# A CHARACTER THEORY FOR PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

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Abstract. In this paper, we construct a character theory for projective representations of finite groups. Consequently, we compute the number of distinct irreducible projective representations (up to isomorphism) of a finite group with a given associated Schur multiplier and deduce properties on the degrees of such projective representations.

## 1. Introduction

Throughout this paper, except in Subsection 3.3, $G$ is a finite group. As in [3, Definition 2] and [5], a *projective representation* $(\pi, V, \alpha)$ of $G$ over $\mathbb{C}$ of *degree* $n$ is a map $\pi : G \to \mathrm{GL}(V)$ such that $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$ for all $x, y \in G$, where $V$ is an $n$-dimensional vector space over $\mathbb{C}$, $\alpha : G \times G \to \mathbb{C}^{\times}$ is the associated multiplier ([3, Definition 1]).

Denote by $\mathrm{Rep}_G^{\alpha}$ the set of projective representations of $G$ with multiplier $\alpha$. We say that the projective representation $(\pi, V, \alpha)$ is *unitary* if the multiplier $\alpha$ is unitary, i.e., there exists a number $N$ with $\alpha(x, y)^N = 1$ for any $x, y \in G$.

For a multiplier $\alpha$, denote by $[\alpha]$ the image of $\alpha$ in $H^2(G, \mathbb{C}^{\times})$. For any $\alpha$, there exists a unitary multiplier $\alpha'$ with $[\alpha'] = [\alpha]$ ([2, Section 1]). Since there is an equivalence between $\mathrm{Rep}_G^{\alpha}$ and $\mathrm{Rep}_G^{\alpha'}$ ([3, Remark 5] or [11]), to study $\mathrm{Rep}_G^{\alpha}$, we may assume that $\alpha$ is unitary. In Section 2, we develop a character theory for unitary projective representations of finite groups by exploiting the analogy with the character theory of linear representations of finite groups.

Moreover, by the standard averaging argument, a projective representation in $\mathrm{Rep}_G^{\alpha}$ decomposes as a direct sum of irreducible ones ([3, Definition 8]). To understand $\mathrm{Rep}_G^{\alpha}$, it suffices to understand the irreducible objects in it. We compute the number of distinct irreducible projective representations in $\mathrm{Rep}_G^{\alpha}$ (Proposition 2.6) and prove some properties of the degrees of these irreducible objects (Theorem 3.5 and 3.9).

The theory of projective representations of finite groups has a long history ([1], [5], [6], [7], [8], [9], etc.). Some of the results in this paper have been proved before. The author claims no originality of those results. See the survey paper [3] for more discussion on the history and a more complete list of references.

Nevertheless, the treatment in this paper is different and induces new results. One main feature is that the *representation groups* $G^*$ ([3, Definition 12]) play no roles here. If we

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take $\alpha$ to be the trivial multiplier, we recover the properties of linear representations of finite groups.

## 2. The character theory

Fix $\alpha$ a unitary multiplier of $G$. Let $(\pi, V, \alpha)$ be a projective representation of $G$. The *character* of $(\pi, V, \alpha)$ $\chi_\pi : G \to \mathbb{C}$ is defined by the equation

$$\chi_\pi(g) = \mathrm{Tr}(\pi(g)) \text{ for all } g \in G.$$

Since $\alpha$ is unitary, $\pi(g)^M = \mathrm{id}_V$ for some number $M$. A simple computation shows that $\chi_\pi(g^{-1}) = \alpha(g, g^{-1})\overline{\chi_\pi(g)}$, where $^-$ denotes the complex conjugation. This property is the main reason that we only consider characters of unitary projective representations.

By tracking through the role of the multiplier $\alpha$ and following the argument in the theory of linear representations of finite groups, we may prove many properties of projective representations which are similar to those of linear representations. In the following, we explain this strategy. If the proofs are straightforward generalizations from the linear representations case, we skip the details and refer to the corresponding parts in the book [10].

2.1. **Basic properties.** Let $(\pi, V, \alpha)$ and $(\pi', W, \alpha)$ be two projective representations of $G$ with the same multiplier $\alpha$. A linear map $\varphi : V \to W$ is called a *$G$-morphism* or a *map of projective representations* if for any $g \in G$ and $v \in V$, $\varphi(\pi(g)v) = \pi'(g)(\varphi(v))$. Write $\mathrm{Hom}_G(V, W)$ for the set of all $G$-morphisms from $V$ to $W$. First, the Schur's Lemma is true for projective representations.

**Lemma 2.1** (Schur's Lemma)**.** *If $V$ and $W$ are irreducible projective representations of $G$ in $\mathrm{Rep}_G^\alpha$ and $\varphi : V \to W$ is a map of projective representations, then*

(1) *Either $\varphi$ is an isomorphism or $\varphi = 0$.*
(2) *If $V = W$, then $\varphi = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.*

If $\phi$ and $\psi$ are two $\mathbb{C}$-valued functions on $G$, define

$$(\phi, \psi) = \frac{1}{|G|}\sum_{g\in G} \phi(g)\overline{\psi(g)}.$$

This is a scalar product, i.e., it is linear in $\phi$, semi-linear in $\psi$, and $(\phi, \phi) > 0$ for all $\phi \neq 0$. Applying Schur's Lemma, we have the orthogonality relations for characters ([10, Section 2.3]).

**Proposition 2.2.** *Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ be two unitary projective representations of $G$ with the same multiplier. Then*

(1) *If $\chi_{\pi_1} = \chi_{\pi_2}$, then $\pi_1 \cong \pi_2$.*
(2) $(\chi_{\pi_1}, \chi_{\pi_2}) = \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi_1, \pi_2)$.
(3) *If $G$ is abelian, then all finite dimensional irreducible projective representations of $G$ with multiplier $\alpha$ have the same degree. Denote this number by $d_G^\alpha$.*

*Proof.* We prove the last assertion. Assume that $\alpha$ is unitary. Let $\pi_i$ be an irreducible projective representation of $G$ with multiplier $\alpha$ and character $\chi_i$ $(i = 1, 2)$. Let $\bar\pi_2$ be the projective representation of $G$ defined by $\bar\pi_2(g) = \overline{\pi_2(g)}$. Here $\bar x$ is the complex conjugation of $x$. Then the associated multiplier of $\bar\pi_2$ is $\bar\alpha = \alpha^{-1}$ since $\alpha$ is unitary.

The character of $\bar{\pi}_2$ is $\bar{\chi}_2$. Consider the projective representation $\pi_1 \otimes \bar{\pi}_2$. The associated multiplier is $\alpha \cdot \alpha^{-1} = 1$. Because $G$ is abelian, there exists a one-dimensional linear representation $\tau : G \to \mathbb{C}^\times$, such that $\dim_{\mathbb{C}} \mathrm{Hom}_G(\tau, \pi_1 \otimes \bar{\pi}_2) \geq 1$, i.e., the number $\frac{1}{|G|}\sum_{g\in G} \tau(g)\overline{\chi_1(g)\bar{\chi}_2(g)}$ is a positive integer. Thus

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi_2, \bar{\tau} \otimes \pi_1) = \frac{1}{|G|}\sum_{g\in G} \chi_2(g)\overline{\chi_1(g)\bar{\tau}(g)}$$

is a positive integer. Moreover, both $\pi_2$ and $\bar{\tau} \otimes \pi_1$ are irreducible and thus they are isomorphic. The claim follows. □

Let $R$ be the $\alpha$-*regular representation* of $G$. It has a basis $(e_g)_{g\in G}$ such that $R(h)(e_g) = \alpha(h,g)e_{hg}$. It is easy to see that $\mathrm{Tr}(R(h)) = 0$ if $h \neq 1$, $\mathrm{Tr}(R(1)) = \deg R = |G|$. As a consequence of Proposition 2.2, we have a decomposition of $R$ ([10, Section 2.4]).

**Proposition 2.3.** *As an object in* $\mathrm{Rep}_G^\alpha$, *$R$ decomposes as*

$$R \cong \bigoplus_{\pi\in\mathrm{Rep}_G^\alpha \ \textit{irreducible}} \pi^{\oplus \deg \pi}.$$

**Definition 2.4.** A function $f : G \to \mathbb{C}$ is called an $\alpha$-*class function* if for all $g, h \in G$,

$$f(hgh^{-1}) = \frac{\alpha(h,h^{-1})}{\alpha(h,gh^{-1})\alpha(g,h^{-1})} f(g) = \frac{\alpha(h,h^{-1})}{\alpha(h,g)\alpha(hg,h^{-1})} f(g).$$

Let $\mathbb{H}_\alpha$ denote the space of $\alpha$-class functions on $G$. The characters of projective representations belong to $\mathbb{H}_\alpha$. Let $g \in G$. We say that $g$ is an $\alpha$-*element* if $\frac{\alpha(h,h^{-1})}{\alpha(h,gh^{-1})\alpha(g,h^{-1})} = 1$ for all elements in $C_G(g) = \{h \in G \mid hg = gh\}$.

*Remark* 2.5. From the definition, it is easy to check that $g$ is an $\alpha$-element if and only if $\alpha(g,h) = \alpha(h,g)$ for all $h \in C_G(g)$. Note that $C_G(xgx^{-1}) = xC_G(g)x^{-1}$. Then if $g \in G$ is an $\alpha$-element, so are the conjugates of $g$. This follows from the equation

$$\begin{aligned}\alpha(xgx^{-1}, xhx^{-1}) &= \frac{\alpha(x, ghx^{-1})\alpha(gx^{-1}, xhx^{-1})}{\alpha(x, gx^{-1})} \\ &= \frac{\alpha(x, ghx^{-1})}{\alpha(x, gx^{-1})}\frac{\alpha(gx^{-1}, x)\alpha(g, hx^{-1})}{\alpha(x, hx^{-1})} \\ &= \frac{\alpha(x, ghx^{-1})\alpha(gx^{-1}, x)}{\alpha(x, gx^{-1})\alpha(x, hx^{-1})}\frac{\alpha(g,h)\alpha(gh, x^{-1})}{\alpha(h, x^{-1})} \\ &= \frac{\alpha(x, ghx^{-1})\alpha(g,h)\alpha(gh, x^{-1})\alpha(x, x^{-1})}{\alpha(x, gx^{-1})\alpha(x, hx^{-1})\alpha(h, x^{-1})\alpha(g, x^{-1})}.\end{aligned} \tag{2.1}$$

Therefore, $\dim_{\mathbb{C}} \mathbb{H}_\alpha = l_\alpha$, where $l_\alpha$ is the number of the conjugacy classes of $G$ which consists of $\alpha$-elements.

We have the following result ([10, Section 2.5]).

**Proposition 2.6.** *Let $\alpha$ be a unitary multiplier. The characters $(\chi_i)$ of irreducible projective representations in $\mathrm{Rep}_G^\alpha$ form an orthonormal basis of $\mathbb{H}_\alpha$. In particular, the number of irreducible projective representations with associated multiplier $\alpha$ (up to isomorphism) is equal to* $\dim_{\mathbb{C}} \mathbb{H}_\alpha = l_\alpha$.

2.2. **Induced projective representations.** Fix a unitary multiplier $\alpha$ of $G$. Let $H \subset G$ be a subgroup of $G$. Denote by $\alpha_H : H \times H \to \mathbb{C}^\times$ the restriction of $\alpha$. Let $(\mathfrak{p}, W, \alpha_H)$ be a projective representation of $H$. Let $V$ be the vector space

$$V = \{f : G \to W \mid f(hg) = \alpha(hg, g^{-1})\mathfrak{p}(h)f(g) \text{ for all } h \in H, g \in G\}.$$

We define a map $\pi : G \to \mathrm{GL}(V)$ by the equation $(\pi(g)f)(g') = \alpha(g', g)f(g'g)$. It is easy to check that $\pi$ is a projective representation of $G$ with multiplier $\alpha$, which is called the *induction* of $\mathfrak{p}$ and is denoted by $\mathrm{Ind}_H^G W$ or $\mathrm{Ind}_H^G \mathfrak{p}$. In this case, the character $\chi_\pi$ is determined by the character $\chi_\mathfrak{p}$ ([10, Section 3.3, Exercise 3.3]). First, we have the following lemma.

**Lemma 2.7.** *For any $w \in W$, define $f_w : G \to W$ by*

$$f_w(g) = \begin{cases} \mathfrak{p}(g)w & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

*Then for any $f \in V$, $f = \sum_{Hx \in H\backslash G} \pi(x^{-1})f_{f(x)}$.*

**Proposition 2.8.** *Let $\alpha$ be a unitary multiplier of $G$. Let $(\mathfrak{p}, W, \alpha_H)$ be a projective representation of $H$ with character $\chi_\mathfrak{p}$. Let $(\pi, V, \alpha)$ be the projective representation of $G$ induced from $(\mathfrak{p}, W)$. If $\chi_\pi$ is the character of $G$, then*

$$\chi_\pi(g) = \sum_{\substack{r \in H\backslash G \\ rgr^{-1} \in H}} \frac{\alpha(g, r^{-1})}{\alpha(r^{-1}, rgr^{-1})}\chi_\mathfrak{p}(rgr^{-1}) = \frac{1}{|H|}\sum_{\substack{s \in G \\ sgs^{-1} \in H}} \frac{\alpha(g, s^{-1})}{\alpha(s^{-1}, sgs^{-1})}\chi_\mathfrak{p}(sgs^{-1}).$$

*Proof.* Fix a set $\{x_i\}$ of representatives of the right cosets $H\backslash G$. Define $V' = \oplus_{x_i} W_{x_i}$, where $W_{x_i} = W$ as vector spaces for all $x_i$. Define a map $\pi' : G \to \mathrm{GL}(V')$ by

$$\pi'(g)((w_i)_{w_i \in W_{x_i}}) = \Big(\frac{\alpha(g, x_{\theta(i)}^{-1})}{\alpha(x_i^{-1}, x_i g x_{\theta(i)}^{-1})}\mathfrak{p}(x_i g x_{\theta(i)}^{-1})w_{\theta(i)}\Big). \tag{2.2}$$

Here $\theta(i)$ is the index such that $x_i g \in Hx_{\theta(i)}$. Using the functions in Lemma 2.7, we define a map $F : V' \to V$ by

$$F((w_i)_{w_i \in W_{x_i}}) = \sum_i \pi(x_i^{-1})f_{w_i}.$$

One checks that $F$ is an isomorphism of vector spaces and $\pi(g) \circ F = F \circ \pi'(g)$. Thus $F$ is an isomorphism of projective representations of $G$. The first equality follows by the same argument as in [10, Chap 3, Prop. 12]. The second equality follows from the equation (which can be shown by direct computation with multipliers)

$$\frac{\alpha(g, r^{-1})}{\alpha(r^{-1}, rgr^{-1})}\chi_\mathfrak{p}(rgr^{-1}) = \frac{\alpha(g, s^{-1})}{\alpha(s^{-1}, sgs^{-1})}\chi_\mathfrak{p}(sgs^{-1}),$$

where $r \in G$ with $rgr^{-1} \in H$ and $s \in Hr$. $\square$

*Remark* 2.9. By identifying projective representations in $\mathrm{Rep}_G^\alpha$ with modules over the twisted group algebra $\mathbb{C}[G]_\alpha$ ([2]), the proof of Proposition 2.8 shows that $\mathrm{Ind}_H^G W \cong \mathbb{C}[G]_\alpha \otimes_{\mathbb{C}[H]_{\alpha_H}} W$. Therefore, for any $E$ in $\mathrm{Rep}_G^\alpha$, we have a canonical isomorphism

$$\mathrm{Hom}_H(W, E|_H) = \mathrm{Hom}_G(\mathrm{Ind}_H^G W, E).$$

Starting with Proposition 2.8 and arguing as in [10, Section 7.3, 7.4], we obtain the *Mackey's irreducibility criterion* for projective representations. For $g \in G$, denote by $H_g$ the subgroup $g^{-1}Hg \cap H$. The projective representation $\mathfrak{p}$ of $H$ defines a projective representation $\mathrm{Res}^H_{H_g} \mathfrak{p}$ by restriction to $H_g$.

**Proposition 2.10** (Mackey's criterion)**.** *In order that the induced projective representation $V = \mathrm{Ind}^G_H W$ be irreducible, it is necessary and sufficient that the following two conditions be satisfied:*

(1) *$W$ is irreducible.*
(2) *For each $s \in G - H$, the two representations $\mathfrak{p}^s$ and $\mathrm{Res}^H_{H_s} \mathfrak{p}$ of $H_s$ are disjoint, i.e., $(\chi_{\mathfrak{p}^s}, \chi_{\mathrm{Res}^H_{H_s} \mathfrak{p}})_{H_s} = 0$. Here $\mathfrak{p}^s$ is the projective representation of $H_s$ over $W$ defined by*

$$\mathfrak{p}^s(x) = \frac{\alpha(x, s^{-1})}{\alpha(s^{-1}, sxs^{-1})}\mathfrak{p}(sxs^{-1}) \text{ for } x \in H_s. \tag{2.3}$$

Assume that $H$ is a normal subgroup of $G$. Let $(\mathfrak{p}, W, \alpha_H)$ be a projective representation of $H$. For any $s \in G$, $\mathfrak{p}^s : H \to \mathrm{GL}(W)$ defined by equation (2.3) is a projective representation of $H$ with associated multiplier $\alpha_H$. It is called the *twist* of $\mathfrak{p}$ by $s$. (See for example [3, Lemma 59].)

**Corollary 2.11.** *Suppose that $H$ is a normal subgroup of $G$. In order that $\mathrm{Ind}^G_H \mathfrak{p}$ be irreducible, it is necessary and sufficient that $\mathfrak{p}$ is irreducible and is not isomorphic to any of its twists $\mathfrak{p}^s$ for $s \in G - H$.*

2.3. **On abelian groups.** In this subsection, assume that $G$ is abelian. We describe the number $d^\alpha_G$ in Proposition 2.2(3) more precisely using the results in Subsection 2.2. Let $\alpha$ be a multiplier of group $G$. Let $A \subset G$ be a subgroup. We say that $A$ is *$\alpha$-symmetric* if $\alpha(a, b) = \alpha(b, a)$ for any $a, b \in A$.

**Lemma 2.12.** *If $G$ is abelian and $\alpha$-symmetric, then $\alpha$ is a coboundary.*

*Proof.* Let $\pi$ be any irreducible projective representation of $G$ with multiplier $\alpha$. Then by assumption $\pi(a)\pi(b) = \pi(b)\pi(a)$ for any $a, b \in G$. Therefore, each $\pi(a)$ is an element of $\mathrm{Hom}_G(\pi, \pi)$. By Schur's Lemma, $\pi(a)$ is a scalar, say $\mu(a)$. Then $\alpha(a, b) = \frac{\mu(a)\mu(b)}{\mu(ab)}$ is a coboundary. □

**Lemma 2.13.** *Let $A$ be an $\alpha$-symmetric subgroup of an abelian group $G$. Let $s \in G - A$. If $\alpha(a, s^i) = \alpha(s^i, a)$ for all $a \in A$ and $i \in \mathbb{Z}$, then the subgroup $B = \langle A, s\rangle$ is also $\alpha$-symmetric.*

*Proof.* This follows from the identity $\alpha(as^i, bs^j) = \frac{\alpha(a,b)\alpha(ab,s^{i+j})\alpha(s^i,s^j)}{\alpha(a,s^i)\alpha(s^j,b)}$. □

**Proposition 2.14.** *Let $G$ be an abelian group. Let $\alpha$ be a fixed multiplier of $G$. Let $A$ be a maximal $\alpha$-symmetric subgroup of $G$. Then $d^\alpha_G = (G : A)$.*

*Proof.* Let $\pi$ be an irreducible projective representation of $G$ with unitary multiplier $\alpha$. Consider the restriction $\pi|_A$, it is a projective representation of $A$ with multiplier $\alpha|_A$, which is a coboundary by Lemma 2.12. Thus $\pi|_A = \oplus_{i\in I}\chi_i$ is a finite direct sum of one-dimensional projective representations. Fix one $\chi \in \{\chi_i\}_{i\in I}$ and consider the projective representation $V' = \mathrm{Ind}^G_A \chi$.

First, we show that $V'$ is irreducible. By Corollary 2.11, it suffices to show that $\chi$ is not isomorphic to $\chi^s$ for any $s \in G-A$. Suppose that there exists $s \in G-A$ such that $\chi \cong \chi^s$. From the definition of $\chi^s$, we have $\alpha(a, s^{-1}) = \alpha(s^{-1}, a)$ for any $a \in A$. Inductively, we see that $\alpha(a, s^i) = \alpha(s^i, a)$ for any $a \in A$ and $i \in \mathbb{Z}$. Therefore, by Lemma 2.13, $\langle A, s\rangle$ is an $\alpha$-symmetric subgroup, which contradicts the assumption on $A$. Thus $V'$ is irreducible.

On the other hand, by Remark 2.9, $\mathrm{Hom}_G(V', \pi) = \mathrm{Hom}_A(\chi, \pi|_A)$ has a nontrivial element. So $V' \cong \pi$ and $\deg \pi = (G : A)$. The theorem follows. $\square$

**Corollary 2.15.** *Let $\alpha$ be a multiplier of an abelian group $G$. Then all the maximal $\alpha$-symmetric subgroups of $G$ have the same index in $G$, and this number is less or equal to $\sqrt{|G|}$. In particular, for any abelian group $G$ and $\alpha \in Z^2(G, \mathbb{C}^\times)$, there exists a subgroup $A$ of $G$ with $|A| \geq \sqrt{|G|}$ such that $[\alpha|_A]$ is trivial in $H^2(A, \mathbb{C}^\times)$.*

## 3. The degrees of irreducible projective representations

In this section, we study the degrees of irreducible projective representations in $\mathrm{Rep}_G^\alpha$ using the results in Section 2. First, arguing as in [10, Section 6.5], we show that the degree of an irreducible object in $\mathrm{Rep}_G^\alpha$ divides the order of $G$ (Theorem 3.5). Then by studying the extensions of irreducible projective representations, we prove a stronger version Theorem 3.9.

### 3.1. The structure of $\mathbb{C}[G]_\alpha$.

Since $\mathbb{C}$ is algebraically closed, each skew field or field of finite degree over $\mathbb{C}$ is equal to $\mathbb{C}$. Thus the *twisted group algebra* $\mathbb{C}[G]_\alpha$ ([2]) is a product of matrix algebras $M_{n_i}(\mathbb{C})$. Let $\pi_i : G \to \mathrm{GL}(W_i)$ be the distinct irreducible projective representations of $G$ with associated multiplier $\alpha$ ($i = 1, \ldots, l = l_\alpha$). Let $n_i = \dim W_i$. Then the ring $\mathrm{End}_{\mathbb{C}}(W_i)$ of endomorphisms of $W_i$ is isomorphic to $M_{n_i}(\mathbb{C})$. The map $\pi_i : G \to \mathrm{GL}(W_i)$ extends by linearity to an algebra homomorphism $\Pi_i : \mathbb{C}[G]_\alpha \to \mathrm{End}(W_i)$. We thus obtain a homomorphism

$$\Pi : \mathbb{C}[G]_\alpha \to \prod_{i=1}^{l} \mathrm{End}(W_i) \cong \prod_{i=1}^{l} M_{n_i}(\mathbb{C}),$$

which is an isomorphism of $\mathbb{C}$-algebras ([10, Section 6.2]).

**Lemma 3.1.** [10, Section 6.5] *The homomorphism $\Pi_i$ maps $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ (the center of the twisted group algebra) into the set of homotheties of $W_i$ and defines an algebra homomorphism*

$$\omega_i : \mathrm{Cent}\,.\mathbb{C}[G]_\alpha \to \mathbb{C}.$$

*If $\alpha$ is unitary, $f = \sum_{g\in G} k_g a_g$ is an element of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$, then*

$$\omega_i(f) = \frac{1}{n_i} \mathrm{Tr}_{W_i}(\Pi_i(f)) = \frac{1}{n_i} \sum_{g\in G} k_g \chi_i(g).$$

*Moreover, the family $(\omega_i)_{1\leq i\leq l}$ defines an isomorphism of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ onto the algebra $\mathbb{C}^l$.*

Let $C$ be the set of conjugacy classes of $\alpha$-elements of $G$. For each $c \in C$, fix an element $g_c \in c$. Set

$$e_c = \sum_{h\in G} a_h a_{g_c} a_h^{-1} = \sum_{h\in G} \frac{\alpha(h, g_c)\alpha(hg_c, h^{-1})}{\alpha(h, h^{-1})} a_{hg_ch^{-1}}.$$

It is easy to see that $e_c$ is an element of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$. The elements $(e_c)_{c\in C}$ form a basis of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$.

*Remark* 3.2. The definition of $e_c$ depends on the choice of the fixed element $g_c \in c$. Let $g'_c = sg_cs^{-1} \in c$ be another element and define

$$e'_c = \sum_{h\in G} a_h a_{g'_c} a_h^{-1} = \sum_{h\in G} \frac{\alpha(h, g'_c)\alpha(hg'_c, h^{-1})}{\alpha(h, h^{-1})} a_{hg'_c h^{-1}}.$$

Then $e_c = \frac{\alpha(s, g_cs^{-1})}{\alpha(g_cs^{-1}, s)} e'_c$.

Indeed, let $g = g_c$, to see this, it suffices to prove that

$$\frac{\alpha(hs, g)\alpha(hsg, h^{-1})\alpha(gs^{-1}, s)}{\alpha(hs, (hs)^{-1})} = \frac{\alpha(h, sgs^{-1})\alpha(hsgs^{-1}, h^{-1})\alpha(s, gs^{-1})}{\alpha(h, h^{-1})}.$$

This follows from

$$\begin{aligned} &\alpha(h, sgs^{-1})\alpha(hsgs^{-1}, h^{-1})\alpha(s, gs^{-1})\alpha(hs, (hs)^{-1}) \\ =&\alpha(h, s)\alpha(hs, gs^{-1})\alpha(hsgs^{-1}, h^{-1})\alpha(hs, (hs)^{-1}) \\ =&\alpha(h, s)\alpha(hs, gs^{-1}h^{-1})\alpha(gs^{-1}, h^{-1})\alpha(hs, (hs)^{-1}) \\ =&\alpha(hs, gs^{-1}h^{-1})\alpha(gs^{-1}, h^{-1})\alpha(h, h^{-1})\alpha(s, s^{-1}h^{-1}) \\ =&\alpha(hs, gs^{-1}h^{-1})\alpha(h, h^{-1})\alpha(gs^{-1}, s)\alpha(g, s^{-1}h^{-1}). \end{aligned} \tag{3.1}$$

In particular, in the case $\alpha$ is unitary, the difference between $e_c$ and $e'_c$ is given by a root of unity.

3.2. **Degrees of irreducible projective representations.** Let $(\pi, V, \alpha)$ be a unitary projective representation of $G$ with character $\chi$. Note that our $\alpha$ is unitary, therefore the eigenvalues of $\pi(g)$ are roots of unity. In particular, they are algebraic integers. Thus the value $\chi(g)$, which is the sum of the eigenvalues of $\pi(g)$, is also an algebraic integer.

**Lemma 3.3.** *Let $f = \sum_{g\in G} k_g a_g$ be an element of* $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ *such that $k_g$'s are algebraic integers. Then $f$ is integral over $\mathbb{Z}$. (Note that this makes sense since* $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ *is a commutative ring.)*

*Proof.* By Remark 3.2, we may write $f = \sum_{c\in C} k_c e_c$ for some algebraic integers $k_c$. To show that $f$ is integral over $\mathbb{Z}$, it suffices to show that each $e_c$ is integral over $\mathbb{Z}$. Let $\mathcal{O} = \mathbb{Z}[\mathrm{Im}(\alpha)]$. It is contained in the ring of integers of the field $\mathbb{Q}(\mathrm{Im}(\alpha))$ and thus is finitely generated over $\mathbb{Z}$. Note that $e_ce_d$ is a linear combination with $\mathcal{O}$-coefficients of the $e_c$'s, the subgroup $R = \oplus_{c\in C}\mathcal{O}\cdot e_c$ is a subring of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ and it is finitely generated over $\mathbb{Z}$. Every element in $R$ is integral over $\mathbb{Z}$. The claim follows. □

**Lemma 3.4.** *Let $(\pi_i, W_i, \alpha)$ be an irreducible unitary projective representation of $G$ with degree $n_i$ and character $\chi_i$. Let $f = \sum_{g\in G} k_g a_g$ be an element of* $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ *such that $k_g$'s are algebraic integers. Then the number $\frac{1}{n_i}\sum_{g\in G} k_g\chi_i(g)$ is an algebraic integer.*

*Proof.* By Lemma 3.1, this number is the image of $f$ under the homomorphism

$$\omega_i : \mathrm{Cent}\,.\mathbb{C}[G]_\alpha \to \mathbb{C}.$$

As $f$ is integral over $\mathbb{Z}$ by Lemma 3.3, the same is true for its image under $\omega_i$. □

**Theorem 3.5.** *The degrees of the irreducible projective representations of $G$ divide the order of $G$.*

*Proof.* It suffices to prove this for unitary irreducible projective representations. Let $\chi$ be the character of such a projective representation with multiplier $\alpha$. First, we show that the element $\sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})a_g$ is an element of Cent $.\mathbb{C}[G]_\alpha$. It suffices to show that $a_h(\sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})a_g) = (\sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})a_g)a_h$ for any $h\in G$. This is equivalent to

$$\begin{aligned} &\alpha(hgh^{-1},hg^{-1}h^{-1})^{-1}\chi(hg^{-1}h^{-1})\alpha(hgh^{-1},h) = \alpha(g,g^{-1})^{-1}\chi(g^{-1})\alpha(h,g) \\ &\Leftrightarrow \alpha(hgh^{-1},hg^{-1}h^{-1}) = \frac{\alpha(hgh^{-1},h)\alpha(h,h^{-1})\alpha(g,g^{-1})}{\alpha(h,g^{-1}h^{-1})\alpha(g^{-1},h^{-1})\alpha(h,g)} \\ &\Leftrightarrow \alpha(hgh^{-1},h)\alpha(h,gh^{-1})\alpha(g,h^{-1}) = \alpha(h,h^{-1})\alpha(h,g) \quad \text{(by equation (2.1))}, \end{aligned} \tag{3.2}$$

which is easy to see since $\alpha$ is a multiplier.

Now applying Lemma 3.4 to the element $\sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})a_g$, the number

$$\frac{1}{n_i}\sum_{g\in G}k_g\chi_i(g) = \frac{1}{n_i}\sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})\chi(g) = \frac{|G|}{n_i}(\chi,\chi) = \frac{|G|}{n_i}$$

is an algebraic integer. Therefore $n_i \mid |G|$. The claim follows. □

**Corollary 3.6.** *Let $G$ be a finite group of order $N$. Let $l^c$ be the number of conjugacy classes of $G$. If the equation*

$$N = n_1^2 + \cdots + n_m^2$$

*has no solution with $m\in\mathbb{Z}_{\geq 1}$, $m\leq l^c$, $n_i\in\mathbb{Z}_{\geq 2}$ and $n_i \mid N$ $(1\leq i\leq m)$, then $H^2(G,\mathbb{C}^\times)=0$.*

Let $\alpha$ be a multiplier of $G$. Let $A$ be a normal subgroup of $G$. Let $\mathfrak{p}: A\to \mathrm{GL}(W)$ be a projective representation of $A$ with multiplier $\alpha$. Define

$$I_{\mathfrak{p}} = \{g\in G:\ \mathfrak{p}^g\cong\mathfrak{p}\}.$$

It is easy to see that $I_{\mathfrak{p}}$ is a subgroup of $G$ and $A$ is a normal subgroup of $I_{\mathfrak{p}}$.

**Lemma 3.7.** *([3, Theorem 62]) Let $(\mathfrak{p},W,\alpha)$ be an irreducible projective representation of $A$. One can extend $W$ to a projective representation $\mathfrak{p}'$ of $I_{\mathfrak{p}}$ with some multiplier $\beta$ such that*

1. *$\mathfrak{p}'(g)\mathfrak{p}(h)\mathfrak{p}'(g)^{-1} = \mathfrak{p}^g(h)$ for all $g\in I_p$ and $h\in A$.*
2. *$\mathfrak{p}'(h)=\mathfrak{p}(h)$ for all $h\in A$.*
3. *$\mathfrak{p}(h)\mathfrak{p}'(g)=\alpha(h,g)\mathfrak{p}'(hg)$.*

The following lemma corresponds to [10, Chap. 8, Proposition 24].

**Lemma 3.8.** *Let $A$ be a normal subgroup of $G$ and $\pi: G\to\mathrm{GL}(V)$ be an irreducible projective representation of $G$. Then*

1. *either there exist a subgroup $H$ of $G$, unequal to $G$ and containing $A$, and an irreducible projective representation $\mathfrak{p}$ of $H$ such that $\pi$ is induced from $\mathfrak{p}$;*
2. *or else the restriction $\mathrm{Res}_A^G\pi$ is isotypic, i.e., it is a direct sum of isomorphic projective representations of $A$.*

**Theorem 3.9.** *Let $A$ be a normal subgroup of $G$. Let $d_A$ be the least common multiple of the degrees of the irreducible projective representations of $A$. (Note that $d_A \mid |A|$.) Then the degree of each irreducible projective representation $\pi$ of $G$ divides the number $d_A \cdot (G : A)$.*

*Proof.* We prove this theorem by induction on the order of $G$. In case (1) of Lemma 3.8, by induction, the degree of $\mathfrak{p}$ divides $d_A \cdot (H : A)$. Therefore, the degree of $\pi$ divides $(G : H) d_A \cdot (H : A) = d_A \cdot (G : A)$.

In case (2) of Lemma 3.8, assume that $V|_A = W^{\oplus k}$ for an irreducible projective representation $(\mathfrak{p}, W)$ of $A$. Thus any twist of $\mathfrak{p}$ is isomorphic to $\mathfrak{p}$. By Lemma 3.7, we may extend $W$ to a projective representation $\mathfrak{p}'$ of $G$ with associated multiplier $\beta$. Define $W' = \mathrm{Hom}_A(W, V) = \{f : W \to V \mid f(\mathfrak{p}(a)w) = \pi(a)f(w)\}$. Define $\mathfrak{q} : G \to \mathrm{GL}(W')$ via the equation

$$(\mathfrak{q}(g)f)(w) = \pi(g)f(\mathfrak{p}'(g)^{-1}w).$$

(1) By Lemma 3.7, $\mathfrak{p}'(g)^{-1}\mathfrak{p}(a) = \frac{\alpha(a,g)}{\alpha(g,g^{-1}ag)}\mathfrak{p}(g^{-1}ag)\mathfrak{p}'(g)^{-1}$. One has

$$\begin{aligned}(\mathfrak{q}(g)f)(\mathfrak{p}(a)w) &= \pi(g)f(\mathfrak{p}'(g)^{-1}\mathfrak{p}(a)w)\\ &= \pi(g)\frac{\alpha(a,g)}{\alpha(g,g^{-1}ag)}f(\mathfrak{p}(g^{-1}ag)\mathfrak{p}'(g)^{-1}w)\\ &= \pi(g)\frac{\alpha(a,g)}{\alpha(g,g^{-1}ag)}\pi(g^{-1}ag)f(\mathfrak{p}'(g)^{-1}w)\\ &= \pi(a)\pi(g)f(\mathfrak{p}'(g)^{-1}w) = \pi(a)(\mathfrak{q}(g)f)(w).\end{aligned} \tag{3.3}$$

Thus $\mathfrak{q}(g)f \in W'$ and $\mathfrak{q}$ is well-defined.

(2) For any $g_1, g_2 \in G$,

$$\begin{aligned}(\mathfrak{q}(g_1g_2)f)(w) &= \pi(g_1g_2)f(\mathfrak{p}'(g_1g_2)^{-1}w)\\ &= \alpha(g_1,g_2)^{-1}\beta(g_1,g_2)\pi(g_1)\pi(g_2)f(\mathfrak{p}'(g_2)^{-1}\mathfrak{p}'(g_1)^{-1}w)\\ &= \alpha(g_1,g_2)^{-1}\beta(g_1,g_2)(\mathfrak{q}(g_1)\mathfrak{q}(g_2)f)(w).\end{aligned} \tag{3.4}$$

Thus $\mathfrak{q}$ is a projective representation of $G$ over $W'$ with multiplier $\alpha\beta^{-1}$.

Consider the natural map

$$W \otimes_{\mathbb{C}} W' \to V,$$

it is easy to check that it is an isomorphism of projective $G$-representations. Furthermore, since $V$ is irreducible, $W'$ is also irreducible as a projective representation of $G$. On the other hand, if $g \in A$, then $\mathfrak{q}(g)$ acts as scalar. Thus $W'$ has a structure as an irreducible projective representation of $G/A$. Therefore, $\deg W' \mid (G : A)$ by Theorem 3.5. Thus $\deg V \mid d_A \cdot (G : A)$. The theorem follows. □

The same argument proves the following result.

**Theorem 3.10.** *Let $\alpha$ be a multiplier of $G$. Let $A$ be a normal subgroup of $G$. Let $d_A^\alpha$ be the least common multiple of the degrees of the irreducible projective representations of $A$ with associated multiplier $\alpha$. Then the degree of each irreducible projective representation $\pi$ of $G$ with multiplier $\alpha$ divides the number $d_A^\alpha \cdot (G : A)$.*

*In particular, if $\alpha = 1$ and $A$ is an abelian normal subgroup of $G$, then the degree of each irreducible linear representation of $G$ divides the number $(G : A)$.*

**Corollary 3.11.** *Let $A$ be a cyclic normal subgroup of $G$. Then the degree of each irreducible projective representation $\pi$ of $G$ divides the number $(G : A)$.*

*Proof.* Since $A$ is cyclic, $H^2(G, \mathbb{C}^\times) = 0$. Therefore $d_A = 1$ and the claim follows. □

The above results have useful applications. We explain the idea in the following simple but nontrivial example.

**Example 3.12.** Let $G = D_{2m}$ be the dihedral group of order $2m$. Let $C_m$ be the normal subgroup of $G$ generated by an element of order $m$. By Corollary 3.11, the degree of each irreducible projective representation of $G$ divides 2. Similarly as in Corollary 3.6, we obtain the fact that $H^2(D_{2m}, \mathbb{C}^\times) = 0$ if $m$ is odd.

Assume now that $m$ is even. Let $\alpha$ be a multiplier of $D_{2m}$ such that $[\alpha]$ is nontrivial. (For example, $m = 4$, $H^2(D_8, \mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$.) Then every irreducible projective representation of $D_{2m}$ with multiplier $\alpha$ has degree 2 and there are $m/2$ of them up to isomorphism. (Note that in this case the number of conjugacy classes of $G$ is $m/2 + 3$.) By the proof of Theorem 3.9, all these irreducible projective representations are induced from one-dimensional projective representations of $C_m$ with multiplier $\alpha_{C_m}$.

3.3. **Remarks on compact groups.** In the following, $G$ is a compact topological group. Fix a Haar measure $\int_G \cdot \,\mathrm{d}\, g$ on $G$.

A *projective representation* $(\pi, V, \alpha)$ of $G$ over $\mathbb{C}$ is a continuous map $\pi : G \to \mathrm{U}(V)$ such that $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$ for all $x, y \in G$, where $\alpha$ is a multiplier on $G$ with $|\alpha(x, y)| = 1$ for any $x, y \in G$, $V$ is a Hilbert space, $\mathrm{U}(V)$ is the space of unitary operators from $V$ to $V$. Here *continuous* means that the map $(g, v) \mapsto \pi(g)v$ is a continuous map from $G \times V$ to $V$.

Most of the properties of projective representations of finite groups carry over to finite dimensional projective representations of compact groups. The strategy in [10, Section 4.3] applies to the projective representations case.

Let $(\pi, V, \alpha)$ be a finite dimensional projective representation of $G$. Let $\langle , \rangle$ be a $G$-invariant Hermitian inner product on $V$, which exists by the averaging argument. Given $v, w \in V$, the function $f : g \mapsto \langle \pi(g)v, w\rangle$ is a *matrix coefficient* of $\pi$. Let $\mathcal{A}_\alpha(G)$ be the space spanned by all matrix coefficients of finite dimensional irreducible projective representations of $G$ with multiplier $\alpha$. The following result can be proved by the same strategy as for linear representations (see for example [4]) with an extra attention on the multiplier $\alpha$.

**Theorem 3.13** (Peter-Weyl Theorem)**.** *Let $L^2(G)$ be the space of measurable functions on $G$ with $\int_G |f(g)|^2 \,\mathrm{d}\, g < \infty$. Then $\mathcal{A}_\alpha(G)$ is dense in $L^2(G)$.*

As a consequence, every irreducible projective representation of $G$ is finite dimensional and the character theory provides a tool to study $\mathrm{Rep}_G^\alpha$ for compact groups $G$ as well.

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