PROJECTIVE REPRESENTATION FRAMES AND PHASE RETRIEVAL

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Abstract. We consider the problem of characterizing projective representations that admit frame vectors with maximal spanning property, a property that allows an algebraic recovering for the phase-retrieval problem. For a given multiplier μ of a finite abelian group G, we show that the representation dimension of any irreducible μ-projective representation of G is exactly the rank of the symmetric multiplier matrix associated with μ. With the help of this result we prove that every irreducible μ-projective representation of a finite abelian group G admits a frame vector, and obtain a complete characterization for all such frame vectors. Consequently the complement of the set of all the maximal spanning frame vectors for any projective unitary representation of any finite abelian group is Zariski-closed. These generalize some of the recent results in recent literature about phase-retrieval with Gabor (or STFT) measurements.

1. Introduction

A finite sequence $F = \{f_i\}_{i=1}^N$ of vectors in a finite dimensional (real or complex) Hilbert space H is called a *frame* for H if there are two constants $0 < C_1 \leq C_2$ such that

$$C_1\|f\|^2 \leq \sum_{i=1}^{N} |\langle f, f_i\rangle|^2 \leq C_2\|f\|^2$$

holds for every $f \in H$. Equivalently, a finite sequence is a frame for H if and only if it is a spanning set of H. A frame $F = \{f_i\}_{i=1}^N$ is called C-tight if $C_1 = C_1 = C$ and Parseval if $C_1 = C_2 = 1$.

Like bases, frames are used for signal decomposition and reconstruction through their dual frames in applications. Define $\Theta_F : H \to \mathbb{C}^N$(or $\mathbb{R}^N$) by

$$\Theta_F(f) = \sum_{i=1}^{N}\langle f, f_i\rangle e_i, \quad \text{for all f } \in H,$$

where $\{e_i\}_{i=1}^N$ is the standard orthonormal basis for $\mathbb{C}^N$ (or $\mathbb{R}^N$). Then Θ_F is the analysis operator of F and its synthesis operator is given by $\Theta_F^*(e_i) = f_i$. The frame operator is $S = \Theta_F^*\Theta_F$. Clearly S is a positive and invertible operator on H and satisfies the condition:

$$Sf = \sum_{i}\langle f, f_i\rangle f_i, \quad \forall f \in H.$$

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Replacing f by $S^{-1}f$ or applying S^{-1} to both sides of the above identity we get the reconstruction formula:

$$f = \sum_{i=1}^{N} \langle f, S^{-1}f_i\rangle f_i = \sum_{i=1}^{N} \langle f, f_i\rangle S^{-1}f_i, \quad \text{for all } f \in H.$$

The sequence $\{S^{-1}f_i\}_{i=1}^N$ is called the *standard or canonical dual* of F. In addition to the standard dual, when $\dim H = n < N$ there exist infinitely many frames $\tilde{F} = \{\tilde{f}_i\}_{i=1}^N$ that also give us a reconstruction formula:

$$f = \sum_{i=1}^{N} \langle f, \tilde{f}_i\rangle f_i = \sum_{i=1}^{N} \langle f, f_i\rangle \tilde{f}_i, \qquad \text{i.e., } \Theta_{\tilde{F}}^*\Theta_F = I.$$

Any frame $\tilde{F} = \{\tilde{f}_i\}_{i=1}^N$ yielding the above reconstruction formula is called an *alternate dual frame* or just a *dual frame* for F. The mixed Gramian matrix for two finite sequences $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^M$ is the $N \times M$ matrix $\Theta_Y\Theta_X^* = [\langle\, x_i\,,\, y_j\,\rangle]$.

The Phase retrieval problem considers recovering a signal of interest from magnitudes of its linear or nonlinear measurements and it arises in various fields of science and engineering, such as X-ray crystallography, coherent diffractive imaging, optics and more. Balan, Bodmann, Casazza and Edidin are the first ones who initiated the investigation of the phase retrieval problem by using linear measurements against a frame (c.f. [4, 5, 6, 7, 8]). For linear measurements with a frame $\{f_i\}_{i=1}^N$, it asks to reconstruct f from its intensity measurements $\{|\langle f, f_i\rangle|\}_{i=1}^N$. Clearly the intensity measurements are the same for both f and λf for every unimodular scalar λ. Therefore the phase retrieval problem asks to recover f up to an unimodular scalar.

Definition 1.1. A frame $\{f_i\}_{i=1}^N$ for H is called *phase retrievable* if the induced quotient map $\mathcal{A} : H/\mathbb{T} \to \mathbb{F}^N$ defined by $\mathcal{A}(f/\mathbb{T}) = \{|\langle f, f_i\rangle|\}_{i=1}^N$ is injective, where $\mathbb{T} = \{\lambda \in \mathbb{F} : |\lambda| = 1\}$.

Phase retrieval is impossible without injective intensity measurements. Balan, Casazza and Edidin obtained the following important characterizations of phase retrievable frames [4, 5, 6] .

Theorem 1.1. Let $\{f_i\}_{i=1}^N$ be a frame for H. If $\{f_i\}_{i=1}^N$ is phase retrievable, then it satisfies the complement property, i.e., for every $\Omega \subseteq \{1, ..., N\}$, either $\{f_i\}_{i\in\Omega}$ or $\{f_i\}_{i\in\Omega^c}$ spans $\mathbb{F}^n$. The complementary property is also sufficient when $\mathbb{F} = \mathbb{R}$, but not sufficient in the complex case $\mathbb{F} = \mathbb{C}$.

Theorem 1.2. Every generic frame $\{f_1, ..., f_N\}$ for $\mathbb{F}^n$ is phase retrievable if $N \geq 2n - 1$ in the real case $\mathbb{F} = \mathbb{R}$ or $N \geq 4n - 2$ in the complex case $\mathbb{F} = \mathbb{C}$.

With the above results, construction or design for phase retrievable frames seems not a difficult problem. For example, if $N \geq 2n - 1$, then every n-independent (or full spark) frame (i.e., every n-vectors in the frame are linearly independent) has this property, and consequently it is phase retrievable in the real case. We refer to a recent paper [11] for a comprehensive discussion on complex phase retrievable frames.

The main issue to the phase retrieval problem lies in the recovering algorithms due to the nonlinearity nature of the map $\mathcal{A}$. We refer to [1, 3, 9, 10, 11, 12, 14, 15, 17, 18, 19, 23] for detailed accounts on some recent developments and various kinds of approaches for the phase retrieval problem. In some special cases a linear reconstruction is also possible. For example, phase retrieval can be formulated as rank-one matrix recovery (phase-lifting) problem if a phase retrievable frame has the *maximal span property,* i.e., the span of $\{f_i \otimes f_i\}$ contains all the rank-one Hermitian operators [7, 8]. In this case, $\{f_i \otimes f_i\}$ form a frame for the Hilbert space $\mathbb{H}^n$ (the space of the linear span of all the $n \times n$ Hermitian matrices) equipped with the Hilbert-Schmidt inner product $\langle A, B\rangle = \mathrm{Tr}(AB^*)$. Let $\{A_i\}_{i=1}^N$ be a dual frame for $\{f_i \otimes f_i\}_{i=1}^N$. Then for every $x \in H$ we have

$$x \otimes x = \sum_{i=1}^{N} \langle x \otimes x, f_i \otimes f_i\rangle A_i = \sum_{i=1}^{N} |\langle x, f_i\rangle|^2 A_i,$$

and so x can be reconstructed (up to a multiple of unimodule scalar) by factorizing the above right hand side rank-one matrix.

Let G be the Gramian of the sequence $\{f_i \otimes f_i\}_{i=1}^N$, i.e. $G_{ij} = \mathrm{Tr}(f_i \otimes f_i \cdot f_j \otimes f_j) = |\langle f_i, f_j\rangle|^4$. An explicit formula for one of the choices of a dual $\{A_i\}_{i=1}^N$ is obtained by Balan, Bodmann, Casazza, Eddin in [7, 8].

Theorem 1.3. If $\{f_i\}_{i=1}^N$ is a $\frac{N}{n}$-tight frame with the maximal span property, then

$$x \otimes x = \sum_{i=1}^{N} |\langle x, f_i\rangle|^2 R_i,$$

where $R_i = \sum_{j=1}^{N} Q_{ij}(f_j \otimes f_j)$ and Q is the pseudo-inverse of G (i..e. $GQG = G$).

Let $k = n(n+1)/2$ or n^2 depending on whether H is real or complex. Then $\dim \mathbb{H}^n = k$, and hence $N \geq k$ if a frame $\{f_i\}_{i=1}^N$ has the maximal span property. A key factor in this reconstruction is the existence and constructions of maximal span frames. The purpose of this note to investigate a special class of frames, namely, projection representation frames, that have the maximal span property. A typical example is the frames obtained by Gabor representations (or STFT measurements):

Example 1.1. Let $H = \mathbb{C}^n$ and $w = (w(0), ..., w(n-1)) \in H$ be a window vector. For $x = (x(0), x(1), ..., x(n-1)) \in H$, the Gabor or STFT measurement of x is given by:

$$X_w(m, k) = \sum_{j=0}^{n-1} x(j)\overline{w(j-m)}e^{-2\pi ikj/n}, \quad (m, k) \in \mathbb{Z}_n \times \mathbb{Z}_n.$$

Let $T, E : \mathbb{C}^n \to \mathbb{C}^n$ be the unitary operators defined by

$$(Tw)(j) = w(j-1); \ (Ew)(j) = e^{2\pi ij/n} w(j).$$

Then $X_w(m, k) = \langle\, x\,,\, E^k T^m w\,\rangle$. The involved mapping $\pi : \mathbb{Z}_n \times \mathbb{Z}_n \to M_{n\times n}(\mathbb{C})$ defined by

$$\pi(m, k) = E^k T^m$$

is the Gabor representation of the group $G = \mathbb{Z}_n \times \mathbb{Z}_n$ on H. So the phase-retrieval problem with Gabor measurements asks to recover x from $|\langle x, E^k T^m w \rangle|$. This example has led extensive research activities in the last few years (c.f. [13, 16, 22, 24, 26, 27]). One of the most basic questions is to characterize the window vector w such that $\{\pi(m,k)w\}_{(m,k)\in G}$ is phase-retrievable or has the maximal span property. A simple characterization for these type of frames was recently obtained by I. Bojarovska and A. Flinth.

Theorem 1.4. [16] Let π be the Gabor representation of $G = \mathbb{Z}_n \times Z_n$ on $\mathbb{C}^n$ and $w \in \mathbb{C}^n$. Then $\{\pi(m,k)w\}_{(m,k)\in G}$ has the maximal span property if and only if $\langle \pi(m,k)w, w \rangle \neq 0$ for every $(m,k) \in G$.

Note that the Gabor representation is a projective representation of the abelian group G. Moreover it is an irreducible representation meaning that $\mathrm{Span}\{\pi(m,k) : (m,k) \in G\} = M_{n\times n}(\mathbb{C})$. This leads to the following natural question for other type of projective representations:

Problem 1.1. *Let π be a projective unitary representation of a finite group G on $\mathbb{C}^n$. Under what condition does π admit a frame vector with the maximal span property? How to characterize all such frame vectors?*

We conjecture that every irreducible projection representation admit a frame with the maximal span property. The main purpose of this paper is to confirm this conjecture for abelian groups, and consequently we get a generalization of the known results for Gabor representations. We achieve this by obtaining a new formula for the representation dimension of irreducible projective representations for finite abelian groups. Let μ be a 2-cocycle (or multiplier) of a finite abelian group G, and let C_μ be its associated symmetric multiplier matrix defined by $C_\mu = [c_{g,h}]$ with $c_{g,h} = \mu(g,h)\overline{\mu(h,g)}$.

Theorem 1.5. If π is an irreducible μ-projective representation of a finite abelian group G on an n-dimensional complex Hilbert space H, then $\mathrm{rank}(C_\mu) = n^2$.

With the help of the above result we shall prove the following result.

Theorem 1.6. Suppose that π is a μ-projective unitary representation for a finite abelian group G on an n-dimensional complex Hilbert space H. If π is irreducible, the π admits a frame vector with the maximal span property. Moreover, $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property if and only if $\langle \pi(g)\xi, \xi \rangle \neq 0$ for any $g \in G$.

2. Group representation frames

A *projective unitary representation* π for a group G is a mapping $g \mapsto \pi(g)$ from G into the group $U(H)$ of all the unitary operators on a separable Hilbert space H such that $\pi(g)\pi(h) = \mu(g,h)\pi(gh)$ for all $g,h \in G$, where $\mu(g,h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. This function $\mu(g,h)$ is then called a *multiplier of* π. In this case we also say that π is a μ-projective unitary representation. It is clear from the definition that we have

(i) $\mu(g_1, g_2g_3)\mu(g_2,g_3) = \mu(g_1g_2,g_3)\mu(g_1,g_2)$ for all $g_1,g_2,g_3 \in G$,
(ii) $\mu(g,e) = \mu(e,g) = 1$ for all $g \in G$, where e denotes the group unit of G.

Any function $\mu : G \times G \to \mathbb{T}$ satisfying (i) – (ii) above will be called a *multiplier* or 2-*cocycle* of G. It follows from (i) and (ii) that we also have
(iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ holds for all $g \in G$.

The set of all possible multipliers on G can be given an abelian group structure by defining the product of two multipliers as their pointwise product. The resulting group we denote by $Z^2(G, \mathbb{T})$. The set of all the mutipliers α satisfying

$$\alpha(g, h) = \beta(gh)\beta(g)^{-1}\beta(h)^{-1}$$

for arbitrary function $\beta : G \to \mathbb{T}$ such that $\beta(e) = 1$ forms a subgroup $B(G, \mathbb{T})$ of $Z^2(G, \mathbb{T})$, and the quotient group $H^2(G, \mathbb{T}) = Z^2(G, \mathbb{T})/B^2(G, \mathbb{T})$ is the second cohomology group of G.

Similar to the group unitary representation case, the *left* and *right regular projective representations* with a prescribed multiplier μ for G can be defined by

$$\lambda_g \chi_h = \mu(g, h)\chi_{gh}, \quad (h \in G),$$

and

$$\rho_g \chi_h = \mu(h, g^{-1})\chi_{hg^{-1}}, \quad (h \in G),$$

where $\{\chi_g : g \in G\}$ is the standard orthonormal basis for $\ell^2(G)$.

Let π be a projective group representation. A vector ξ is called a π-*maximal spanning frame vector* if $\{\pi(g)\xi\}_{g\in G}$ has the maximal spanning property. We will use $\mathcal{M}_\pi$ to denote the set of all π-maximal spanning frame vectors. The following elementary result shows that if $\mathcal{M}_\pi$ is non empty, then every generic representation frame has the maximal span property.

Theorem 2.1. If $\mathcal{M}_\pi$ is not empty, then it is an open dense subset of H. In fact $\mathcal{M}_\pi^c$ is Zariski-closed.

Proof. We can assume that $H = \mathbb{C}^n$ or $\mathbb{R}^n$. Let $\{B_1, ..., B_k\}$ be a basis for $\mathbb{H}^n$. For each $g \in G$ and $x = (x_1, ..., x_n) \in H$, write

$$\pi(g)x \otimes \pi(g)x = \sum_{i=1}^{k} c_{i,g}(x)B_i.$$

where the coefficients $c_{i,g}(x)$ can be obtained by solving a system of linear equations and so they are quadratic polynomial of x_j when $\mathbb{F} = \mathbb{R}$ and in the complex case they are polynomials of u_i, v_i with u_i, v_i being the real and imaginary part of x_i. Let $C(x) = [c_{i,g}]_{k\times|G|}$, and P_Λ be the determinate of the submatrix of $C(x)$ consisting of the g-th columns for $g \in \Lambda$, where Λ is any subset of G with cardinality k. Then again each P_Λ is a polynomial of x_j or polynomial of u_i and v_i in the complex case. Clearly $x \in \mathcal{M}_\pi$ if and only if $\mathrm{rank}(C(x)) = k$. This implies that

$$\mathcal{M}_\pi = H \setminus \cap_{\Lambda\subset G, |\Lambda|=k} Z(P_\Lambda),$$

where $Z(P) = \{x \in H, P(x) = 0\}$.

Since $\mathcal{M}_\pi$ is nonempty, there exist $x \in H$ and Λ such that $P_\Lambda(x) \neq 0$. Thus P_Λ is a nonzero polynomial, and therefore we get that $H \setminus \mathcal{M}_\pi$ is Zariski-closed and so $\mathcal{M}_\pi$ is open and dense in H. $\square$

In this paper we are interested in establishing sufficient and/or necessary conditions on π such that $\mathcal{M}_\pi$ is not empty. We make the following formal conjecture:

Conjecture 1. Every irreducible μ-projective representation π admits a maximal spanning frame vector.

Let π be a projective unitary representation of G on a Hilbert space $H := \mathbb{C}^n$. For each $\xi \in H$, consider the matrix

$$A(\xi) = [a_{g,h}(\xi)]_{G\times G}$$

with $a_{g,h}(\xi) = \langle\, \pi(h)\pi(g)\xi\,,\, \pi(g)\xi\,\rangle$. We first establish the following sufficient condition:

Lemma 2.2. If there exists $\xi \in H$ such that $A(\xi)$ has rank n^2 (where $n = \dim H$), then π is irreducible and $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property.

Proof. Let $X = \{\pi(g)\}_{g\in\Lambda}$ and $Y = \{\pi(g)\xi\otimes\pi(g)\xi\}_{g\in G}$ be two sequences in $B(H)$ equipped with the trace inner product. Note that the mixed Gramiann matrix $\Theta_Y\Theta_X^*$ is exactly the matrix $A(\xi)$ which is assumed to have rank n^2. Thus $\mathrm{rank}(\Theta_Y) \geq n^2$ and $\mathrm{rank}(\Theta_X) \geq n^2$. Since we also have $\mathrm{rank}(\Theta_Y) \leq n^2$ and $\mathrm{rank}(\Theta_X) \leq n^2$, we get that $\mathrm{rank}(\Theta_Y) = n^2 = \mathrm{rank}(\Theta_X)$, which implies that π is irreducible and $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property. $\square$

3. Proofs of the main results

In this section we prove the main results of this paper and discuss a few consequences and examples related to the main theorem.

Lemma 3.1. Suppose that π is a μ-projective unitary representation for a finite group G on an n-dimensional complex Hilbert space H. Then there exists $\xi \in H$ such $\langle\, \pi(g)\xi\,,\, \xi\,\rangle \neq 0$ for all $g \in G$. Moreover, the set of all such vectors ξ is open and dense in H.

Proof. We may assume that $H = \mathbb{C}^n$. By the Baire-Category theorem it suffices to prove that for each $g \in G$, the set $\{\xi \in \mathbb{C}^n : \langle\, \pi(g)\xi\,,\, \xi\,\rangle \neq 0\}$ is open and dense in $\mathbb{C}^n$. Since $\langle\, \pi(g)\xi\,,\, \xi\,\rangle$ is a quadratic polynomial of ξ, we only need to point out that this is a nonzero polynomial. Indeed, if $\langle\, \pi(g)\xi\,,\, \xi\,\rangle = 0$ for all $\xi \in \mathbb{C}^n$, then we have $\pi(g) = 0$, which is a contradiction. $\square$

Lemma 3.2. Suppose that π is a μ-projective unitary representation for an abelian group G. If there exists $\xi \in H$ such that $\{\pi(g)\xi\}_{g\in G}$ has the maximal spanning property, then $\langle\, \pi(g)\xi\,,\, \xi\,\rangle \neq 0$ for any $g \in G$.

Proof. Since $\{\pi(g)\xi\}_{g\in G}$ has the maximal spanning property we have that $span\{\pi(g)\xi \otimes \pi(g)\xi : g \in G\} = B(H)$. So if $\langle\pi(h)\xi, \xi\rangle = 0$ for some $h \in G$, then for every $g \in G$ have

$$\begin{aligned} |\langle\, \pi(h)\pi(g)\xi\,,\, \pi(g)\xi\,\rangle| &= |\langle\, \pi(g)^{-1}\pi(h)\pi(g)\xi\,,\, \xi\,\rangle| = |\langle\, c\cdot\pi(g^{-1}hg)\xi\,,\, \xi\,\rangle| \\ &= |\langle\, \pi(h)\xi\,,\, \xi\,\rangle| = 0. \end{aligned}$$

Here $c = \mu(g, g^{-1})^{-1}\mu(g^{-1}, h)\mu(g^{-1}h, g) \in \mathbb{T}$. Thus $Tr(\pi(h)(\pi(g)\xi \otimes \pi(g)\xi)) = 0$, and so $\pi(h) = 0$ which leads to a contradiction. $\square$

Lemma 3.3. [2] Let μ be a multiplier for an abelian group G. Then all the irreducible μ-projective representations have the same representation dimension.

The following result tells us that the representation dimension of the irreducible μ-projective representations is uniquely determined by the rank of its symmetric multiplier matrix. Let μ be a multiplier for an abelian group G. Recall that the *symmetric multiplier matrix* is defined by $C_\mu = [c_{g,h}]$ with $c_{g,h} = \mu(g,h)\overline{\mu(h,g)}$. The following lemma clearly covers Theorem 1.5.

Lemma 3.4. Suppose that π is a μ-projective unitary representation for an abelian group G on an n-dimensional Hilbert space $H = \mathbb{C}^n$. Then $\mathrm{rank}(C_\mu) \le n^2$. Moreover, π is an irreducible μ-representation if and only if $\mathrm{rank}(C_\mu) = n^2$.

Remark 3.1. While it is well known that all the irreducible μ-projective representations have the same representation dimension, in general it is not easy to find an explicit formula for the dimension [2]. The above result seems to be new and it does provide us a very easy way to compute the representation dimension for any give multiplier μ.

Proof. By Lemma 3.1, there exists $\eta \in \mathbb{C}^n$ such that $\langle \pi(g)\eta, \eta\rangle \ne 0$ for any $g \in G$. Let $\Theta_1 : M_{n\times n}(\mathbb{C}) \to \ell^2(G)$ be the analysis operator for $\{\pi(g)\}_{g\in G}$, and $\Theta_2 : M_{n\times n}(\mathbb{C}) \to \ell^2(G)$ be the analysis operator for $\{\pi(g)\eta \otimes \pi(g)\eta\}_{g\in G}$. Then we have

$$\Theta_2\Theta_1^* = [\langle\, \pi(g)\pi(h)\eta\,,\, \pi(h)\eta\,\rangle]_{G\times G}.$$

Note that

$$\langle\, \pi(g)\pi(h)\eta\,,\, \pi(h)\eta\,\rangle = c_{g,h}\langle\, \pi(g)\eta\,,\, \eta\,\rangle.$$

and $\langle \pi(g)\eta, \eta\rangle \ne 0$ for every $g \in G$. So we get that

$$\mathrm{rank}(C_\mu) = \mathrm{rank}(\Theta_2\Theta_1^*) \le \mathrm{rank}(\Theta_1) = \dim_{\mathbb{C}} \mathrm{Span}\{\pi(g) : g \in G\} \le n^2.$$

Now assume that $\mathrm{rank}(C_\mu) = n^2$. Then the above inequality implies that $\dim_{\mathbb{C}} \mathrm{Span}\{\pi(g) : g \in G\} = n^2$, and thus π is irreducible. Conversely, let us assume that π is irreducible. We prove that $\mathrm{rank}(C_\mu) = n^2$.

We first introduce a couple of notations. Let $\hat{G}$ be the dual group of G, and $\bar{\pi} : g \mapsto \overline{\pi(g)}$, the complex conjugation of $\pi(g)$. Then $\bar{\pi}$ is a projective representation with multiplier $\bar{\mu}$. Consider the group representation $\pi \otimes \bar{\pi} : g \mapsto \pi(g) \otimes \overline{\pi(g)}$. Then it is a projective representation with multiplier $\mu\bar{\mu} = 1$, and so it is a group representation. Hence $\pi \otimes \bar{\pi}$ can be decomposed as a direct sum of one-dimensional group representations of G and moreover, each one dimensional representation of G appears at most once in the direct sum decomposition of $\pi \otimes \bar{\pi}$. Let $T_\mu = \{\chi \in \hat{G} : \chi \subset \pi \otimes \bar{\pi}\}$. Then T_μ is a subgroup of $\hat{G}$. Define

$$G_\mu = T_\mu^\perp = \{g \in G : \chi(g) = 1, \forall \chi \in T_\mu\}.$$

Note that $|T_\mu| = \dim\ H \times \dim\ H = n^2$. Thus $[G : G_\mu] = |T_\mu| = n^2$.

Since G is abelian, it is easy to verify that $c : G \times G \to \mathbb{T}$ defined by $c(g,h) = c_{gh} = \mu(g,h)\overline{\mu(h,g)}$ is a bi-homomorphism, i.e., $c(gg', h) = c(g,h)c(g', h)$ and $c(g, hh') = c(g,h)c(g,h')$ for all $g, g', h, h' \in G$. This induces a homomorphism $\lambda_\mu : G \to \hat{G}$. By [21, Proposition 2.4] we know that

$$G_\mu = \mathrm{Ker}(\lambda_\mu) = \{g \in G : \lambda_\mu(g) = 1\}.$$

Therefore we get

$$|\lambda_\mu(G)| = [G : \mathrm{Ker}(\lambda_\mu)] = n^2.$$

Recall that the characters of G are linearly independent. Since each row of the symmetric multiplier matrix C_μ defines a character of G by $h \mapsto c(g,h)$, the rank of C_μ is equal to the number of different characters that appear in the rows of C_μ. By the definition of λ_μ, this number is exactly the cardinality of the image of λ_μ. This implies that $\mathrm{rank}(C_\mu) = |\lambda_\mu(G)| = n^2$ as claimed. $\square$

Corollary 3.5. Let μ be a multiplier of an abelian group G and $n^2 = \mathrm{rank}(C_\mu)$. Then every n-dimensional μ-projective representation π of G is irreducible.

Proof. Let σ be an irreducible subrepresentation of π on a d-dimensional π-invariant subspace. Then, by Lemma 3.4, the representation dimension of σ is equal to $\mathrm{rank}(C_\mu) = d^2$. This implies that $d = n$ and thus $\sigma = \pi$. Therefore π is irreducible. $\square$

Proof of Theorem 1.6:

Assume that π is an irreducible μ-projective representation of G on $H = \mathbb{C}^n$. By Lemma 3.2 we know that if $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property, then $\langle\, \pi(g)\xi\,,\, \xi\,\rangle \neq 0$ for every $g \in G$. Therefore, to complete the proof, it suffices to show that $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property when $\langle\, \pi(g)\xi\,,\, \xi\,\rangle \neq 0$ for every $g \in G$.

Let Θ_1 and $\Theta_2 : M_{n\times n}(\mathbb{C}) \to \ell^2(G)$ be the analysis operators defined in the proof of Lemma 3.4. Then we know that $\mathrm{rank}(\Theta_2\Theta_1^*) = \mathrm{rank}(C_\mu) = n^2$. Since π is irreducible, we get that $\mathrm{rank}(\Theta_1^*) = n^2$. This implies that $\mathrm{rank}(\Theta_2) = n^2$ since we also have $\mathrm{rank}(\Theta_2) \leq n^2$. Therefore $\{\pi(g)\xi \otimes \pi(g)\xi : g \in G\}$ spans $M_{n\times n}(\mathbb{C})$, i.e., $\{\pi(g)\xi\}_{g\in G}$ has the maximal span property. $\square$

Clearly Theorem 1.4 is a special case of Theorem 1.6, and the Gabor representation is a special irreducible projective unitary representation for the group $\mathbb{Z}_n \times \mathbb{Z}_n$. The following example presents us all the possible irreducible projective unitary representations for the group $\mathbb{Z}_n \times \mathbb{Z}_n$.

Example 3.1. Let $G = \mathbb{Z}_n \times \mathbb{Z}_n$ and $H^2(G, \mathbb{T})$ be the second cohomology group. Then $H^2(G, \mathbb{T}) \cong \mathbb{Z}_n$. Let $\xi = e^{2\pi i/n}$. Let $\alpha \in Z^2(G, \mathbb{T})$ be given by

$$\alpha((m,k),(m',k')) = \xi^{-mk'}.$$

Then $[\alpha] \in H^2(G, \mathbb{T})$ is a generator. To understand all the projective representations of G, it suffices to understand the α^a-projective representations of G for each $a \in \{0,\ 1,\ \dots,\ n-1\}$. Denote by IrrRep_G^a the set of isomorphic classes of irreducible α^a-projective representations of G.

Fix $a \in \{0,\ 1,\ \dots,\ n-1\}$. Let $a' = n/\tilde{a}$, where $\tilde{a}$ denotes the greatest common factor of n and a. Let $H = \langle a' \rangle \times \mathbb{Z}_n \triangleleft G$. Then

$$\alpha^a(x,y) = \alpha^a(y,x) = 1 \text{ for any } x,\ y \in H.$$

In particular, H is α^a-symmetric. It is also easy to check that H is maximal α^a-symmetric. Hence the objects in IrrRep_G^a all have dimension $[G : H] = a'$ and there are $\tilde{a}^2$ objects in IrrRep_G^a (cf. [20, Section 2.3]).

The irreducible projective representations of G are induced from one-dimensional linear representations of H (cf. [20, Proposition 2.14]). Let $u \in \mathbb{Z}_{\tilde{a}}$ and $v \in \mathbb{Z}_n$. Let $\chi_{u,v} : H \to \mathbb{C}^\times$ be the one-dimensional linear representation given by $(m,k) \mapsto \xi^{mu+kv}$. Let

$\pi_{u,v} = \alpha^a \operatorname{Ind}_H^G \chi_{u,v}$. Here the induction is with respect to α^a (cf. [20, Section 2.2]). Then $\pi_{u,v} \in \operatorname{IrrRep}_G^a$. One may check that $\pi_{u,v} \cong \pi_{u',v'}$ if and only if $v \equiv v' \pmod{\tilde{a}}$.

In matrix form, we may describe $\pi_{u,v}$ as follows. Let V be a $\mathbb{C}$-vector space with dimension a'. Fix $\{e_0, e_1, \ldots, e_{a'-1}\}$ a basis of V. Define $\pi_{u,v}((1,0))$ and $\pi_{u,v}((0,1))$ by

$$\pi_{u,v}((1,0))e_i = \xi^u e_{i+1}, \quad \pi_{u,v}((0,k))e_i = \chi_{u,v}^{(i,0)}((0,k))e_i \text{ for } 0 \le i \le a'-1.$$

Here the subscript i is considered modulo a', $\chi_{u,v}^{(i,0)}$ is the α^a-twist of $\chi_{u,v}$ by $(i,0)$ (cf. [20, Proposition 2.10]). More precisely, as $a' \times a'$ matrices,

$$\pi((0,k)) = \tilde{E}^k, \quad \pi((m,0)) = \xi^{mu}\tilde{T}^m,$$

where

$$\tilde{E} = \operatorname{diag}(\xi^v, \xi^{v+a}, \ldots, \xi^{v+(a'-1)a}),$$

$$\tilde{T} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{a' \times a'}. \tag{3.1}$$

If we take $u = v = 0$, $a = 1$, then it is the representation in Example 1.1. With this description, one may find all $w \in V$ with the maximal span property by applying Theorem 3.1.

This example can be further generalized by considering the tensor product of group representations.

Example 3.2. Let G_1 and G_2 be two finite groups. Let $\alpha_i \in Z^2(G_i, \mathbb{T})$ be a 2-cocycle of G_i $(i = 1, 2)$. Define a map $\alpha_1 \times \alpha_2 : (G_1 \times G_2) \times (G_1 \times G_2) \to \mathbb{T}$ by

$$(\alpha_1 \times \alpha_2)((g_1, g_2), (h_1, h_2)) = \alpha_1(g_1, h_1)\alpha_2(g_2, h_2) \text{ for all } g_1, h_1 \in G_1,\ g_2, h_2 \in G_2.$$

It is easy to check that $\alpha_1 \times \alpha_2 \in Z^2(G_1 \times G_2, \mathbb{T})$. Let (π_1, V_1, α_1) and (π_2, V_2, α_2) be projective representations of G_1 and G_2 respectively. Define a map $\pi_1 \times \pi_2 : G_1 \times G_2 \to \mathrm{GL}(V_1 \otimes V_2)$ by

$$\pi_1 \times \pi_2((g_1, g_2)) = \pi_1(g_1) \otimes \pi_2(g_2) \text{ for all } g_1 \in G_1,\ g_2 \in G_2.$$

Then $(\pi_1 \times \pi_2, V_1 \otimes V_2, \alpha_1 \times \alpha_2)$ is a projective representation of $G_1 \times G_2$. If moreover π_1 and π_2 are unitary projective representations, then so is $\pi_1 \times \pi_2$. In this situation, we have

(1) if π_i is irreducible $(i = 1, 2)$, then $\pi_1 \times \pi_2$ is an irreducible projective representation of $G_1 \times G_2$;
(2) each irreducible projective representation of $G_1 \times G_2$ with multiplier $\alpha = \alpha_1 \times \alpha_2$ is isomorphic to a representation $\pi_1 \times \pi_2$, where π_i is an irreducible projective representation of G_i with multiplier α_i $(i = 1, 2)$.

It is easy to prove the following result.

Proposition 3.6. If $\xi_i \in H_i$ is a π_i-maximal spanning frame vector $(i = 1, 2)$, then $\xi_1 \otimes \xi_2$ is a $(\pi_1 \times \pi_2)$-maximal spanning frame vector.

If $G = \prod_{n_i}(\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i})$ with n_i pairwise coprime, then $H^2(G, \mathbb{T}) = \prod_i H^2(\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}, \mathbb{T})$. Hence every element of $H^2(G, \mathbb{T})$ is represented by a cocycle in $\prod_i Z^2(\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}, \mathbb{T})$. Then for projective representations of G, we obtain all the vectors with maximal span property by the last example.

4. Remarks and further problems

For an abelian group, all the irreducible μ-projective unitary representations have the same representations dimension n, and we have proved that $\mathrm{rank}(C_\mu) = n^2$ which is a key ingredient in the proof of our main theorem. However, this is not the same case anymore for non-abelian groups since irreducible representations with respect to the same multiplier could have different representation dimensions.

Question 4.1. *Let μ be a multiplier for a finite non-abelian group G such that all the irreducible μ-projective unitary representations have the same representations dimension n. Is it still true that* $\mathrm{rank}(C_\mu) = n^2$*?*

Example 4.1. Consider the metacyclic groups of type $G = \mathbb{Z}_m \ltimes \mathbb{Z}_p$ with p a prime. Fix a presentation of G

$$G = \langle a, b \mid a^m = 1, b^p = 1, bab^{-1} = a^r \rangle,$$

where $r \in \mathbb{Z}_{\geq 0}$ and $r^p \equiv 1 \pmod{m}$. By [25, 2.11.3 Theorem],

$$H^2(G, \mathbb{C}^\times) = \begin{cases} 0 & \text{if } p \nmid (m, r-1), \\ \mathbb{Z}_p & \text{if } p \mid (m, r-1). \end{cases}$$

In the following, we assume that $p \mid (m, r-1)$. Fix ζ a primitive l-th root of unity, where $l = (m, 1 + r + \cdots + r^{p-1})$. Define $\alpha : G \times G \to \mathbb{T}$ by

$$\alpha(a^i b^j, a^{i'} b^{j'}) = \begin{cases} 1 & \text{if } j = 0, \\ \zeta^{i'(1+r+\cdots+r^{j-1})} & \text{otherwise.} \end{cases}$$

By [25, 2.11.1 Lemma and 2.11.3 Theorem], this α is a well-defined element in $Z^2(G, \mathbb{T})$ and it represents a generator of $H^2(G, \mathbb{T})$. If we arrange the elements of G in the order as $1,\ a,\ a^2,\ \ldots,\ a^{m-1},\ b,\ ab,\ \ldots,\ a^{m-1}b,\ \ldots,\ b^{p-1},\ ab^{p-1},\ \ldots,\ a^{m-1}b^{p-1}$, by writing down C_α explicitly, one sees that C_α is given by

$$\begin{pmatrix} A & X_1^{-1}A & \cdots & X_{p-1}^{-1}A \\ AX_1 & X_1^{-1}AX_1 & \cdots & X_{p-1}^{-1}AX_1 \\ \cdots & \cdots & \cdots & \cdots \\ AX_{p-1} & X_1^{-1}AX_{p-1} & \cdots & X_{p-1}^{-1}AX_{p-1} \end{pmatrix}_{p \times p},$$

where A is the $m \times m$ matrix with all entries equal to 1, and

$$X_i = \mathrm{diag}(1, \zeta^{1+r+\cdots+r^{i-1}}, \cdots, \zeta^{(m-1)(1+r+\cdots+r^{i-1})}).$$

Note that the rank of

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & \zeta & \cdots & \zeta^{m-2} & \zeta^{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \zeta^{1+r+\cdots+r^{p-2}} & \cdots & \zeta^{(m-2)(1+r+\cdots+r^{p-2})} & \zeta^{(m-1)(1+r+\cdots+r^{p-2})} \end{pmatrix}_{p\times m},$$

is p by Vandermonde. The rank of C_α is p^2. On the other hand, by [20, Corollary 3.11], every irreducible α-projective representation of G has dimension p. Hence Question 4.1 has an affirmative answer in this case.

Moreover, all the irreducible α-projective representations of G admit maximal span vectors. Indeed, let $\pi : G \to \mathrm{GL}(V)$ be an irreducible α-projective representation. By Lemma 2.2, it suffices to show that there exists an element $\xi \in V$ with $\mathrm{rank}(A(\xi)) = p^2$.

Let $m' = m/p$. Then $m \mid m'(r-1)$. The subgroup $K = \langle a^{m'}, b\rangle$ of G is abelian and $K \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Furthermore, the restriction $\alpha|_{K\times K} \in Z^2(K, \mathbb{T})$ represents a generator of $H^2(K, \mathbb{T}) \cong \mathbb{Z}_p$. Hence $\pi|_K : K \to \mathrm{GL}(V)$ is an irreducible projective representation of K by Corollary 3.5. For any $\xi \in V$, we have

$$\mathrm{rank}(\langle \pi(g)\pi(h)\xi, \pi(h)\xi\rangle)_{g,h\in G} \geq \mathrm{rank}(\langle \pi(g)\pi(h)\xi, \pi(h)\xi\rangle)_{g,h\in K}.$$

The claim follows from the abelian case.

One of our ultimate goals is to confirm our Conjecture 1 that every irreducible μ-representation admits a maximal spanning frame vector. By Lemma 2.2 we know that π will have a maximal spanning frame vector ξ if $\mathrm{rank}(A(\xi)) = n^2$, where n is the representation dimension and $A(\xi)$ is the mixed Gramian of $\{\pi(g)\xi \otimes \pi(g)\xi\}_{g\in G}$ and $\{\pi(g)\}_{g\in G}$. So we make the following conjecture:

Conjecture 2. Let π be an irreducible μ-projective unitary representation of a finite group G on an n-dimensional Hilbert space H. Then there exists a vector $\xi \in H$ such that $\mathrm{rank}(A(\xi)) = n^2$.

We point out that in the abelian group case, $\mathrm{rank}(A(\xi)) = \mathrm{rank}(C_\mu)$ when $\langle\, \pi(g)\xi\,,\, \xi\,\rangle \neq 0$ for every $g \in G$, and thus the above conjecture is true in this case. So far we only considered irreducible projective unitary representations with maximal spanning frame vectors. It is natural to ask what can happen with reducible representations and phase-retrievable representation frames.

Question 4.2. Let π be an irreducible μ-projective unitary representation of a finite group G on an n-dimensional Hilbert space H.

(i) Under what condition does π admit a phase-retrievable frame vector ξ? (i.e., $\{\pi(g)\xi\}$ is a phase-retrievable frame).

(ii) If π admits a maximal spanning frame vector, must π be irreducible?

Finally let us look at Conjecture 1 from a slightly different angle. Let π be a finite-dimensional μ-projective representation of a finite group G. We say that π is *cyclic* if there exists a non-zero vector $v \in H$ such that $\pi(G)v$ generate H. So π is irreducible if and only if every non-zero v is a cyclic vector. Assume that $\pi : G \to \mathrm{GL}(H)$ is cyclic. Then H is a cyclic $\mathbb{C}[G]_\mu$-module, where $\mathbb{C}[G]_\mu$ is the twisted group algebra. Assume that $H = \mathbb{C}[G]_\mu.v$. Define

$$\begin{aligned} \mathbb{C}[G]_\mu &\to H \\ g &\mapsto g.v \end{aligned}$$

It is surjective and $H \cong \mathbb{C}[G]_\mu/I$ for some ideal I. Conversely, assume that H is a $\mathbb{C}[G]_\mu$-module with the form $\mathbb{C}[G]_\mu/I$. Then the element $1 + I \in \mathbb{C}[G]_\mu/I$ is a cyclic element. We obtain the following lemma.

Lemma 1. *A μ-projective representation $\pi : G \to \mathrm{GL}(H)$ is cyclic if and only if $H \cong \mathbb{C}[G]_\mu/I$ as $\mathbb{C}[G]_\mu$-modules, where I is an ideal of $\mathbb{C}[G]_\mu$.*

For a μ-projective representation $\pi : G \to \mathrm{GL}(H)$, define the adjoint representation $\mathrm{Ad}\,\pi : G \to \mathrm{GL}(\mathrm{End}(H))$ by $\mathrm{Ad}\,\pi(g).m = \pi(g)^{-1}m\pi(g)$. Note that $\mathrm{Ad}(\pi)$ is a linear representation.

Proposition 2. *With the notation as above, if π is irreducible, then $\mathrm{Ad}\,\pi$ is cyclic.*

Proof. This follows from the following two facts.

(1) We have an isomorphism of $\mathbb{C}[G]$-modules
$$\mathbb{C}[G] \to \bigoplus_W W^{\oplus \dim W}.$$
Here W runs through equivalent classes of irreducible linear representations of G.

(2) Consider the adjoint representation of the Lie group $\mathrm{GL}_n(\mathbb{C})$. The adjoint representation of $\mathrm{GL}_n(\mathbb{C})$ on its Lie algebra $\mathfrak{g} = M_n(\mathbb{C})$ is given by conjugation, which is isomorphic to $V \otimes V^*$, where $V = \mathbb{C}^n$ and $\mathrm{GL}_n(\mathbb{C})$ acts on V in the obvious way.

Back to our situation, from (2), one sees that $\mathrm{Ad}\,\pi \cong H \otimes H^*$. Let U be an irreducible linear representation of G such that $\mathrm{Hom}_G(U, H \otimes H^*)$ is not trivial. Then by [20, Proposition 2.2],

$$\dim \mathrm{Hom}_G(U, H \otimes H^*) = \langle \chi_U, \chi_H \chi_{H^*} \rangle = \langle \chi_U \chi_H, \chi_H \rangle = \dim \mathrm{Hom}_G(H \otimes U, H) \leq \dim U,$$

where the second Hom_G means homomorphism of μ-projective representations, the last inequality follows from the irreducibility of H. Therefore, by (1), $H \otimes H^*$ is a direct summand of $\mathbb{C}[G]$. Hence $\mathrm{Ad}\,\pi \cong H \otimes H^*$ is cyclic. □

Question 4.3. If π is irreducible, does $\mathrm{Ad}\,\pi$ admit a cyclic vector of the form $\xi \otimes \xi$?

An affirmative answer to the above question will also confirm conjecture 1.

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