MOD p REPRESENTATIONS OF LOCAL DIVISION ALGEBRAS OVER p-ADIC FIELDS

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Abstract. In this paper, we study the mod p representations of $D^\times$, where D is a division algebra over a finite extension F of $\mathbb{Q}_p$. Moreover, in the case D is a quaternion algebra, we construct a bijection between rank two mod p representations of $D^\times$ and rank two mod p representations of G_F. We also study the deformations of mod p representations of $D^\times$.

1. Introduction and notation

Let $p > 2$ be a prime number. Let F be a finite extension of $\mathbb{Q}_p$. Let D be a division algebra over F. In this note, we classify the irreducible mod p representations of $D^\times$. This is Theorem 2.9. Using this classification, we shall see that there is a bijection between n-dimensional irreducible mod p representations of $D^\times$ and n-dimensional irreducible mod p representations of the absolute Galois group $G_F = Gal(\bar{F}/F)$. This bijection is explained in Subsection 2.4. If furthermore D is a quaternion algebra, we classify all rank two mod p representations of $D^\times$ and extend the above bijection by computing certain Ext^1 groups. See Theorem 3.1 and Theorem 3.4.

In this paper, $\mathbb{F}$ denotes a finite extension of $\mathbb{F}_p$. We assume that $\mathbb{F}$ is sufficiently large in the sense that it is the coefficient ring for all the representations we consider in the following and it contains all the images of embeddings $k \hookrightarrow \bar{\mathbb{F}}$ where k is the residue field of F or D. We write $R^\times$ for the group of invertible elements of a ring R.

2. mod p representations of $D^\times$

2.1. Properties of local division algebras.

Recall that F is a finite field extension of $\mathbb{Q}_p$. We let

$$\begin{aligned}
&\mathcal{O}_F = \text{the valuation ring in } F,\\
&\mathfrak{p}_F = \text{the maximal ideal of } \mathcal{O}_F,\\
&k_F = \mathcal{O}_F/\mathfrak{p}_F, \ \text{the residue field of } F,\\
&q = p^f = q_F = |k_F|, \ \text{the cardinality of } k_F.
\end{aligned}$$

The unit group $\mathcal{O}_F^\times$ has a filtration

$$\mathcal{O}_F^\times \supset 1 + \mathfrak{p}_F \supset 1 + \mathfrak{p}_F^2 \supset \cdots .$$

We also write

$$v_F : F^\times \to \mathbb{Z}$$

for the canonical surjective valuation of F.

Let D be a finite dimensional central F-division algebra with $\dim_F D = n^2$. The homomorphism $v_F : F^\times \to \mathbb{Z}$ extends to a surjective homomorphism

$$v_D : D^\times \to \mathbb{Z}$$

which is indeed a valuation. Extend v_D to D by defining $v_D(0) = \infty$. Let

$$\begin{aligned}
\mathcal{O}_D &= \{x \in D \mid v_D(x) \geq 0\}, \\
\mathfrak{p}_D &= \{x \in D \mid v_D(x) \geq 1\}, \\
\mathfrak{p}_D^i &= \{x \in D \mid v_D(x) \geq i\}, \\
k_D &= \mathcal{O}_D/\mathfrak{p}_D.
\end{aligned}$$

Lemma 2.1. *With the above notation,*

(1) $\mathcal{O}_D$ *is a ring, and it is the unique maximal order in* D.
(2) $\mathfrak{p}_D$ *is the unique maximal ideal of* $\mathcal{O}_D$. *Moreover, any left (or right)* $\mathcal{O}_D$ *lattice spanning* D *over* F *is of the form* $\mathfrak{p}_D^i$ *for some uniquely determined* $i \in \mathbb{Z}$.
(3) *The residue ring* k_D *is a field, and indeed an extension of* k_F *of degree* n.

We have a chain of subgroups

$$\mathcal{O}_D^\times \supset 1 + \mathfrak{p}_D \supset 1 + \mathfrak{p}_D^2 \supset \cdots,$$

each of them is compact, open, and normal in $D^\times$. We have canonical isomorphisms

$$\begin{aligned}
&\mathcal{O}_D^\times / 1 + \mathfrak{p}_D \cong k_D^\times, \\
&1 + \mathfrak{p}_D^i / 1 + \mathfrak{p}_D^{i+1} \cong \mathfrak{p}_D^i / \mathfrak{p}_D^{i+1},
\end{aligned}$$

and therefore, for $i \geq 1$, $1 + \mathfrak{p}_D^i / 1 + \mathfrak{p}_D^{i+1}$ is an elementary abelian p-group of order q^n. In particular, we have a short exact sequence

$$1 \to 1 + \mathfrak{p}_D \to \mathcal{O}_D^\times \to k_D^\times \to 1 \tag{2.1}$$

with the kernel $1 + \mathfrak{p}_D$ a pro-p group.

We fix uniformizers ω_F and ω_D for $\mathcal{O}_F$ and $\mathcal{O}_D$ respectively.

Lemma 2.2. *With the above notation,*

(1) $\mathfrak{p}_F \mathcal{O}_D = \mathfrak{p}_D^n$.
(2) $F^\times \mathcal{O}_D^\times$ *is a subgroup of* $D^\times$ *with index* n. *Therefore,* $D^\times$ *is compact modulo its center* $F^\times$.
(3) $D^\times$ *acts on* $k_D^\times$ *by conjugation. In particular, we have*

$$\omega_D x \omega_D^{-1} = x^q, \quad x \in k_D.$$

2.2. **1-dimensional representations.** Write $\mathrm{Nrd} : D^\times \to F^\times$ for the reduced norm. We have a short exact sequence

$$1 \to D^\times_{\mathrm{Nrd}=1} \to D^\times \xrightarrow{\mathrm{Nrd}} F^\times \to 1.$$

Just as in the GL_2 case, $D^\times_{\mathrm{Nrd}=1}$ is the commutator of $D^\times$, every character of $D^\times$ factor through Nrd. Therefore, there is a one to one correspondence between characters of $D^\times$ and characters of $F^\times$.

Lemma 2.3. *The following diagram is commutative.*

$$\begin{tikzcd} \mathcal{O}_D^\times \arrow[r] \arrow[d, "\mathrm{Nrd}"'] & k_D^\times \arrow[d, "\mathrm{Norm}"] \\ \mathcal{O}_F^\times \arrow[r] & k_F^\times \end{tikzcd}$$

where $\mathrm{Norm} : k_D^\times \to k_F^\times$ *is the map* $x \mapsto x^{1+q+\cdots+q^{n-1}}$.

Proof. Easy to check from the definition. □

Since $1+\mathfrak{p}_D$ is a pro-p group, every mod p representation of $1+\mathfrak{p}_D$ is trivial. Therefore, every representation of $\mathcal{O}_D^\times$ factor through $\pi : \mathcal{O}_D^\times \to k_D^\times$. In particular, every irreducible mod p representation of $\mathcal{O}_D^\times$ is 1-dimensional. We may (and will) identify the mod p characters of $\mathcal{O}_D^\times$ and of $k_D^\times$. We say a character $\chi : k_D^\times \to \mathbb{F}^\times$ or $\chi : \mathcal{O}_D^\times \to \mathbb{F}^\times$ of *order* a if $\chi = \chi^{q^a}$ and a is the smallest positive integer satisfies this condition. Note that if *chi* is of order a, then $a|n$.

2.3. **Higher dimensional representations.** We show that every irreducible mod p representation of $D^\times$ has dimension dividing n. We start with the following lemma.

Lemma 2.4. *A smooth irreducible admissible representation of* $D^\times$ *over* E *always has a central character.*

Proof. Let π be such a representation and $H \subset \mathcal{O}_D^\times$ be an open subgroup such that $\pi^H \neq 0$. Because π is admissible, π^H has finite dimension over E, and there is $v \in \pi^H$ such that $F^\times$ acts on v by multiplication by a character. As π is irreducible, $\pi = \langle D^\times v \rangle$. The lemma follows since $F^\times$ is the center of $D^\times$. □

Let π be a b-dimensional irreducible representation of $D^\times$. By the above lemma, we may assume that π has trivial central character. Then $\pi|_{F^\times \mathcal{O}_D^\times}$ decomposes to a direct sum of b 1-dimensional representations of $F^\times \mathcal{O}_D^\times$, say, $\pi|_{F^\times \mathcal{O}_D^\times} = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_b$. (Indeed, each subquotient of $\pi|_{F^\times \mathcal{O}_D^\times}$ is a representation of $\mathcal{O}_D^\times$. Therefore, they factor through $\mathcal{O}_D^\times \to k_D^\times$.) By Frobenius reciprocity, π is a subquotient of $\mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \rho_1$. By Lemma 2.2, π is at most n dimensional. As the representation $\pi^\Delta := \pi(\omega_D \cdot \omega_D^{-1})$ is isomorphic to π, and $\rho^\Delta = \rho^q$ by Lemma 2.2, we have $\{\rho_i\}_{i=1}^n = \{\rho_i^q\}_{i=1}^n$.

We construct all irreducible representations of $D^\times$ from character of $\mathcal{O}_D^\times$. Let $\chi : k_D^\times \to \mathbb{F}^\times$ be a character of $k_D^\times$. We may and will consider it as a character of $\mathcal{O}_D^\times$ via the map $\mathcal{O}_D^\times \to k_D^\times$. Extending χ to a character of $F^\times \mathcal{O}_D^\times$ by triviality, we then obtain a representation $\pi_\chi = \mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi$ of $D^\times$.

Lemma 2.5. $\pi_\chi = \mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi$ *is reducible if and only if* $\chi(x) = \chi(x^{q^a})$ *for some integer* $a|n$ *and* $a < n$.

Proof. We know that $\{\omega_D^i\}_{i=0}^{n-1}$ is a set of representatives of $D^\times / F^\times \mathcal{O}_D^\times$. The representation $\mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi$ is reducible if and only if there exists an positive integer a such that $\chi \cong \chi^{\omega_D^a}$, i.e., $\chi \cong \chi^{q^a}$. The integer a must divide n since $\chi \cong \chi^{q^n}$. □

Remark 2.6. Assume that χ is a character of order a with $a|n$, then χ extends to a character of $F^\times\mathcal{O}_D^\times\langle\omega_D^a\rangle$ by letting $\chi(\omega_D^a)=1$. Thus

$$\pi_\chi \cong (\mathrm{Ind}_{F^\times\mathcal{O}_D^\times\langle\omega_D^a\rangle}^{D^\times}\chi)^{n/a}.$$

In the above formula, $\mathrm{Ind}_{F^\times\mathcal{O}_D^\times\langle\omega_D^a\rangle}^{D^\times}\chi$ is an a-dimensional irreducible representation of $D^\times$.

Lemma 2.7. *Assume that χ_i and χ' are two characters of $k_D^\times$. Then $\pi_\chi\cong\pi_{\chi'}$ if and only if $\chi'=\chi^{q^a}$ for some integer $a|n$.*

Proof. First, if $\chi'=\chi^{q^a}$, it is easy to check that $\pi_{\chi'}=\pi_\chi^{\omega_D^a}\cong\pi_\chi$. On the other hand, if $\pi_{\chi'}\cong\pi_\chi$, then

$$\pi_\chi|_{F^\times\mathcal{O}_D^\times}\cong\pi_{\chi'}|_{F^\times\mathcal{O}_D^\times}.$$

Thus $\{\chi^{q^i}\}=\{(\chi')^{q^i}\}$. The lemma follows. □

Fix an embedding $k_D^\times\to\bar{\mathbb{F}}_p^\times$. Let χ_i be the character of $k_D^\times$ with the form $x\mapsto x^i$. Let $Q=1+q+\cdots+q^{n-1}$.

Lemma 2.8. *With the above notation, we have*

$$(\mathrm{Ind}_{F^\times\mathcal{O}_D^\times}^{D^\times}\chi)\otimes(\mu\circ\mathrm{Nrd})\cong\mathrm{Ind}_{F^\times\mathcal{O}_D^\times}^{D^\times}(\chi\cdot\chi_Q).$$

Here, $\mu: F^\times\to k_F^\times$ is the residue map with $\mu(\omega_F)=1$.

Proof. By Lemma 2.3, $\chi\otimes(\mu\circ\mathrm{Nrd})\cong\chi\cdot\chi_Q$, the lemma follows. □

From the above discussion, we have the following theorem.

Theorem 2.9. *The irreducible mod p representations of $D^\times$ are the following:*

(1) *the one-dimensional representations $\kappa\circ\mathrm{Nrd}$,*
(2) *the a-dimensional representations*

$$(\mathrm{Ind}_{F^\times\mathcal{O}_D^\times}^{D^\times}\chi)\otimes(\kappa\circ\mathrm{Nrd}),$$

where χ is a level a character, $a|n$.

Here $\kappa: F^\times\to\mathbb{F}^\times$ is a character of $F^\times$.

2.4. **Galois representations.** In this section, we recall the classification of n-dimensional irreducible mod p representations of G_F.

Let ρ be a n-dimensional irreducible representation of G_F. Let I and I^w be the inertia and the wild inertia subgroup of G_F. Note that they are actually normal subgroups. Let ρ^{I^w} be the subspace of ρ where I^w acts trivially. For $w\in I^w$, $g\in G_F$, $v\in\rho^{I^w}$, $wgv=g(g^{-1}wg)v=gv$. So ρ^{I^w} is a subrepresentation of ρ. On the other hand, I^w is a pro-p group, ρ^{I^w} is not trivial. Therefore, $\rho^{I^w}=\rho$ and $\rho|_I$ factor through I/I^w. Thus $\rho|_I$ is the direct sum of n fundamental characters $\psi_1\oplus\cdots\oplus\psi_n$. Let $f\in G_F$ be a lift of the Frobenius element $(x\mapsto x^q)$. As $\rho^f:=\rho(f\cdot f^{-1})$ is isomorphic to ρ, we have $\rho|_I\cong\rho^f|_I$. That is

$$\psi_1\oplus\cdots\oplus\psi_n=\psi_1^q\oplus\cdots\oplus\psi_n^q.$$

Arguing as before, we see that $\{\psi_i\}=\{\psi^{q^i}\}$ for some level n fundamental character ψ.

By the above discussion, the n-dimensional irreducible mod p representations of $D^\times$ are parameterized by order n characters $k_D^\times \to \mathbb{F}^\times$ and the characters $F^\times \to \mathbb{F}^\times$. The n-dimensional irreducible mod p representations of G_F are parameterized by level n characters $I \to \mathbb{F}^\times$ and characters $G_F \to \mathbb{F}^\times$. If we fix an embedding $k_D^\times \hookrightarrow \mathbb{F}^\times$ and the Artin map $F^\times \cong G_F^{ab}$, we obtain a natural bijection between the n-dimensional irreducible mod p representations of $D^\times$ and of G_F.

3. Quaternion algebra case

3.1. **Two dimensional mod p representations of $D^\times$.** Let D be a quaternion algebra over F. Then we have the following result.

Theorem 3.1. *Let $\pi : D^\times \to GL_2(E)$ be a continuous representation, then ρ is of one of the following forms:*

(1) *π is irreducible and*

$$\pi \cong (\mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi_i) \otimes (\kappa \circ \mathrm{Nrd}),$$

where $1 \le i < q$. In this case,

$$\pi|_{\mathcal{O}_D^\times} \cong \chi_i \oplus \chi_i^q.$$

(2) *π is reducible and*

$$\pi|_{\mathcal{O}_D^\times} \cong \begin{pmatrix} \chi_{q+1}^a & * \\ 0 & \chi_{q+1}^b \end{pmatrix},$$

where a and b are two integers. In this case, we may write

$$\pi|_{\mathcal{O}_D^\times} \cong \begin{pmatrix} \chi_{q+1}^c & * \\ 0 & 1 \end{pmatrix} \otimes \eta$$

for some η extends to $D^\times$ and some integer c with $0 \le c \le q-2$.

Proof. We only have to prove part (2). Since π is reducible, hence in particular $\pi|_{\mathcal{O}_D^\times}$ is reducible. Assume that it is of the form $\begin{pmatrix} \delta_1 & * \\ 0 & \delta_2 \end{pmatrix}$ where δ_1 and δ_2 are continuous characters of $\mathcal{O}_D^\times$ that extend to $D^\times$. Therefore, $\delta_i = \delta_i^q$ and each δ_i factors through $\mathcal{O}_D^\times \to k_D^\times \to k_F^\times$. The result follows. □

Lemma 3.2. *We have an isomorphism*

$$\mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi_i \cong (\mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi_{q+1-i}) \otimes (\mu \circ \mathrm{Nrd})^{i-1}.$$

Proof. It suffices to check that $x^i = x^{q(q+1)-i+(q+1)(i-1)}$ for $x \in k_D^\times$, which is clear. □

The following result is well known, see for example Proposition 2.7 and Corollary 2.9 of [4].

Theorem 3.3. *Let $\rho : G_F \to GL_2(E)$ be a continuous representation, let $w_1, w_2 : I \to E^\times$ be fundamental characters of level q and q^f respectively. Then ρ is of one of the following form:*

(1) *ρ is irreducible and:*

$$\rho|_I \cong \begin{pmatrix} w_2^a & 0 \\ 0 & w_2^{qa} \end{pmatrix} \otimes \eta$$

for some character η that extends to G_F and some integers $1 \leq a < q$.

(2) *ρ is reducible and:*

$$\rho|_I \cong \begin{pmatrix} w_1^c & * \\ 0 & 1 \end{pmatrix} \otimes \eta$$

for some character η that extends to G_F and some integers $0 \leq c \leq q-2$.

We can define a bijection between rank two semisimple mod p representations of $D^\times$ and rank two semisimple mod p representations of G_F as follows.

Irreducible representations:

$$(\mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi_a) \otimes (\eta \circ \mathrm{Nrd}) \leftrightarrow \rho|_I \cong \begin{pmatrix} w_2^a & 0 \\ 0 & w_2^{qa} \end{pmatrix} \otimes \eta$$

Reducible representations:

$$\pi|_{\mathcal{O}_D^\times} \cong \begin{pmatrix} \chi_{q+1}^c & * \\ 0 & 1 \end{pmatrix} \otimes (\eta \circ \mathrm{Nrd}) \leftrightarrow \rho|_I \cong \begin{pmatrix} w_1^c & * \\ 0 & 1 \end{pmatrix} \otimes \eta$$

Here $\eta : F^\times \to \mathbb{F}^\times$, and we consider it also as a character of G_F via local class field theory.

We can extend this bijection to a bijection between all two dimensional mod p representations of $D^\times$ and G_F. First note that we may extend w_1 to a character of G_F by letting $w_1(\mathrm{Frob}) = 1$, where Frob is a lifting of the Frobenius element. We may also extend χ_{q+1} to a character of $D^\times$ by letting $\chi_{q+1}(\omega_D) = 1$. We have the following result.

Theorem 3.4. *For $1 \leq c \leq q-1$, there exists a bijection $H^1(G_F, w_1^c) \cong H^1(D^\times, \chi_{q+1}^c)$. Therefore, there is a bijection between two dimensional mod p representations of $D^\times$ and two dimensional mod p representations of G_F, under which the irreducible ones corresponds to irreducible ones.*

Proof. We have the following short exact sequence

$$0 \to I \to G_F \to \hat{\mathbb{Z}} \to 0,$$

which induces

$$0 \to I/I^w \to G_F/I^w \to \hat{\mathbb{Z}} \to 0.$$

Since I^w is a pro-p group, it acts trivially on w_1.

$$H^1(G_F, w_1^c) \cong H^1(G_F/I^w, w_1^c).$$

The exact sequence of low degree terms in Lyndon-Hochschild-Serre spectral sequence is

$$\begin{aligned} 0 &\to H^1(\hat{\mathbb{Z}}, (w_1^c)^{I/I^w}) \to H^1(G_F/I^w, w_1^c) \to H^1(I/I^w, w_1^c)^{\hat{\mathbb{Z}}} \\ &\to H^2(\hat{\mathbb{Z}}, (w_1^c)^{I/I^w}) \to H^2(G_F/I^w, w_1^c). \end{aligned} \tag{3.1}$$

Similarly, let $\Gamma = D^\times/1 + \omega_D \mathcal{O}_D$, we also have

$$H^1(D^\times, \chi_{q+1}) \cong H^1(\Gamma, \chi_{q+1})$$

and the following exact sequence

$$1 \to k_D^\times = (\mathcal{O}_D/\omega_D)^\times \to \Gamma \xrightarrow{v_D} \mathbb{Z} \to 0$$

The Lyndon-Hochschild-Serre spectral sequence in this case gives us the following exact sequence

$$\begin{aligned} 0 &\to H^1(\mathbb{Z}, (\chi^c_{q+1})^{k_D^\times}) \to H^1(\Gamma, \chi^c_{q+1}) \to H^1(k_D^\times, \chi^c_{q+1})^{\mathbb{Z}} \\ &\to H^2(\mathbb{Z}, (\chi^c_{q+1})^{k_D^\times}) \to H^2(\Gamma, \chi^c_{q+1}). \end{aligned} \tag{3.2}$$

If we identity $w_1^c = \chi^c_{q+1}$ as vector spaces, then the Frob-action on w_1^c corresponds to the ω_D-action on χ^c_{q+1} and the I/I^w-action on w_1^c corresponds to the $k_D^\times$-action on χ^c_{q+1}. Thus $H^j(\hat{\mathbb{Z}}, (w_1^c)^{I/I^w}) \cong H^j(\mathbb{Z}, (\chi^c_{q+1})^{k_D^\times})$ for $j = 1, 2$. On the other hand, $H^1(I/I^w, w_1^c)^{\hat{\mathbb{Z}}} \cong H^1(k_D^\times, \chi^c_{q+1})^{\mathbb{Z}}$ since $\mathrm{Frob} \cdot g \cdot \mathrm{Frob}^{-1} = g^q$ for any $g \in G_F$ and $\omega_D \cdot x \cdot \omega_D^{-1} = x^q$ for any $x \in D$. The lemma follows by five-lemma. □

3.2. **Some remarks.** We make some remarks on representations of $D^\times$.

3.2.1. *Representations of $D^\times$ in characteristic zero.* Let $\pi : D^\times \to GL(V)$ be an irreducible representation of $D^\times$ over a complex vector space. By the same argument as that of Lemma 2.4, π admits a central character. On the other hand, we know that $D^\times/F^\times$ is compact. Therefore π is finite dimensional and $\mathrm{Ker}\,\pi$ contains an open normal subgroup of $D^\times$. In particular, π is trivial on a unit group $U_D^m = 1 + \mathfrak{p}_D^m \subset D^\times$ for some $m \geq 0$. Starting with this fact, we may parameterize all complex irreducible representations of $D^\times$. See Section 54 of [5] for this parametrization.

Let K be a sufficiently large finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. Let V be a vector space over K and $\pi : D^\times \to GL(V)$ an irreducible smooth admissible representation of $D^\times$. Similarly, π admits a central character. On the other hand, we know that $D^\times/F^\times$ is compact. Therefore π is finite dimensional. Yet in this case, it is not true that π is trivial on a unit group $U_D^m = 1 + \mathfrak{p}_D^m \subset D^\times$ for some $m \geq 0$. This is because that the topology on $1 + \mathfrak{p}_D$ and the topology on $1 + \mathfrak{p}_K$ are *compatible* in some sense. One trivial example is the reduced norm $\mathrm{Nrd} : D^\times \to F^\times$ whose kernel is $D^\times_{\mathrm{Nrd}=1}$.

3.2.2. *A mod p Jacquet-Langlands correspondence for $GL_2(\mathbb{Q}_p)$.* In this subsection, we assume that $F = \mathbb{Q}_p$. Let Z denote the center of $GL_2(\mathbb{Q}_p)$, let r be an integer with $0 \leq r \leq p-1$ and $\mathrm{Symm}^r E^2$ be the representation of $GL_2(\mathbb{Z}_p)$ (via the natural projection $GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{F}_p)$) and extend it to $Z \cdot GL_2(\mathbb{Z}_p)$ by letting p act trivially. For convenience, we write $\sigma_r = \mathrm{Symm}^r E^2$. Denote

$$c\text{-}\mathrm{Ind}_{GL_2(\mathbb{Z}_p)Z}^{GL_2(\mathbb{Q}_p)} \sigma_r$$

the E-vector space of functions $f : GL_2(\mathbb{Q}_p) \to \mathrm{Symm}^r E^2$ with compact support modulo Z and $f(kg) = \sigma_r(k) f(g)$ (for $k \in GL_2(\mathbb{Z}_p)Z$ and $g \in GL_2(\mathbb{Q}_p)$). It is a $GL_2(\mathbb{Q}_p)$ representation with right regular action.

Lemma 3.5. $\mathrm{End}_{GL_2(\mathbb{Q}_p)}(c\text{-}\mathrm{Ind}_{GL_2(\mathbb{Z}_p)Z}^{GL_2(\mathbb{Q}_p)} \sigma_r) \simeq E[T]$.

This lemma is a special case of Proposition 8 of [2]. See Section 3.1 of [2] for the definition of the operator T and more details. The following theorem is proved in [2] and [3].

Theorem 3.6. *The smooth irreducible admissible mod p representations of $GL_2(\mathbb{Q}_p)$ over E are the following:*
(1) the one dimensional representations $\chi \circ \det$
(2) the representations

$$(c\text{-}\mathrm{Ind}_{GL_2(\mathbb{Z}_p)Z}^{GL_2(\mathbb{Q}_p)} \sigma_r/(T-\lambda)) \otimes (\chi \circ \det)$$

for $0 \leq r \leq p-1$, $\lambda \in E^\times$ *and* $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$
(3) the representations

$$\mathrm{Ker}(c\text{-}\mathrm{Ind}_{GL_2(\mathbb{Z}_p)Z}^{GL_2(\mathbb{Q}_p)} 1/(T-1) \twoheadrightarrow 1) \otimes (\chi \circ \det)$$

(4) the representations

$$(c\text{-}\mathrm{Ind}_{GL_2(\mathbb{Z}_p)Z}^{GL_2(\mathbb{Q}_p)} \sigma_r/T) \otimes (\chi \circ \det).$$

Definition 3.7. Let $r \in \{0, ..., p-1\}$, $\kappa : \mathbb{Q}_p^\times \to \bar{\mathbb{F}}_p^\times$. With the above notation, we give the following modulo p Jacquet-Langlands correspondence:
(1)

$$\kappa \circ \mathrm{Nrd} \leftrightarrow \mathrm{Ker}(c\text{-}\mathrm{Ind}_{GL_2(\mathbb{Z}_p)Z}^{GL_2(\mathbb{Q}_p)} 1/(T-1) \twoheadrightarrow 1) \otimes (\kappa \circ \det),$$

(2)

$$(\mathrm{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \chi_{r+1}) \otimes (\kappa \circ \mathrm{Nrd}) \leftrightarrow (c\text{-}\mathrm{Ind}_{GL_2(\mathbb{Z}_p)Z}^{GL_2(\mathbb{Q}_p)} \sigma_r/T) \otimes (\kappa \circ \det).$$

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