\mathcal{O}$-display $\mathcal{P}$ of level n over a k-algebra R there is a unique morphism $V^\sharp : P \to P^{(1)} = W_{\mathcal{O},n}(R) \otimes_{F_n^f, W_{\mathcal{O},n}(R)} P$ with $V^\sharp(F_1(x)) = 1 \otimes x$ for all $x \in Q$. The proof of this is similar to the one of Lemma 2.2. $V^\sharp$ is compatible with truncation. We call $\mathcal{P}$ *nilpotent*, if there is an m, such that the m-th fold iteration of $V^\sharp$, i.e., the composite morphism $P \to P^{(1)} \to \ldots \to P^{(m)}$, is zero. Because $I_{\mathcal{O},R,m}$ is nilpotent, $\mathcal{P}$ is nilpotent if and only if its truncation to level 1 is nilpotent. An f-$\mathcal{O}$-display over R is nilpotent if and only if all its truncations are nilpotent.

Lemma 4.5. *(Cf. [12, Lemma 3.17].) There is a unique reduced closed substack $f - \mathrm{nDisp}_{\mathcal{O},n} \subset f - \mathrm{Disp}_{\mathcal{O},n}$ such that the geometric points of $f - \mathrm{nDisp}_{\mathcal{O},n}$ are precisely the nilpotent truncated f-$\mathcal{O}$-displays of level n. We have the Cartesian diagram*

$$\begin{tikzcd} f - \mathrm{nDisp}_{\mathcal{O},n+1} \arrow[r] \arrow[d] & f - \mathrm{nDisp}_{\mathcal{O},n} \arrow[d] \\ f - \mathrm{Disp}_{\mathcal{O},n+1} \arrow[r] & f - \mathrm{Disp}_{\mathcal{O},n} \end{tikzcd}$$

and the morphism $f - \mathrm{nDisp}_{\mathcal{O},n} \to f - \mathrm{Disp}_{\mathcal{O},n}$ is of finite presentation. In particular, $f - \mathrm{nDisp}_{\mathcal{O},n+1} \to f - \mathrm{nDisp}_{\mathcal{O},n}$ is smooth and essentially surjective on R-valued points for every R.

Proof. The diagram is the obvious generalization of [12, Lemma 3.17]. The last assertion follows from Lemma 4.4. $\square$

4.2. Applications to f-$\mathcal{O}$-displays

Proposition 4.6. *(Cf. [10, Proposition 1.2].) Let $f \geq 1$ and $\mathcal{O}$ be given. For any positive integer h there is a sequence of finitely generated reduced k-algebras $B_1 \to B_2 \to \ldots$ with faithfully flat smooth maps and a nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ of rank h over $B = \bigcup B_i$ with the property that for any other nilpotent f-$\mathcal{O}$-display $\mathcal{P}'$ over a reduced k-algebra R and* *of rank h, there exist a sequence $R \to S_1 \to S_2 \to \dots$ of faithfully flat étale k-algebra morphisms and a k-algebra morphism $B \to S = \bigcup S_i$ such that $\mathcal{P}_S \cong \mathcal{P}'_S$.*

Proof. We construct recursively an infinite commutative diagram

$$\begin{tikzcd} Y_1 \arrow[d] & Y_2 \arrow[l] \arrow[d] & Y_3 \arrow[l] \arrow[d] & \dots \arrow[l] \\ f-\mathrm{nDisp}_{\mathcal{O},1} & f-\mathrm{nDisp}_{\mathcal{O},2} \arrow[l] & f-\mathrm{nDisp}_{\mathcal{O},3} \arrow[l] & \dots, \arrow[l] \end{tikzcd}$$

where $Y_m = \operatorname{Spec} B_m$ for a finitely generated k-algebra B_m, such that $Y_1 \to f-\mathrm{nDisp}_{\mathcal{O},1}$ and $Y_{m+1} \to \chi_{m+1} = f - \mathrm{nDisp}_{\mathcal{O},m+1} \times_{f-\mathrm{nDisp}_{\mathcal{O},m}} Y_m$ are smooth presentations. By Lemma 4.5 the morphisms $B_m \to B_{m+1}$ are faithfully flat and smooth. We have a canonical nilpotent f-$\mathcal{O}$-display $\mathcal{P}$ over $B = \varinjlim B_m$.

A nilpotent f-$\mathcal{O}$-display $\mathcal{P}'$ over a reduced k-algebra R is equivalent to a compatible system of morphisms $\operatorname{Spec} R \to f-\mathrm{nDisp}_{\mathcal{O},m}$. For $\operatorname{Spec} S_1 = \operatorname{Spec} R \times_{f-\mathrm{nDisp}_{\mathcal{O},1}} Y_1$, there is a natural map $\operatorname{Spec} S_1 \to \chi_2$. For $m \geq 2$ we have for $\operatorname{Spec} S_m = \operatorname{Spec} S_{m-1} \times_{\chi_m} Y_m$ a natural map $\operatorname{Spec} S_m \to \chi_{m+1}$. Hence we obtain compatible isomorphisms $\tau_n(\mathcal{P})_S \cong \tau_n(\mathcal{P}')_S$ over $S = \bigcup S_n$, where τ_n should be the truncation morphisms. Therefore $\mathcal{P}_S \cong \mathcal{P}'_S$. Because a surjective smooth morphism has a section étale locally, we may replace the S_n by an étale system. $\square$

Definition 4.7. (Cf. [10, Definition 5.4].) A nilpotent f-$\mathcal{O}$-display over a k-algebra R is *of reduced type* if all its truncations are in $f - \mathrm{nDisp}_{\mathcal{O},m}$.

Proposition 4.8. *(Cf. [10, Lemma 5.5].) A nilpotent f-$\mathcal{O}$-display over a k-algebra R is of reduced type, if and only if there are k-algebra morphisms $R \to S \leftarrow A$ with A reduced, $S = \bigcup S_i$ for a system of étale faithfully flat k-algebra morphisms $R \to S_1 \to S_2 \to \dots$, and the base change of this f-$\mathcal{O}$-display to S descends to A.*

Proof. This follows from the definition and Proposition 4.6. $\square$

Definition 4.9. We call a faithfully flat morphism of $\mathcal{O}'$-algebras $R \to S$ an *admissible covering*, if $S \otimes_R S$ is reduced.

All assertions we will need about admissible coverings can be found in [10, Chapter 3], where the ring morphisms have to be replaced by $\mathcal{O}'$-algebra morphisms. The proof of [10, Proposition 3.4] depends on [10, Lemma 3.3], which is not correct. In [12, Section 8.2] it is clarified, how to prove the Proposition without using this Lemma.

Proposition 4.10. *(Cf. [10, Proposition 4.4, Lemma 6.1].) Let $\mathcal{O} \to \mathcal{O}'$ be an unramified / totally ramified extension. Assume that $\Omega_1(\mathcal{O}, \mathcal{O}'), \Omega_2(\mathcal{O}, \mathcal{O}'), \Gamma_1(\mathcal{O}, \mathcal{O}')$, or $\Gamma_2(\mathcal{O}, \mathcal{O}')$ is* *fully faithful for all $\mathcal{O}'$-algebras with π' nilpotent in them, then the respective functor is an equivalence for all such algebras.*

Proof. It remains to show that $\Omega_1(\mathcal{O},\mathcal{O}')$, $\Omega_2(\mathcal{O},\mathcal{O}'), \Gamma_1(\mathcal{O},\mathcal{O}')$ resp. $\Gamma_2(\mathcal{O},\mathcal{O}')$ is essentially surjective for all $\mathcal{O}'$-algebras R with π' nilpotent in R. We treat only the $\Omega_1(\mathcal{O},\mathcal{O}')$-case, since the others follow analogously.

First we show the assertion for all reduced k-algebras R, where k is the residue field of $\mathcal{O}'$. Let $\mathcal{P}$ be a nilpotent f-$\mathcal{O}$-display over R. With $R \to S \leftarrow B$ given as in Proposition 4.6, $\mathcal{P}_S$ descends to B. Since $R \to S$ is an admissible covering, it is enough to show that $\Omega_1(\mathcal{O},\mathcal{O}')$ is essentially surjective over B (Proposition 2.11).

When k' is an uncountable algebraically closed field of characteristic p extending k, then $B \to B \otimes_k k'$ is an admissible covering and we may apply [10, Proposition 3.2] to $B \otimes_k k' = \bigcup B_i \otimes_k k'$. So we may reduce to the base ring $\prod (B \otimes_k k')_{\mathfrak{m}}$, where the product runs through all maximal ideals $\mathfrak{m}$ of $B \otimes_k k'$. We may reduce to $(B \otimes_k k')_{\mathfrak{m}}$, since nilpotent f-$\mathcal{O}$-displays are compatible with arbitrary products of reduced local $\mathcal{O}'$-algebras. The residue field of $(B \otimes_k k')_{\mathfrak{m}}$ is k' by [10, Lemma 4.3] and we may apply [10, Proposition 3.4] to consider just the completion of $(B \otimes_k k')_{\mathfrak{m}}$, which is a reduced complete local ring with perfect residue field. The assertion then follows from Theorem 3.30.

Now we consider general $\mathcal{O}'$-algebras R with π' nilpotent in R. By Proposition 3.33, it suffices to treat the case, where R is a k-algebra. Let $\mathcal{P}$ be a nilpotent f-$\mathcal{O}$-display over R. Because $f-\mathrm{Disp}_{\mathcal{O},1}$ is of finite type and $f-\mathrm{nDisp}_{\mathcal{O},1} \to f-\mathrm{Disp}_{\mathcal{O},1}$ is finitely presented, $\mathcal{P}$ is of reduced type modulo a nilpotent ideal. We may assume by Proposition 3.33 that $\mathcal{P}$ is of reduced type. Now let $R \to S \leftarrow A$ be as in Proposition 4.8. Because $\Omega_1(\mathcal{O},\mathcal{O}')$ is fully faithful, it suffices to show that $\mathcal{P}_S$ lies in the image of $\Omega_1(\mathcal{O},\mathcal{O}')$, which holds, since A is reduced and $\mathcal{P}_S$ descends to A. Therefore the proposition follows. □

5. Dieudonné $\mathcal{O}$-displays

Let R be a complete Noetherian local $\mathcal{O}$-algebra with perfect residue field of characteristic p and $p \geq 3$. For each G, a π-divisible $\mathcal{O}$-module over R, there is a connected-etale sequence

$$0 \to G^0 \to G \to G^{et} \to 0, \tag{5.1}$$

where G^0 is connected and G^{et} is etale (cf. [17]). The connected part can be described by nilpotent $\mathcal{O}$-displays by Theorem 1.1. In this section, we introduce Dieudonné $\mathcal{O}$-displays and describe the sequence (5.1) in a similar manner. Since we define things in an appropriate way, most of the results in [22] and [9] can be generalized to Dieudonné $\mathcal{O}$-displays. We explain the main results and constructions in the following and refer to [22] and [9] for the detail proofs.

5.1. Basic properties of Dieudonné $\mathcal{O}$-displays

Definition 5.1. A *Dieudonné $\mathcal{O}$-display* over R is a quadruple $(P, Q, F, \dot{F})$, where P is a finitely generated free $\widehat{W}_{\mathcal{O}}(R)$-module, $Q \subset P$ is a $\widehat{W}_{\mathcal{O}}(R)$-submodule, and $F : P \to P$, $F_1 : Q \to P$ are ^{F}-linear maps, such that

(1) $\widehat{I}_{\mathcal{O},R}P \subset Q \subset P$ and P/Q is a free R-module.
(2) $F_1 : Q \to P$ is an ^{F}-linear epimorphism.
(3) For any $x \in P$ and $w \in \widehat{W}_{\mathcal{O}}(R)$, we have

$$F_1({}^V wx) = wFx.$$

Denote by $\mathrm{Ddisp}_{\mathcal{O}}/R$ the category of Dieudonné $\mathcal{O}$-displays over R. If $\mathcal{O} = \mathbb{Z}_p$, then a Dieudonné $\mathcal{O}$-display is the same as a Dieudonné display as defined in [22, Definition 1]. See Section 1.2.1 for the precise definition of $\widehat{W}_{\mathcal{O}}$ and $\widehat{I}_{\mathcal{O}}$.

Let $S \to R$ be an $\mathcal{O}$-algebra morphism. The *base change* of a Dieudonné $\mathcal{O}$-display with respect to $S \to R$ is defined by the same formulae as in Definition 2.5 by replacing $W_{\mathcal{O}}$ by $\widehat{W}_{\mathcal{O}}$.

We construct a functor from the category of Dieudonné $\mathcal{O}$-displays over R to the category of $\mathcal{O}$-displays over R. Let $\mathcal{P} = (P, Q, F, F_1)$ be a Dieudonné $\mathcal{O}$-display over R, define $\mathcal{F}(\mathcal{P}) = (P', Q', F, F_1)$, where $P' = W_{\mathcal{O}}(R) \otimes_{\widehat{W}_{\mathcal{O}}(R)} P$, $Q' = \mathrm{Ker}(W_{\mathcal{O}}(R) \otimes_{\widehat{W}_{\mathcal{O}}(R)} P \to P/Q)$, and the operators $F : P' \to P'$ and $F_1 : Q' \to P$ are defined as follows:

- $F(\xi \otimes x) = {}^F\xi \otimes Fx$ $\xi \in W(R)$, $x \in P$
- $F_1(\xi \otimes y) = {}^F\xi \otimes F_1 y$ $\xi \in W(R)$, $y \in Q$
- $F_1({}^V\xi \otimes x) = \xi \otimes Fx$ $\xi \in W(R)$, $x \in P$.

It is easy to check that $\mathcal{F}(\mathcal{P})$ is an $\mathcal{O}$-display over R. A Dieudonné $\mathcal{O}$-display $\mathcal{P}$ over R is *nilpotent* if $\mathcal{F}(\mathcal{P})$ is a nilpotent $\mathcal{O}$-display. Following the same argument in [22, Section 2], we have the following result.

Proposition 5.2. *(Cf. [22, Theorem 5 and Corollary 6].) Let R be a Noetherian complete local $\mathcal{O}$-algebra with perfect residue field of characteristic p. The following three categories are equivalent.*

- *The category of nilpotent Dieudonné $\mathcal{O}$-displays over R.*
- *The category of nilpotent $\mathcal{O}$-displays over R.*
- *The category of π-divisible formal $\mathcal{O}$-modules over R.*

Definition 5.3. Let $\mathcal{P} = (P, Q, F, F_1)$ be a Dieudonné $\mathcal{O}$-display over R. We say that $\mathcal{P}$ is *etale*, if $Q = P$. We say that $\mathcal{P}$ is *multiplicative*, if $Q = \widehat{I}_{\mathcal{O},R}P$.

Applying [22, Lemmas 6, 8 and 9] to the case $A = \widehat{W}_{\mathcal{O}}(R)$, $\tau = {}^F$, $M = P$, $\phi = F : P \to P$, we may rewrite [22, Propositions 16 and 17] for Dieudonné $\mathcal{O}$-displays over R.

Proposition 5.4. *Let $\mathcal{P} = (P, Q, F, F_1)$ be a Dieudonné $\mathcal{O}$-display over R.*

(1) *There is a morphism $\mathcal{P} \to \mathcal{P}^{et}$ to an etale Dieudonné $\mathcal{O}$-display over R, such that any other morphism to an etale Dieudonné $\mathcal{O}$-display $\mathcal{P} \to \mathcal{P}_1$ factors uniquely through $\mathcal{P}^{et}$. Moreover $\mathcal{P}^{et}$ has the following properties:*
   - *the induced map $P \to P^{et}$ is surjective;*
   - *let P^{nil} be the kernel of $P \to P^{et}$. Then $(P^{nil}, P^{nil} \cap Q, F, F_1)$ is a nilpotent Dieudonné $\mathcal{O}$-display over R.*

(2) *There is a morphism from a multiplicative Dieudonné $\mathcal{O}$-display over R $\mathcal{P}^{mult}$ to $\mathcal{P}$, such that any other morphism from a multiplicative Dieudonné $\mathcal{O}$-display $\mathcal{P}_1 \to \mathcal{P}$ factors uniquely through $\mathcal{P}^{mult}$. Moreover $\mathcal{P}^{mult}$ has the following properties:*
   - *the induced map $P^{mult} \to P$ is injective and $P^{mult} \cap Q = \widehat{I}_{\mathcal{O},R} P^{mult}$;*
   - *$(P/P^{mult}, Q/\widehat{I}_{\mathcal{O},R} P^{mult}, F, F_1)$ is an F-nilpotent Dieudonné $\mathcal{O}$-display over R.*

5.2. Dieudonné $\mathcal{O}$-displays and π-divisible $\mathcal{O}$-modules

The same argument as in [22, Section 4] shows that there is an equivalence between Dieudonné $\mathcal{O}$-displays and π-divisible $\mathcal{O}$-modules. We sketch the main ideas in the following.

Let R be an Artinian local $\mathcal{O}$-algebra with perfect residue field k. We will denote by $\bar{R}$ the unramified extension of R with residue field $\bar{k}$. We write $\Gamma = \mathrm{Gal}(\bar{k}/k)$ for the Galois group. Then Γ acts continuously on the discrete module $\bar{R}$.

Let H be a finitely generated $\mathcal{O}$-module. Assume that we are given an action of Γ on H, which is continuous with respect to the π-adic topology on H. The action of Γ on $\widehat{W}_{\mathcal{O}}(\bar{R})$ and H induces an action on $\widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\mathcal{O}} H$. We set

$$P(H) = (\widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\mathcal{O}} H)^{\Gamma}.$$

As remarked in [22], one can show by reduction to the case $R = k$ that $P(H)$ is a finitely generated free $\widehat{W}_{\mathcal{O}}(R)$-module and that the natural map

$$\widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\widehat{W}_{\mathcal{O}}(R)} P(H) \to \widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\mathcal{O}} H$$

is an isomorphism. We define an etale Dieudonné $\mathcal{O}$-display over R

$$\mathcal{P}(H) = (P(H), Q(H), F, F_1)$$

where $Q(H) = P(H)$ and F_1 is induced by the map

$$\widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\mathcal{O}} H \to \widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\mathcal{O}} H$$
$$w \otimes h \mapsto {}^{F}w \otimes h.$$

Conversely, if $\mathcal{P}$ is an etale Dieudonné $\mathcal{O}$-display over R, we define $H(\mathcal{P})$ to be the kernel of the homomorphism of $\mathcal{O}$-modules

$$F_1 - \mathrm{id} : \widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\widehat{W}_{\mathcal{O}}(R)} P \to \widehat{W}_{\mathcal{O}}(\bar{R}) \otimes_{\widehat{W}_{\mathcal{O}}(R)} P,$$

which is an $\mathcal{O}[\Gamma]$-module. We see that the category of etale Dieudonné $\mathcal{O}$-displays over R is equivalent to the subcategory of the category of continuous $\mathcal{O}[\Gamma]$-modules, of which the objects are free and finitely generated over $\mathcal{O}$.

Proposition 5.5. *(Cf. [22, Proposition 18].) Let $\mathcal{P} = (P, Q, F, F_1)$ be a nilpotent Dieudonné $\mathcal{O}$-display over R. Let us denote by $C_{\bar{R}}$ the cokernel of the map $F_1 - \mathrm{id} : Q_{\bar{R}} \to P_{\bar{R}}$ with its natural structure of a Γ-module. Then we have a natural isomorphism*

$$\mathrm{Hom}_{\Gamma,\mathcal{O}}(H, C_{\bar{R}}) \cong \mathrm{Ext}^1(\mathcal{P}(H), \mathcal{P}).$$

Let H be a continuous $\mathcal{O}[\Gamma]$-module, which is free and finitely generated over $\mathcal{O}$. We define a Barsotti–Tate group

$$\mathrm{BT}_{\mathcal{O}}(H) = \varinjlim_n G_n,$$

where G_n is the finite etale group scheme corresponds to the finite Γ-module $\pi^{-n}H/H$.

Proposition 5.6. *(Cf. [22, Proposition 19].) Let H be as above and let G be a formal π-divisible $\mathcal{O}$-module over R. Then there is a canonical isomorphism*

$$\mathrm{Hom}_{\Gamma,\mathcal{O}}(H, G(\bar{R})) \cong \mathrm{Ext}^1(\mathrm{BT}_{\mathcal{O}}(H), G).$$

Combine the above results, we obtain the following result, which corresponds to [22, Theorem 20].

Theorem 5.7. *Let R be a Noetherian complete local $\mathcal{O}$-algebra with perfect residue field of characteristic p. There is a functor $\mathrm{BT}_{\mathcal{O}}$ from the category of Dieudonné $\mathcal{O}$-displays over R to the category of π-divisible $\mathcal{O}$-modules over R which is an equivalence of categories. On the subcategory of nilpotent Dieudonné $\mathcal{O}$-displays, this is the equivalence in Proposition 5.2.*

We explain that the equivalence in Theorem 5.7 is compatible with duality. First, we recall the definition of dual Dieudonné $\mathcal{O}$-displays. Let $\mathcal{G}_m = (\widehat{W}_{\mathcal{O}}(R), \widehat{I}_{\mathcal{O},R}, {}^{F}, {}^{V^{-1}})$. If $\mathcal{P}$ and $\mathcal{P}'$ are Dieudonné $\mathcal{O}$-displays over R, a bilinear form $\alpha : \mathcal{P} \times \mathcal{P}' \to \mathcal{G}_m$ is a $\widehat{W}_{\mathcal{O}}(R)$-bilinear map $\alpha : P \times P' \to \widehat{W}_{\mathcal{O}}(R)$ such that

$$ {}^{V}(\alpha(F_1x, F_1'x')) = \alpha(x, x') $$

for $x \in Q$, $x' \in Q'$. This implies the following equations:

(1) $\alpha(F_1x, F'y') = {}^{F}(\alpha(x, y'))$
(2) $\alpha(Fy, F_1'x') = {}^{F}(\alpha(y, x'))$
(3) $\alpha(Fy, F'y) = \pi^{F}(\alpha(y, y'))$

for $x \in Q$, $x' \in Q'$, $y \in P$, $y \in P'$. Let $\mathrm{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{G}_m)$ denote the abelian group of bilinear forms.

Definition 5.8. Let $\mathcal{P}$ be a Dieudonné $\mathcal{O}$-display over R. The contravariant functor $\mathcal{P}' \mapsto \mathrm{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{G}_m)$ is represented by a Dieudonné $\mathcal{O}$-display $\mathcal{P}^t$ over R, called the *dual* of $\mathcal{P}$.

More precisely, $\mathcal{P}^t$ can be described as follows. Let $\mathcal{P} = (P, Q, F, F_1)$, then

$$ \mathcal{P}^t = (P^\vee, \tilde{Q}, F', F_1') $$

where $\tilde{Q} = \{x \in P^\vee \mid x(Q) \subset \widehat{I}_{\mathcal{O},R}\}$ and $M^\vee = \mathrm{Hom}_{\widehat{W}_{\mathcal{O}}(R)}(M, \widehat{W}_{\mathcal{O}}(R))$ for any $\widehat{W}_{\mathcal{O}}(R)$-module M. If $P = L \oplus T$ is a normal decomposition and $Q = L \oplus \widehat{I}_{\mathcal{O},R}T$, then $P^\vee = L^\vee \oplus T^\vee$ and $\tilde{Q} = \widehat{I}_{\mathcal{O},R}L^\vee \oplus T^\vee$. In particular, taking duals sends etale Dieudonné $\mathcal{O}$-displays to multiplicative Dieudonné $\mathcal{O}$-displays and vice versa.

Let $\mathbb{G}_{\mathcal{O}}$ be the $\mathcal{O}$-module attached to $\mathcal{G}_m$. Let $\mathbb{G}_{\mathcal{O}}[\pi^n]$ be the π^n-torsion of $\mathbb{G}_{\mathcal{O}}$. For a π-divisible $\mathcal{O}$-module G over R, the *Serre $\mathcal{O}$-dual* (or *special $\mathcal{O}$-dual*) $G^\vee$ of G is defined in the same way as the Serre dual of G, by using $\mathbb{G}_{\mathcal{O}}$ and $\mathbb{G}_{\mathcal{O}}[\pi^n]$ instead of $\mathbb{G}_m$ and $\mu_{p^n} = \mathbb{G}_m[p^n]$.

Theorem 5.9. *With the notation as above, for every Dieudonné $\mathcal{O}$-display $\mathcal{P}$ over R, the two $\mathcal{O}$-modules $\mathrm{BT}_{\mathcal{O}}(\mathcal{P}^t)$ and $\mathrm{BT}_{\mathcal{O}}(\mathcal{P})^\vee$ are isomorphic.*

Proof. The proof is the same as the proof of [9, Theorem 3.4]. We explain the main steps in the following.

Let $\mathcal{C}_R$ be the category of all R-algebras S with the properties: (1) the nilradical $\mathcal{N}(S)$ is a nilpotent ideal; (2) $\mathfrak{m}S \subset \mathcal{N}(S)$; (3) $S/\mathcal{N}(S)$ is a union of finite dimensional k-algebras. Let $\tilde{\mathcal{C}}_R$ be the category of abelian sheaves on $\mathcal{C}_R^{op}$ for the flat topology, i.e., coverings are faithfully flat R-algebra homomorphisms in $\mathcal{C}_R$. Suppose that R is an $\mathcal{O}$-algebra. Let L be the fraction field of $\mathcal{O}$. Let $\mathcal{P} = (P, Q, F, F_1)$ be a Dieudonné $\mathcal{O}$-display over R. Then base change of Dieudonné $\mathcal{O}$-displays makes P and Q into abelian sheaves on $\mathcal{C}_R^{op}$. If $\mathcal{P} = (P, Q, F, F_1)$ is a Dieudonné $\mathcal{O}$-display over R, we define

$$ Z(\mathcal{P}) = [Q \xrightarrow{F_1 - \mathrm{id}} P] $$

as a complex in $\tilde{\mathcal{C}}_R$ in degree 0, 1 and

$$\mathrm{BT}_{\mathcal{O}}^{L}(\mathcal{P}) = Z(\mathcal{P}) \otimes^{L} L/\mathcal{O}$$

in the derived category $D(\tilde{\mathcal{C}}_R)$.

Arguing as in [9, Theorem 1.7], the functor $\mathrm{BT}_{\mathcal{O}}^{L}$ induces an equivalence between the category $\mathrm{Ddisp}_{\mathcal{O}}/R$ and the category of π-divisible $\mathcal{O}$-modules over R. This equivalence coincides with the equivalence defined by the functor $\mathrm{BT}_{\mathcal{O}}$ in Theorem 5.7. Note that $\mathrm{Hom}(\widehat{W}_{\mathcal{O}}, \mathbb{G}_{\mathcal{O}})$ is the Cartier $\mathbb{E}_{\mathcal{O}}$-module of the formal group $\mathbb{G}_{\mathcal{O}}$. By Proposition 2.18, for any $R \in \mathrm{Nil}_{\mathcal{O}}$ and $B \in \mathrm{Nil}_R$, the morphism

$$W_{\mathcal{O}}(R) \times \widehat{W}_{\mathcal{O}}(B) \xrightarrow{\text{mult}} \widehat{W}_{\mathcal{O}}(B) \to \mathbb{G}_{\mathcal{O}}(B)$$

induces an isomorphism

$$W_{\mathcal{O}}(R) \xrightarrow{\sim} \mathrm{Hom}(\widehat{W}_{\mathcal{O}}, \mathbb{G}_{\mathcal{O}}). \tag{5.2}$$

Now the same argument of [9, Theorem 3.4] goes through in the case of Dieudonné $\mathcal{O}$-displays and the theorem follows. □

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