A SHORT NOTE ON THE BREUIL-MÉZARD CONJECTURE

CHUANGXUN CHENG, HENDRIK VERHOEK

1. Introduction

In this short note, we explain the proof of the Breuil-Mézard conjecture and the geometric Breuil-Mézard conjecture in the case $K = \mathbb{Q}_p$ and $n = 2$ following [1, Section 2] and [3, Section 3]. In order to illustrate the ideas of the proof, we focus on this specific case and make certain technical assumptions. The notions on automorphic multiplicities and cycles are introduced in the previous talks, so we will use these notions without repeating the definitions.

Fix a prime number $p > 3$, a finite extension $\mathbb{F}/\mathbb{F}_p$, and a continuous representation $\bar{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$. Let E be a finite totally ramified extension of $W(\mathbb{F})[1/p]$ with ring of integers $\mathcal{O}$ and uniformiser π. We assume that E is sufficiently large; in particular, we assume that $\sharp\mathbb{F} > 5$, so that $\mathrm{PGL}_2(\mathbb{F})$ is a simple group. Let $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ be an inertial type, i.e., a representation with open kernel which extends to $W_{\mathbb{Q}_p}$. Let ϵ and ω be the p-adic cyclotomic character and the mod p cyclotomic character respectively. Fix integers a, b with $b \geq 0$ and a character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that $\overline{\psi\epsilon} = \det \bar{r}$. We will also write ψ for the character $(\mathbb{A}_F^\infty)^\times/F^\times \to \mathcal{O}^\times$ corresponding to ψ via class field theory, which we normalize so that uniformisers correspond to geometric Frobenius elements. We let $R^{\square,\psi}(a,b,\tau,\bar{r})$ and $R_{cr}^{\square,\psi}(a,b,\tau,\bar{r})$ be the framed deformation $\mathcal{O}$-algebra which are universal for framed deformations of $\bar{r}$ with determinant $\psi\epsilon$, and are potentially semistable (respectively potentially crystalline) with Hodge-Tate weights $(a, a+b+1)$ and inertia type τ. Let $\sigma(tau)$ and $\sigma^{cr}(\tau)$ denote the finite-dimensional irreducible E-representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ corresponding to τ via Henniart's inertia local Langlands correspondence. We set $\sigma(a,b,\tau) = \sigma(\tau) \otimes_E \det^a \mathrm{Symm}^b E^2$ and $\sigma^{cr}(a,b,\tau) = \sigma^{cr}(\tau) \otimes_E \det^a \mathrm{Symm}^b E^2$. We let $L_{a,b,\tau}$ (respectively $L^{cr}_{a,b,\tau}$) be a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice in $\sigma(a,b,\tau)$ (respectively $\sigma^{cr}(a,b,\tau)$.) Write $\sigma_{m,n}$ for the representation $\det^m \otimes \mathrm{Symm}^n \mathbb{F}^2$ of $\mathrm{GL}_2(\mathbb{F}_p)$, $0 \leq m \leq p-2$, $0 \leq n \leq p-1$. Then we may write

$$(L_{a,b,\tau} \otimes_{\mathcal{O}} \mathbb{F})^{ss} \xrightarrow{\sim} \oplus_{m,n} \sigma_{m,n}^{a_{m,n}},$$

and

$$(L^{cr}_{a,b,\tau} \otimes_{\mathcal{O}} \mathbb{F})^{ss} \xrightarrow{\sim} \oplus_{m,n} \sigma_{m,n}^{a^{cr}_{m,n}},$$

for some integers $a_{m,n}$ and $a^{cr}_{m,n}$. The Breuil-Mézard conjecture is the following.

Conjecture 1.1. *There are integers $\mu_{m,n}(\bar{r})$ depending only on m, n, and $\bar{r}$ such that for any a, b, τ,*

$$e(R^{\square,\psi}(a,b,\tau,\bar{r})/\pi) = \sum_{m,n} a_{m,n}\mu_{m,n}(\bar{r}),$$

and

$$e(R_{cr}^{\square,\psi}(a,b,\tau,\bar{r})/\pi) = \sum_{m,n} a_{m,n}^{cr}\mu_{m,n}(\bar{r}),$$

The geometric version of the Breuil-Mézard conjecture is the following.

Conjecture 1.2. *There are cycles $\mathcal{C}_{m,n}(\bar{r})$ depending only on m, n, and $\bar{r}$ such that for any a, b, τ,*

$$Z(R^{\square,\psi}(a,b,\tau,\bar{r})/\pi) = \sum_{m,n} a_{m,n}\mathcal{C}_{m,n}(\bar{r})$$

and

$$Z(R_{cr}^{\square,\psi}(a,b,\tau,\bar{r})/\pi) = \sum_{m,n} a_{m,n}^{cr}\mathcal{C}_{m,n}(\bar{r})$$

Remark 1.3. As remarked in [3], or by [3, Lemma 4.3.1], the truth of the conjectures is independent of the choice of ψ. We may assume that ψ is crystalline. If the Conjecture 1.1 is true for all a, b, τ, then $\mu_{m,n}(\bar{\rho}) = 0$ unless $\det \bar{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{2m+n+1}$. Furthermore, if $\det \rho|_{I_{\mathbb{Q}_p}} = \omega^{2m+n+1}$, we must have

$$\mu_{m,n}(\bar{\rho}) = e(R_{cr}^{\square,\psi}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi),$$

where $\tilde{m}$ is chosen so that $\psi|_{I_{\mathbb{Q}_p}} = \epsilon^{2\tilde{m}+n+1}$.

Similarly, if the Conjecture 1.2 is true for all a, b, τ, then we must have

$$\mathcal{C}_{m,n}(\bar{r}) = Z(R_{cr}^{\square,\psi}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi).$$

The main goal of this note is to explain the proofs of the following two theorems.

Theorem 1.4 (Kisin). *If $\bar{r} \not\sim \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any χ, then Conjecture 1.1 holds for $\bar{r}$.*

Theorem 1.5 (Emerton, Gee). *If $\bar{r} \not\sim \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any χ, then Conjecture 1.2 holds for $\bar{r}$.*

The proofs of these two results are similar. We first realize local representations globally using potential modularity theorems. Then we deduce the theorems using the global argument made in [1]. For the first step, we recall the following result.

Proposition 1.6. *Let $\bar{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. Then there is a totally real field F and a continuous irreducible representation $\bar{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ such that*

1. *p splits completely in F;*
2. *$\bar{\rho}$ is totally odd;*
3. *$\bar{\rho}(G_F) = \mathrm{GL}_2(\mathbb{F})$;*
4. *if $v \nmid p$ is a place of F then $\bar{\rho}|_{G_{F_v}}$ is unramified;*
5. *if $v \mid p$ is a place of F then $\bar{\rho}|_{G_{F_v}} \cong \bar{r}$;*
6. *$[F : \mathbb{Q}]$ is even;*
7. *$\bar{\rho}$ is modular.*

Proof. This is [3, Proposition 3.2.1]. $\square$

This proposition finishes the first step. In the rest of this paper, we concentrate on the global argument. In Section 2, we recall the definitions of quaternionic forms and Hecke operators. We also prove some properties of these objects which are needed in the argument. In Section 3, we explain the global patching. We construct an important object M_∞ and state the relation of M_∞ and the Breuil-Mézard conjecture. In Section 4, we sketch the proofs of the mains results.

2. Quaternionic forms and basic properties

2.1. **Definition of quaternionic forms.** Let F be a totally real field such that $[F:\mathbb{Q}]$ is even. Let D be a quaternion algebra over F which is ramified at all the infinite places of F and at finite places in the set Σ. We assume that $\Sigma\cap\{v : v \mid p\}=\emptyset$. Fix $\mathcal{O}_D$ a maximal order of D and isomorphisms $(\mathcal{O}_D)_v := \mathcal{O}_D\otimes_{\mathcal{O}_F}\mathcal{O}_{F_v}\xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$ for each finite place $v\notin\Sigma$. For each finite place v of F, fix a uniformizer π_v of F_v. Write $\Sigma_p=\Sigma\cup\{v : v\mid p\}$. Let $U=\prod_v U_v\subset(D\otimes_F\mathbb{A}_F^\infty)^\times$ be an open compact subgroup contained in $\prod_v(\mathcal{O}_D)_v^\times$. Assume that $U_v=(\mathcal{O}_D)_v^\times$ for all $v\in\Sigma_p$. In particular, $U_v=\mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v\mid p$.

Let A be a topological $\mathbb{Z}_p$-algebra. For each $v\mid p$, we fix a continuous representation $\sigma_v : U_v\to\mathrm{Aut}(W_{\sigma_v})$ on a finite free A-module. We write $W_\sigma=\otimes_{v|p}W_{\sigma_v}$ and denote by $\sigma:\prod_{v|p}U_v\to\mathrm{Aut}(W_\sigma)$ the corresponding representation. We regard σ as being a representation of U by letting U_v act trivially if $v\nmid p$. Let $\psi:(\mathbb{A}_F^\infty)^\times/F^\times\to A^\times$ be a continuous character such that for any place v of F, σ on $U_v\cap\mathcal{O}_{F_v}^\times$ is given by multiplication by ψ. Then W_σ becomes a $U(\mathbb{A}_F^\infty)^\times$-module if we let $(\mathbb{A}_F^\infty)^\times$ act on W_σ via ψ.

Let $S_{\sigma,\psi}(U,A)$ denote the set of continuous functions

$$f : D^\times\backslash(D\otimes_F\mathbb{A}_F^\infty)^\times\to W_\sigma$$

such that

- $f(gu)=\sigma(u)^{-1}f(g)$ for $u\in U$, $g\in(D\otimes_F\mathbb{A}_F^\infty)^\times$,
- $f(gz)=\psi(z)^{-1}f(g)$ for $z\in(\mathbb{A}_F^\infty)^\times$, $g\in(D\otimes_F\mathbb{A}_F^\infty)^\times$.

We may write $(D\otimes_F\mathbb{A}_F^\infty)^\times=\coprod_{i\in I}D^\times t_iU(\mathbb{A}_F^\infty)^\times$ for some $t_i\in(D\otimes_F\mathbb{A}_F^\infty)^\times$ and some finite index set I, then we have isomorphism

$$S_{\sigma,\psi}(U,A)\xrightarrow{\sim}\oplus_{i\in I}W_\sigma^{(U(\mathbb{A}_F^\infty)^\times\cap t_i^{-1}D^\times t_i)/F^\times}$$
$$f\mapsto\{f(t_i)\}_{i\in I}.$$

We assume that U satisfies the following condition

$$(U(\mathbb{A}_F^\infty)^\times\cap t^{-1}D^\times t)/F^\times=1 \text{ for all } t\in(D\otimes_F\mathbb{A}_F^\infty)^\times. \tag{2.1}$$

Under this condition, we have the following lemma.

Lemma 2.1. *With the notation as above,*

(1) *if B is an A-algebra, then*

$$S_{\sigma,\psi}(U,A)\otimes_A B\xrightarrow{\sim}S_{\sigma\otimes_A B,\psi}(U,B)$$

is an isomorphism.

(2) *$S_{\sigma,\psi}(U,A)$ is a finite projective A-module, and the functor $W_\sigma\mapsto S_{\sigma,\psi}(U,A)$ is exact in W_σ.*

2.2. **Hecke operators.** Let S be the union of the primes in Σ_p and the primes v of F such that $U_v \in D_v^\times$ is not maximal compact. We assume that for $v \in S\backslash\Sigma_p$, $U_v \subset \mathrm{GL}_2(\mathcal{O}_{F_v})$ is contained in the matrices whose reduction modulo π_v is upper triangular and contains those whose reduction is upper triangular unipotent.

Let $\mathbb{T}_{S,\mathcal{O}}^{univ} = \mathcal{O}[T_v, S_v, U_{\pi_w}]_{v\notin S, w\in S\backslash\Sigma_p}$ be a commutative polynomial ring in the indicated formal variables. We consider the left action of $(D\otimes_F \mathbb{A}_F^\infty)^\times$ on W_σ-valued functions on $(D\otimes_F \mathbb{A}_F^\infty)^\times$ given by $(gf)(z) = f(zg)$. Then $S_{\sigma,\psi}(U,\mathcal{O})$ becomes a $\mathbb{T}_{S,\mathcal{O}}^{univ}$-module with S_v acting via the double coset $U_v \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U_v$, T_v acting via the double coset $U_v \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U_v$, and U_{π_w} acting via the double coset $U_w \begin{pmatrix} \pi_w & 0 \\ 0 & 1 \end{pmatrix} U_w$. These operators do not depend on the choice of π_v. We write $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$ or simply $\mathbb{T}_{\sigma,\psi}(U)$ for the image of $\mathbb{T}_{S,\mathcal{O}}^{univ}$ in $\mathrm{End}\, S_{\sigma,\psi}(U,\mathcal{O})$.

Definition 2.2. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{S,\mathcal{O}}^{univ}$. We say that $\mathfrak{m}$ is in the support of (σ,ψ) if $S_{\sigma,\psi}(U,\mathcal{O})_\mathfrak{m}$ is non-zero. We say that $\mathfrak{m}$ is *Eisenstein* if $T_v - 2 \in \mathfrak{m}$ for all but finitely many primes which split in some fixed abelian extension of F.

2.3. **Some properties.** Let Q be a finite set of primes of F which is disjoint from S, and for each $v \in Q$ fix a quotient Δ_v of $(\mathcal{O}_{F_v}/\pi_v)^\times$ of p-power order. Write $\Delta = \prod_{v\in Q}\Delta_v$. Define open compact subgroups U_Q and U_Q^- of U by setting $(U_Q)_v = (U_Q^-)_v = U_v$ if $v \notin Q$, and setting

$$(U_Q^-)_v = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) :\ g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\pi_v}\}$$

and

$$(U_Q)_v = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (U_Q^-)_v :\ ad^{-1} \mapsto 1 \in \Delta_v\}$$

if $v \in Q$.

By definition, $\Delta \xrightarrow{\sim} U_Q^-/U_Q$ and it acts naturally on $S_{\sigma,\psi}(U_Q,\mathcal{O})$ via the right multiplication of U_Q^- on $D^\times\backslash(D\otimes_F \mathbb{A}_F^\infty)^\times$. For each $h \in \Delta$ we denote by $\langle h\rangle$ the corresponding operator.

Lemma 2.3. *With the notation as above,*

(1) *the operator* $\sum_{h\in\Delta}\langle h\rangle$ *on* $S_{\sigma,\psi}(U_Q,\mathcal{O})$ *induces an isomorphism*

$$\sum_{h\in\Delta}\langle h\rangle : S_{\sigma,\psi}(U_Q,\mathcal{O})_\Delta \xrightarrow{\sim} S_{\sigma,\psi}(U_Q^-,\mathcal{O}).$$

(2) $S_{\sigma,\psi}(U_Q,\mathcal{O})$ *is a finite projective* $\mathcal{O}[\Delta]$*-module.*

Proof. By definition, we have

$$\dim S_{\sigma,\psi}(U_Q,\mathcal{O}) = [U_Q^- : U_Q]\dim S_{\sigma,\psi}(U_Q^-,\mathcal{O}).$$

Thus property (2) follows from property (1). For $t \in (D\otimes_F \mathbb{A}_F^\infty)^\times$, we have

$$(U_Q(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = (U_Q^-(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times.$$

Hence it suffices to show that Δ acts freely on $D^\times\backslash(D\otimes_F \mathbb{A}_F^\infty)^\times/U_Q(\mathbb{A}_F^\infty)^\times$. Suppose $u \in U_Q^-$ fixes one of these double cosets, then there exists $t \in (D\otimes_F \mathbb{A}_F^\infty)^\times$ and $v \in U_Q(\mathbb{A}_F^\infty)^\times$ such that

$$uv^{-1} \in U_Q^-(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t = F^\times.$$

Hence $u \in U_Q$. □

Fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}^{univ}_{S,\mathcal{O}}$ such that $\mathfrak{m}$ is induced by a maximal ideal of $\mathbb{T}_{\sigma,\psi}(U)$, and for $v \in Q$ the Hecke polynomial $X^2 - T_vX + \mathrm{Norm}(v)S_v$ has distinct roots in $\mathbb{T}^{univ}_{S,\mathcal{O}}/\mathfrak{m}$. After increasing $\mathbb{F}$, we may assume that $\mathfrak{m}$ has residue field $\mathbb{F}$, and then each of these polynomials has two distinct roots α_v, $\beta_v \in \mathbb{F}$.

Write $S_Q = S \cup Q$. Let $\mathfrak{m}_Q$ denote the ideal $\mathbb{T}^{univ}_{S_Q,\mathcal{O}}$ generated by $\mathfrak{m} \cap \mathbb{T}^{univ}_{S_Q,\mathcal{O}}$ and the elements $U_{\pi_v} - \tilde{\alpha}_v$, where $v \in Q$ and $\tilde{\alpha}_v \in \mathcal{O}$ is any lifting of α_v.

Lemma 2.4. *The ideal $\mathfrak{m}_Q$ induces proper, maximal ideals in $\mathbb{T}_{\sigma,\psi}(U_Q)$ and $\mathbb{T}_{\sigma,\psi}(U_Q^-)$. If $\alpha_v\beta_v \neq \mathrm{Norm}(v)^{\pm 1}$ and $\mathrm{Norm}(v) \equiv 1 \pmod{\pi}$ for all $v \in Q$, then the natural map*

$$S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}} \to S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q} \tag{2.2}$$

is an isomorphism of $\mathbb{T}^{univ}_{S_Q,\mathcal{O}}$-modules.

Proof. The first claim follows from the fact that the Hecke polynomial $X^2 - T_vX + \mathrm{Norm}(v)S_v$ acting on $S_{\sigma,\psi}(U,\mathcal{O}) \subset S_{\sigma,\psi}(U_Q^-,\mathcal{O})$ vanishes at $X = U_{\pi_v}$. To see the second claim, it suffices to consider the case when Q consists of a single element v. We have the following map

$$\begin{aligned} S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}} \oplus S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}} &\to S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}} \\ (f_1,f_2) &\mapsto \pi_1^*f_1 + \pi_2^*f_2, \end{aligned} \tag{2.3}$$

where π_1^* and π_2^* are induced by Id_2 and $\begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix}$ respectively. By next lemma, this is an isomorphism. Moreover, we have the following identity

$$(U_{\pi_v} - B_v)(\pi_1^*f_1 + \pi_2^*f_2) = (U_{\pi_v} - B_v)(\pi_1^*(f_1 + B_vf_2)). \tag{2.4}$$

To see this, it suffices to show

$$\begin{aligned} (U_{\pi_v} - B_v)(\pi_2^*f_2) &= (U_{\pi_v} - B_v)(\pi_1^*(B_vf_2))) \\ &= \pi_1^*(T_vB_vf_2) - \pi_2^*(B_vf_2) - B_v\pi_1^*B_vf_2 \end{aligned} \tag{2.5}$$

This follows from the fact that $B_v^2 - T_vB_v + \mathrm{Norm}(v)S_v = 0$. By identity (2.4), the map $S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}} \to S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q}$ is surjective. Since both sides are finite free $\mathcal{O}$-modules of the same rank, it is an isomorphism. □

Lemma 2.5. *The map (2.3) is an isomorphism.*

Proof. Since both sides of (2.3) are finite free $\mathcal{O}$-modules, it suffices to show that the map is an isomorphism modulo π. First, we have the following identities.

- $T_v = \pi_{1*}\pi_2^*$
- $\mathrm{Norm}(v) + 1 = \pi_{1*}\pi_1^*$
- $U_{\pi_v}(\pi_1^*(f_1)) = \pi_1^*(T_v(f_1)) - \pi_2^*(f_1)$
- $U_{\pi_v}(\pi_2^*(f_2)) = \mathrm{Norm}(v)S_v\pi_1^*(f_2)$

- $U_{\pi_v}^2 - T_v U_{\pi_v} + \mathrm{Norm}(v) S_v = 0$

Assume that $\pi_1^*(f_1) = \pi_2^*(f_2) \pmod{\pi}$, then

$$\begin{aligned} &\pi_{1*}\pi_1^*(f_1) = \pi_{1*}\pi_2^*(f_2) \pmod{\pi} \\ \Rightarrow &((\mathrm{Norm}(v)+1)f_1 = T_v f_2 \pmod{\pi} \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} ((\mathrm{Norm}(v)+1)T_v f_1 - T_v f_1 &= \pi_{1*}(U_{\pi_v}(\pi_1^*(f_1))) \\ &= \pi_{1*}(U_{\pi_v}(\pi_2^*(f_2))) = \mathrm{Norm}(v) S_v (\mathrm{Norm}(v)+1) f_2 \pmod{\pi} \end{aligned} \tag{2.7}$$

Therefore,

$$\begin{aligned} &\mathrm{Norm}(v) T_v f_1 = \mathrm{Norm}(v) S_v (\mathrm{Norm}(v)+1) f_2 \pmod{\pi} \\ \Rightarrow &T_v^2 f_1 = S_v(\mathrm{Norm}(v)+1) T_v f_2 = S_v (\mathrm{Norm}(v)+1)^2 f_1 \pmod{\pi} \\ \Rightarrow &(T_v^2 - 4S_v) f_1 = 0 \pmod{\pi} \end{aligned} \tag{2.8}$$

Since $\alpha_v \neq \beta_v$, so $(T_v^2 - 4S_v) \neq 0 \pmod{\pi}$. Thus $f_1 = 0 \pmod{\pi}$, and $f_2 = 0 \pmod{\pi}$. The map modulo π is injective. Hence the map modulo π is an isomorphism and the lemma follows. □

3. Global patching

3.1. **Setup.** Let $\bar{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. Assume that $\bar{\rho}$ satisfies the following conditions.

(1) $\det \bar{\rho} = \psi\epsilon \pmod{\pi}$.
(2) $\bar{\rho}$ is unramified outside Σ_p, and has odd determinant.
(3) The restriction of $\bar{\rho}$ to $G_{F(\zeta_p)}$ is absolutely irreducible.
(4) If $p = 5$ and $\bar{\rho}$ has projective image isomorphic to $P\,\mathrm{GL}_2(\mathbb{F}_5)$, then the kernel of $\mathrm{proj}\bar{\rho}$ does not fix $F(\zeta_5)$.
(5) $S \backslash \Sigma_p$ contains exactly one prime v and U_v consists of matrices with upper triangular, unipotent reduction. Moreover $(1 - \mathrm{Norm}(v)) \in \mathbb{F}^\times$, and the ratio of the eigenvalues of $\bar{\rho}(\mathrm{Frob}_v)$ is not in $\{1, \mathrm{Norm}(v), \mathrm{Norm}(v)^{-1}\}$. Here, Frob_v denotes an arithmetic Frobenius at v. (This condition makes sure that U satisfies condition (2.1).)

Write $V_{\mathbb{F}}$ for the underlying $\mathbb{F}$-vector space of $\bar{\rho}$ and fix a basis for $V_{\mathbb{F}}$. For $v \in \Sigma_p$, we denote by $R_v^\square$ the universal framed deformation $\mathcal{O}$-algebra of $\bar{\rho}|_{G_{F_v}}$, and by $R_v^{\square,\psi}$ the quotient of $R_v^\square$ corresponding to deformations with determinant $\psi\epsilon$. When $\bar{\rho}$ is absolutely irreducible, we denote by $R_{F,S}^\psi$ the quotient of the universal deformation $\mathcal{O}$-algebra of $\bar{\rho}$, corresponding to deformations with determinant $\psi\epsilon$. We denote by $R_{F,S}^{\square,\psi}$ the complete local $\mathcal{O}$-algebra representing the functor which assigns to a local Artinian $\mathcal{O}$-algebra A, the set of isomorphism classes of tuples $\{V_A, \beta_v\}_{v\in\Sigma_p}$, where V_A is a deformation of $V_{\mathbb{F}}$ to A having determinant $\psi\epsilon$, and β_v is a lifting of the chosen basis of $V_{\mathbb{F}}$ to an A-basis of V_A.

For $v \in \Sigma_p$, the functor $\{V_A, \beta_w\}_{w\in\Sigma_p} \mapsto \{V_A, \beta_v\}$ induces the structure of an $R_v^{\square,\psi}$-algebra on $R_{F,S}^{\square,\psi}$. We set $R_{\Sigma_p}^{\square,\psi} = \hat{\otimes}_{\mathcal{O},v\in\Sigma_p} R_v^{\square,\psi}$.

Suppose that $\mathfrak{m} \subset \mathbb{T}^{univ}_{S,\mathcal{O}}$ is a maximal non-Eisenstein ideal with associated representation $\bar{\rho}$. That is, $\mathrm{Tr}\,\bar{\rho}(\mathrm{Frob}_v) = T_v \pmod{\mathfrak{m}}$ for $v \notin S$. We assume that $\mathfrak{m}$ is chosen so that for $v \in S\backslash\Sigma_p$, $U_{\pi_v} \pmod{\mathfrak{m}}$ is equal to one of the eigenvalues of $\bar{\rho}(\mathrm{Frob}_v)$. We assume that $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}} \neq 0$. In this case, by condition (5) above, $S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}$ is a rank two (not necessarily free) $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$-module. The induced map $R^{\psi}_{F,S} \to \mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$ is surjective.

Before we explain the patching procedure, we recall the following result [2, Proposition 3.2.5],.

Proposition 3.1. *Set $g = \dim_{\mathbb{F}} H^1(G_F, ad^0\bar{\rho}(1)) - [F:\mathbb{Q}] + |\Sigma_p| - 1$. For each positive integer n, there exists a finite set of primes Q_n of F, which is disjoint from S, and such that*

1. *If $v \in Q_n$, then $\mathrm{Norm}(v) = 1 \pmod{p^n}$ and $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues.*
2. *$|Q_n| = \dim_{\mathbb{F}} H^1(G_F, ad^0\bar{\rho}(1))$. If $S_{Q_n} = S \cup Q_n$, then as an $R^{\square,\psi}_{\Sigma_p}$-algebra, $R^{\square,\psi}_{F,S_{Q_n}}$ is topologically generated by g elements. In particular, $g \geq 0$.*

3.2. **Patching.** For $n \geq 1$ fix a set Q_n as in Proposition 3.1. Let Δ_v be the maximal p-quotient of $(\mathcal{O}_{F_v}/\pi_v)^{\times}$. Write $\Delta_{Q_n} = \prod_{v\in Q_n} \Delta_v$. For each $v \in Q_n$ we fix a choice of zero of polynomial $X^2 - T_v X + \mathrm{Norm}(v) S_v$ in $\mathbb{F}$, and we denote $\mathfrak{m}_{Q_n} \subset \mathbb{T}^{univ}_{S_{Q_n},\mathcal{O}}$ the corresponding maximal ideal. We apply the discussion of subsection 2.3 to each of these Q_n.

The universal property gives us a map of $\mathcal{O}$-algebras $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}_{\sigma,\psi}(U_{Q_n})_{\mathfrak{m}_{Q_n}}$ such that for $v \notin S_{Q_n}$, the trace of the arithmetic Frobenius Frob_v on the tautological $R^{\psi}_{F,S_{Q_n}}$-representation of G_F maps to T_v. We regard $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ as an $R^{\psi}_{F,S_{Q_n}}$-module via this map. Moreover, $R^{\psi}_{F,S_{Q_n}}$ has a natural structure of $\mathcal{O}[\Delta_{Q_n}]$-algebra so that the induced $\mathcal{O}[\Delta_{Q_n}]$-structure on $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ is the one given in subsection 2.3. By Lemma 2.3, this is a finite free $\mathcal{O}[\Delta_{Q_n}]$-module, whose rank does not depend on n. Denote this rank by r. We set

$$M_n = R^{\square,\psi}_{F,S_{Q_n}} \otimes_{R^{\psi}_{F,S_{Q_n}}} S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$$

for $n \geq 0$, and $S_{Q_0} = S$.

Fix a filtration by $\mathbb{F}$-subspaces

$$0 = L_0 \subset L_1 \subset \cdots \subset L_s = W_\sigma \otimes_{\mathcal{O}} \mathbb{F} = W_{\bar{\sigma}}$$

on $W_{\bar{\sigma}}$ such that L_i is $\mathrm{GL}_2(\mathbb{Z}_p)$-stable, and for $i = 0, 1, \ldots, s-1$, $\sigma_i = L_{i+1}/L_i$ is absolutely irreducible. This induces a filtration of $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}} \otimes_{\mathcal{O}} \mathbb{F}$ whose associated graded pieces are the finite free $\mathbb{F}[\Delta_{Q_n}]$-modules $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$. We denote by

$$0 = M_n^0 \subset M_n^1 \subset \cdots \subset M_n^s = M_n \otimes_{\mathcal{O}} \mathbb{F}$$

the induced filtration in $M_n \otimes_{\mathcal{O}} \mathbb{F}$, obtained by extension of scalars.

Set $j = 4|\Sigma_p| - 1$, $h = |Q_n|$, $d = [F:\mathbb{Q}] + 3|\Sigma_p|$. Then $g = h + j - d$. We fix surjections

$$\mathcal{O}[[y_1, \ldots, y_h]] \to \mathcal{O}[\Delta_{Q_n}]. \tag{3.1}$$

Then by Lemmas 2.3 and 2.4, we have $M_0 \xrightarrow{\sim} M_n/(y_1, \ldots, y_h)$.

The map $R^{\psi}_{F,S_{Q_n}} \to R^{\square,\psi}_{F,S_{Q_n}}$ is formally smooth of relative dimension j. We extend the maps (3.1) to maps

$$\mathcal{O}[[y_1,\dots,y_{h+j}]] \to R^{\square,\psi}_{F,S_{Q_n}} \tag{3.2}$$

in such a way that $R^{\square,\psi}_{F,S_{Q_n}}$ is identified with $R^{\psi}_{F,S_{Q_n}}[[y_{h+1},\dots,y_{h+j}]]$. We also fix surjections of $R^{\square,\psi}_{\Sigma_p}$-algebras

$$R^{\square,\psi}_{\Sigma_p}[[x_1,\dots,x_g]] \to R^{\square,\psi}_{F,S_{Q_n}} \tag{3.3}$$

and a lifting of the maps in (3.2)

$$\mathcal{O}[[y_1,\dots,y_{h+j}]] \to R^{\square,\psi}_{\Sigma_p}[[x_1,\dots,x_g]]. \tag{3.4}$$

We regard each M_n as a $R^{\square,\psi}_{\Sigma_p}[[x_1,\dots,x_g]]$-module via (3.3) and the map $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}_{\sigma,\psi}(U_{Q_n})_{\mathfrak{m}_{Q_n}}$.

For $n \geq 1$, let

$$\mathfrak{c}_n = (\pi^n, (y_1+1)^{p^n-1}-1,\dots,(y_h+1)^{p^n-1}-1, y_{h+1}^{p^n},\dots,y_{h+j}^{p^n}) \subset \mathcal{O}[[y_1,\dots,y_{h+j}]].$$

Then by the proof of [2, Proposition 3.3.1], after replacing the sequence $\{Q_n\}_{n\geq 1}$ by a subsequence, we may assume that there exist maps of $R^{\square,\psi}_{\Sigma_p}[[x_1,\dots,x_g]]$-modules $f_n : M_{n+1}/\mathfrak{c}_{n+1} \to M_n/\mathfrak{c}_n$, which reduce modulo $(y_1,\dots,y_h)+\mathfrak{c}_n$ to the identity on $M_0/\mathfrak{c}_n$. Moreover, if we give $M_n/(\mathfrak{c}_n,\pi)$ the filtration induced by that on $M_n\otimes_{\mathcal{O}}\mathbb{F}$, we may assume that $f_n \pmod{\pi}$ is compatible with filtrations.

Passing to the limit over n, we obtain a map of $R^{\square,\psi}_{\Sigma_p}[[x_1,\dots,x_g]]$-modules

$$M_\infty \to M_\infty/(y_1,\dots,y_h) \xrightarrow{\sim} M_0.$$

Since M_n is a finite free $\mathcal{O}[\Delta_{Q_n}][[y_{h+1},\dots,y_{h+j}]]$-module, $M_n/\mathfrak{c}_n$ is a finite free $\mathcal{O}[[y_1,\dots,y_{h+j}]]/\mathfrak{c}_n$-module, and M_∞ is a finite free $\mathcal{O}[[y_1,\dots,y_{h+j}]]$-module. Moreover, $M_\infty\otimes_{\mathcal{O}}\mathbb{F}$ has a filtration

$$0 = M^0_\infty \subset M^1_\infty \subset \cdots \subset M^s_\infty = M_\infty\otimes_{\mathcal{O}}\mathbb{F}$$

and since M^i_n/M^{i-1}_n is a finite free $\mathbb{F}[\Delta_{Q_n}][[y_{h+1},\dots,y_{h+j}]]$-module, $M^i_\infty/M^{i-1}_\infty$ is a finite free $\mathbb{F}[[y_1,\dots,y_{h+j}]]$-module for $i=1,\dots,s$.

3.3. **Some properties of M_∞.** For each place $v \mid p$ of F, we choose integers a_v, b_v with $b_v \geq 0$, together with an inertial type τ_v. Let $*$ be either *cr* or nothing (the same choice of $*$ being made for all $v \mid p$). We assume that $2a_v+b_v$ is independent of v. Let $\bar{R}^{\square,\psi}_v := R_*(a_v,b_v,\tau_v,\bar{\rho}|_{G_{F_v}})$ be the quotient of $R^{\square,\psi}_v$ as defined in Section 1. The action of $R^{\square,\psi}_v$ on M_n factors through $\bar{R}^{\square,\psi}_v$ follows from the fact that the Galois representations attached to Hilbert modular eigenforms are compatible with the local Langlands correspondence, as well as the compatibility of the local and global Jacquet-Langlands correspondences.

For $v\in\Sigma$, let $\gamma_v : G_{F_v} \to \mathcal{O}^\times$ be an unramified character such that $\gamma_v^2 = \psi|_{G_{F_v}}$ and $\bar{\rho}|_{G_{F_v}}$ is an extension of γ_v by $\gamma_v(1)$. In this case, by [2, Corollary 2.6.7], there is a quotient $\bar{R}^{\square,\psi}_v$ of $R^{\square,\psi}_v$ which is a domain of dimension 4, with $\bar{R}^{\square,\psi}_v$ formally smooth over E, and such that for any finite extension E'/E and a map $x : R^{\square,\psi}_v \to E'$ factors through $\bar{R}^{\square,\psi}_v$ if and only if the corresponding representation V_x is an extension of γ_v by $\gamma_v(1)$. Again, the fact that the action of $R_v^{\square,\psi}$ on M_n factors through $\bar{R}_v^{\square,\psi}$ is a consequence of the compatibility between the local and global Langlands and Jacquet-Langlands correspondence.

We write $\bar{R}_{\Sigma_p}^{\square,\psi} = \hat{\otimes}_{\mathcal{O},v\in\Sigma_p}\bar{R}_v^{\square,\psi}$. The relative dimension of $\bar{R}_v^{\square,\psi}$ over $\mathcal{O}$ is $3+[F_v : \mathbb{Q}_p] = 4$ if $v \mid p$, and 3 if $v \in \Sigma$. In particular, $\bar{R}_{\Sigma_p}^{\square,\psi}$ has relative dimension $[F:\mathbb{Q}]+3|\Sigma_p|$ over $\mathcal{O}$. Define

$$\bar{R}_\infty := \bar{R}_{\Sigma_p}^{\square,\psi}[[x_1,\ldots,x_g]].$$

Remark 3.2. In order to prove the Breui-Mezard conjecture, it suffices to take $\Sigma = \emptyset$. We consider the general case here since it has applications to modularity problems.

Lemma 3.3. *The following conditions are equivalent.*

1. *M_∞ is a faithful $\bar{R}_\infty$-module.*
2. *M_∞ is a faithful $\bar{R}_\infty$-module which has rank 2 at all generic point of $\bar{R}_\infty$.*
3. *$e(\bar{R}_\infty/\pi) = \frac{1}{2}e(M_\infty/\pi, \bar{R}_\infty/\pi)$.*
4. *$e(\bar{R}_\infty/\pi) \le \frac{1}{2}e(M_\infty/\pi, \bar{R}_\infty/\pi)$.*

Moreover, if these conditions hold, and $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ is a deformation of $\bar{\rho}$ such that for $v \in \Sigma_p$, $\rho|_{G_{F_v}}$ is an extension of γ_v by $\gamma_v(1)$ if $v \in \Sigma$, and $\rho|_{G_{F_v}}$ is potentially semi-stable of type (a_v, b_v, τ_v, ψ) if $v \mid p$, then ρ is modular, and arises from an eigenform in $S_{\sigma,\psi}(U,\mathcal{O}) \otimes_{\mathcal{O}} E$.

Proof. This is [1, Lemma 2.2.10]. Write $\mathcal{O}[[\Delta_\infty]] = \mathcal{O}[[y_1,\ldots,y_{h+j}]]$, and denote by $\mathbb{T}_\infty$ the image of $\bar{R}_\infty$ in $\mathrm{End}_{\mathcal{O}[[\Delta_\infty]]}(M_\infty)$. Since M_∞ is finite free over $\mathcal{O}[[\Delta_\infty]]$, $\mathbb{T}_\infty$ is a finite torsion free $\mathcal{O}[[\Delta_\infty]]$-module, and hence all its components have relative dimension $h+j$ over $\operatorname{Spec}\mathcal{O}$. Let Z be such a component, then Z surjects onto $\operatorname{Spec}\mathcal{O}[[\Delta_\infty]]$. This implies that the rank of $M_\infty|_Z$ is 2. (Otherwise, $M_0 = M_\infty \otimes_{\mathcal{O}[[\Delta_\infty]]} \mathcal{O}$ would have a fibre of dimension $a \neq 2$ over some point of $\operatorname{Spec} R_{F,S}^{\psi}[1/p]$, and $S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}$ would have rank $a \neq 2$ over some point of $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$, which is impossible.) Thus, if M_∞ is a faithful $\mathbb{T}_\infty$ module, its rank is exactly two on each component of $\operatorname{Spec}\mathbb{T}_\infty$. This shows (1) $\Leftrightarrow$ (2).

Since $\bar{R}_\infty$ is pure of relative dimension $d+g = h+j$ over $\mathcal{O}$, the inclusion $\operatorname{Spec}\mathbb{T}_\infty \hookrightarrow \operatorname{Spec}\bar{R}_\infty$ identifies $\operatorname{Spec}\mathbb{T}_\infty$ with a union of irreducible components of $\operatorname{Spec}\bar{R}_\infty$. Thus $e(\mathbb{T}_\infty/\pi) \le e(\bar{R}_\infty/\pi)$. The equality holds if and only if the inclusion is an isomorphism.

On the other hand, we have

$$\begin{aligned} e(M_\infty/\pi, \bar{R}_\infty/\pi) &= e(M_\infty/\pi, \mathbb{T}_\infty/\pi) \quad (\text{since } \dim\bar{R}_\infty = \dim\mathbb{T}_\infty) \\ &= 2e(\mathbb{T}_\infty/\pi) \quad (\text{since generically } M_\infty \text{ has rank two.}) \end{aligned}$$

Therefore, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Suppose that the conditions (1)-(4) hold. Then ρ induces a map $\mathbb{T}_\infty = \bar{R}_\infty \to \mathcal{O}$, which kills the ideal $(y_1,\ldots,y_{h+j})$, and hence a map $\xi : \mathbb{T}_\infty/(y_1,\ldots,y_{h+j})[1/p] \to E$. Since M_∞ has positive rank on all components of $\mathbb{T}_\infty$, the fibre of M_0 over the closed point of $\mathbb{T}_\infty/(y_1,\ldots,y_{h+j})[1/p]$ corresponding to ξ is non-empty, and ξ induces a map $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}} \to E$, which corresponds to the required eigenform in $S_{\sigma,\psi}(U,\mathcal{O}) \otimes_{\mathcal{O}} E$. $\square$

4. Proof of the main results

In this section, we sketch the proof of the main results in the potentially semistable case. The proof in the potentially crystalline case is the same. We assume that the finite set Σ is empty and the character ψ is crystalline. For any Serre weight $\sigma_{m,n}$, as in Remark 1.3 we define

$$\mu_{m,n}(\bar{\rho}) = e(R_{cr}^{\square,\psi_{cr}}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi)$$

and

$$\mathcal{C}_{m,n}(\bar{\rho}) = Z(R_{cr}^{\square,\psi_{cr}}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi).$$

First, we calculate the Hilbert-Samuel multiplicities of the modules $M_\infty^i/M_\infty^{i-1}$. For $i = 1, \ldots, s$, write σ_i for the representation L_i/L_{i-1}. Then we may write $\sigma_i = \otimes_{v|p}\sigma_{m_{i,v},n_{i,v}}$ where $(m_{i,v}, n_{i,v}) \in \{0, 1, \ldots, p-2\} \times \{0, 1, \ldots, p-1\}$ and $\sigma_{m_{i,v},n_{i,v}}$ is an irreducible constituent of W_{σ_v}/π. Let $\bar{R}_{v,i}^{\square,\psi} = R_{cr}^{\square,\psi}(\tilde{m}_{i,v}, n_{i,v}, \mathbf{1}, \bar{\rho}|_{G_{F_v}})/\pi$ for $v \mid p$. Define

$$\bar{R}_{\Sigma_p,i}^{\square,\psi}[[x_1, \ldots, x_g]] = \hat{\otimes}_{v\in\Sigma_p}\bar{R}_{v,i}^{\square,\psi}$$

and

$$\bar{R}_\infty^i = \bar{R}_{\Sigma_p,i}^{\square,\psi}[[x_1, \ldots, x_g]].$$

Lemma 4.1. *The action of R_∞ on M^i/M^{i-1} factors through $\bar{R}_\infty^i$.*

Proof. It suffices to show that for a fixed prime $v_0|p$, the action of $R_v^{\square,\psi}/\pi$ on M^i/M^{i-1} factors through $\bar{R}_{v_0,i}^{\square,\psi} = R_{cr}^{\square,\psi}(\tilde{m}_{i,v_0}, n_{i,v_0}, \mathbf{1}, \bar{\rho}|_{G_{F_{v_0}}})/\pi$. Fix i and a prime $v_0|p$. Let

$$\tilde{\sigma}_{i,v_0} = \mathrm{Symm}^{n_{i,v_0}}\,\mathcal{O}^2 \otimes \tilde{\omega}^{m_{i,v_0}} \circ \det.$$

For $v \neq v_0$, let $\tilde{\sigma}_{i,v}$ be a representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ of the form $\mathrm{Symm}^{n_{i,v_0}}\,\mathcal{O}^2 \otimes \tilde{\sigma}_{i,v}^{sm}$, where $\tilde{\sigma}_{i,v}^{sm}$ is a smooth representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ on a finite free $\mathcal{O}$-module, such that $\tilde{\sigma}_{i,v}^{sm}$ has an $\mathcal{O}^\times$-valued central chsracter, and $\tilde{\sigma}_{i,v} \otimes_{\mathcal{O}} \mathbb{F}$ admits $\sigma_{m_{i,v},n_{i,v}}$ as a Jordan-Hölder factor. (Indeed, since $\mathrm{Hom}_{\mathbb{F}}(\mathrm{Symm}^{n_{i,v}}\,\mathbb{F}^2 \otimes \det^{m_{i,v}}, \mathrm{Symm}^{n_{i,v_0}}\,\mathbb{F}^2)$ is a smooth $\mathrm{GL}_2(\mathcal{O}_{F_v})$-representation and can therefore be embedded into a sum of a finite number of copies of the space of smooth $\mathbb{F}$-valued functions on $\mathrm{GL}_2(\mathcal{O}_{F_v})$, the representation $\tilde{\sigma}_{i,v}^{sm}$ exists.) Let $\tilde{\sigma}_i = \otimes_{v|p}\tilde{\sigma}_{i,v}$.

Next we choose a continuous character $\tilde{\psi} : (\mathbb{A}_F^\infty)^\times/F^\times \to \mathcal{O}^\times$ such that $\tilde{\psi} \equiv \psi \pmod{\pi}$. We also choose a compact open subgroup $\tilde{U} = \prod_v \tilde{U}_v \subset U_{Q_n}$ such that $\tilde{U}_v$ is maximal compact for $v \in \Sigma_p$ and for all v the restriction $\tilde{\sigma}_{i,v}|_{\tilde{U}_v\cap\mathcal{O}_{F_v}^\times}$ is given by multiplication by $\tilde{\psi}$.

Let $\tilde{S}$ be the union of the primes in S_{Q_n} and the primes where $\tilde{U}_v$ is not maximal. Denote by $\tilde{\mathfrak{m}}$ the maximal ideal in $\mathbb{T}_{\tilde{S},\mathcal{O}}^{univ}$ corresponding to $\bar{\rho}$. Then

$$M_n^i/M_n^{i-1} = R_{F,S_{Q_n}}^{\square,\psi} \otimes_{R_{F,S_{Q_n}}^\psi} S_{\sigma_i,\psi}(U_{Q_n}, \mathbb{F})_{\mathfrak{m}_{Q_n}}$$

is a subquotient of $R_{F,\tilde{S}}^{\square,\tilde{\psi}} \otimes_{R_{F,\tilde{S}}^{\tilde{\psi}}} S_{\tilde{\sigma}_i,\tilde{\psi}}(\tilde{U}, \mathcal{O})_{\tilde{\mathfrak{m}}} \otimes_{\mathcal{O}} \mathbb{F}$. On the latter module, the action of $R_{v_0}^\square$ factors through $R_{cr}^{\square,\psi}(\tilde{m}_{i,v_0}, n_{i,v_0}, \mathbf{1}, \bar{\rho}|_{G_{F_{v_0}}})$. This proves the lemma with $\psi_{i,v_0} = \tilde{\psi}|_{G_{F_v}}$. On the other hand, $R_{cr}^{\square,\psi}(\tilde{m}_{i,v_0}, n_{i,v_0}, \mathbf{1}, \bar{\rho}|_{G_{F_{v_0}}})/\pi$ is independent of the character ψ_{i,v_0}, the lemma follows. $\square$

Proposition 4.2. *The $\bar{R}_\infty$-module $M_\infty^i/M_\infty^{i-1}$ is non-zero if and only if for each $v \mid p$ we have $\mu_{m_{i,v},n_{i,v}}(\bar\rho|_{G_{F_v}}) \neq 0$. If this condition holds for all $v \mid p$, and for each $v \mid p$ we have $\bar\rho|_{G_{F_v}} \not\sim \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any χ, then*

$$\frac{1}{2} e(M_\infty^i/M_\infty^{i-1}, \bar{R}_\infty/\pi) = \prod_{v|p} \mu_{m_{i,v},n_{i,v}}(\bar\rho|_{G_{F_v}}) := e_{\Sigma_p}. \tag{4.1}$$

Proof. This is [1, Proposition 2.2.14]. For simplicity, we give the proof only for those $\bar\rho$ where $\bar\rho|_{G_{F_v}}$ is not a direct sum of two characters for each $v \mid p$. For the proof for the general case, see [1]. Since M_∞^i is flat over $\mathcal{O}[[\Delta_\infty]]$, the image of $\bar{R}_\infty^i$ in $\mathrm{End}_{\mathcal{O}[[\Delta_\infty]]}(M_\infty^i)$ has relative dimension $h+j$ over $\mathcal{O}$. Thus the support of M_∞^i consists all of $\mathrm{Spec}\,\bar{R}_\infty^i$. Since $\bar{R}_\infty^i$ is irreducible and generically reduced, the same computation as in Lemma 3.3 gives us

$$e(M_\infty^i, \bar{R}_\infty^i) = 2e(\bar{R}_\infty^i) = 2e(\bar{R}_{\Sigma_p,i}^{\square,\psi}) = 2\prod_{v\in\Sigma_p} e(\bar{R}_{v,i}^{\square,\psi}) = 2e_{\Sigma_p}.$$

The proposition follows. □

Theorem 4.3. *Suppose that for each $v \mid p$, $\bar\rho|_{G_{F_v}} \not\sim \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any χ,. Then*

1. *M_∞ is a faithful $\bar{R}_\infty$-module. In particular, the conditions in Lemma 3.3 hold.*
2. *For each $v \mid p$, we have*
$$e(\bar{R}_v^{\square,\psi}/\pi) = \mu(a_v, b_v, \tau_v, \bar\rho|_{G_{F_v}}).$$
In particular, Theorem 1.4 holds.

Proof. (1) Using the results in [1, Section 1.7], which are explained in the previous talks, we have

$$e(\bar{R}_\infty/\pi) = e_\Sigma \prod_{v|p} e(\bar{R}_v^{\square,\psi}/\pi) \leq e_\Sigma \prod_{v|p} \mu(a_v, b_v, \tau_v, \bar\rho|_{G_{F_v}}).$$

On the other hand, by the above proposition, we have

$$\begin{aligned}\frac{1}{2} e(M_\infty/\pi, \bar{R}_\infty/\pi) =& \frac{1}{2}\sum_{i=1}^{s} e(M_\infty^i/M_\infty^{i-1}, \bar{R}_\infty/\pi) \\ =& e_\Sigma \sum_{i=1}^{s}\prod_{v|p} \mu_{m_{i,v},n_{i,v}}(\bar\rho|_{G_{F_v}}) \\ =& e_\Sigma \prod_{v|p} \mu(a_v, b_v, \tau_v, \bar\rho|_{G_{F_v}}).\end{aligned}$$

Hence

$$e(\bar{R}_\infty/\pi) \leq \frac{1}{2} e(M_\infty/\pi, \bar{R}_\infty/\pi).$$

The result follows from Lemma 3.3.

(2) We have inequality

$$e(\bar{R}_v^{\square,\psi}/\pi) \leq \mu(a_v, b_v, \tau_v, \bar\rho|_{G_{F_v}}).$$

If this inequality is strict, then we must have

$$e(\bar{R}_\infty/\pi) < \frac{1}{2}e(M_\infty/\pi, \bar{R}_\infty/\pi).$$

This contradicts to Lemma 3.3. □

Proof of Theorem 1.5. Note that we may assume $\Sigma = \emptyset$. We have

$$Z(\bar{R}_\infty/\pi) = (\prod_{v|p} Z(\bar{R}_v^{\square,\psi}/\pi) \times Z(\operatorname{Spec}\mathbb{F}[[x_1,\ldots,x_g]]).$$

On the other hand,

$$\begin{aligned} Z(\bar{R}_\infty/\pi) = \frac{1}{2}Z(M_\infty/\pi) =& \frac{1}{2}\sum_{i=1}^{s} Z(M_\infty^i/M_\infty^{i-1}) \\ \geq& \sum_{i=1}^{s} Z(\bar{R}_\infty^i/\pi) \\ =& (\prod_{v|p}\sum_{m,n} a_{m,n}\mathcal{C}_{m,n}) \times Z(\operatorname{Spec}\mathbb{F}[[x_1,\ldots,x_g]]). \end{aligned} \tag{4.2}$$

Note that the inequality on cycles gives a corresponding inequality on multiplicities, which is in fact an equality. Therefore, the above inequality is an equality, and we deduce that

$$\prod_{v|p} Z(\bar{R}_v^{\square,\psi}/\pi) = \prod_{v|p}\sum_{m,n} a_{m,n}\mathcal{C}_{m,n}.$$

Therefore

$$Z(\bar{R}_v^{\square,\psi}/\pi) = \sum_{m,n} a_{m,n}\mathcal{C}_{m,n}$$

as required. □

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References

[1] Kisin, Mark *The Fontaine-Mazur conjecture for* GL_2. J.A.M.S. 22(3) (2009) 641-690.

[2] Kisin, Mark *Moduli of finite flat group schemes and modularity.* Annals of Math. 170(3) (2009), 1085-1180.

[3] Emerton, M; Gee, T *A geometric perspective of the Breuil-Mézard conjecture.* To appear, available at `http://www2.imperial.ac.uk/~tsg/`