A CHARACTER THEORY FOR PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

CHUANGXUN CHENG

ABSTRACT. In this paper, we construct a character theory for projective representations of finite groups and deduce some consequences of this theory. In particular, we compute the number of distinct irreducible projective representations (up to isomorphism) of a finite group with a given associated Schur multiplier. We also deduce properties on the degrees of such projective representations. Consequently, for a finite group G, we obtain a sufficient condition for $H^2(G, \mathbb{C}^\times) = 0$. We also study unitary projective representations of compact groups and prove the Peter-Weyl theorem.

1. INTRODUCTION

Throughout this paper, except in Section 7, G is a finite group with identity element 1.

One way to study projective representations of a finite group G is to construct a representation group G^* of G and to show that the projective representations of G correspond to the linear representations of G^*. Then we may obtain properties of projective representations of G by studying the linear representations of G^*. In this paper, we study projective representations without G^*. Moreover, if we take the trivial multiplier, we recover the properties of linear representations of finite groups. An advantage of this approach is that we can generalize some of the results to the case of projective representations of certain infinite groups. In the last section we explain this idea and study unitary projective representations of compact groups.

1.1. **Definitions.** To outline the contents of the paper, we first recall the definitions of Schur multipliers and projective representations.

Definition 1.1. A map $\alpha : G \times G \to \mathbb{C}^\times$ is called a *multiplier* (or a *factor set* or a *2-cocycle*) on G if

(1) $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$.
(2) $\alpha(x, 1) = \alpha(1, x) = 1$ for all $x \in G$.

We say a multiplier is *unitary* if there exists a positive integer N, such that $\alpha(x, y)^N = 1$ for all $x, y \in G$.

The set of all possible multipliers on G has an abelian group structure by defining the product of two multipliers as their pointwise product. We denote this group by $Z^2(G, \mathbb{C}^\times)$.

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Department of Mathematics, Bielefeld University, D-33501 Bielefeld, Germany
Email: ccheng@math.uni-bielefeld.de

There is a special subgroup $B^2(G, \mathbb{C}^\times)$ of $Z^2(G, \mathbb{C}^\times)$ consisting of multipliers α with the form

$$\alpha(x, y) = \frac{\mu(xy)}{\mu(x)\mu(y)},$$

where $\mu : G \to \mathbb{C}^\times$ is an arbitrary function with $\mu(1) = 1$. An element of $B^2(G, \mathbb{C}^\times)$ is called a *2-coboundary*. We denote the quotient group by $H^2(G, \mathbb{C}^\times) = Z^2(G, \mathbb{C}^\times)/B^2(G, \mathbb{C}^\times)$. If α is an element of $Z^2(G, \mathbb{C}^\times)$, we denote its image in $H^2(G, \mathbb{C}^\times)$ by $[\alpha]$.

Definition 1.2. Let V be an n-dimensional vector space over $\mathbb{C}$ $(n < \infty)$. A *projective representation* of G over V is a map $\pi : G \to \mathrm{GL}(V)$ such that $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$ for all $x, y \in G$, where α is the associated multiplier. We denote this projective representation by (π, V, α) or (π, V).

We call a projective representation of G *unitary* if the associated multiplier is unitary. We call the integer n the *degree* of π. If α is the associated multiplier of π, we say that the projective representation (π, V, α) belongs to $[\alpha]$.

Definition 1.3. A *sub projective representation* of a projective representation (π, V) is a vector subspace W of V which is stable under G, i.e., $\pi(g)W \subset W$ for all $g \in G$. A projective representation is called *irreducible* if there is no proper nonzero G-stable subspace W of V.

Let (π, V, α) and (π', W, α) be two projective representations of G with the same multiplier α. A linear map $\varphi : V \to W$ is called a *G-morphism* or a *map of projective representations* if for any $g \in G$ and $v \in V$, $\varphi(\pi(g)v) = \pi'(g)(\varphi(v))$. Write $\mathrm{Hom}_G(V, W)$ for the set of all G-morphisms from V to W.

Two projective representations (π, V) and (π', W) are *equivalent* if there exists a linear isomorphism $\varphi : V \to W$ and a map $\mu : G \to \mathbb{C}^\times$ with $\mu(1) = 1$, such that the following diagram commutes for all $g \in G$.

(1.1)
$$
\begin{array}{ccc}
V & \xrightarrow{\ \varphi\ } & W \\
{\scriptstyle \pi(g)}\big\downarrow & & \big\downarrow{\scriptstyle \mu(g)\cdot\pi'(g)} \\
V & \xrightarrow[\ \varphi\ ]{} & W
\end{array}
$$

By Lemma 2.8, if (π, V, α) and (π', W, β) are equivalent, then $[\alpha] = [\beta]$.

If we fix a basis of V, then we may identify the group $\mathrm{GL}(V)$ and the group $\mathrm{GL}_n(\mathbb{C})$. A projective representation (π, V) gives us a homomorphism of groups $\bar{\pi} : G \to \mathrm{PGL}_n(\mathbb{C})$. On the other hand, for a group homomorphism $\bar{\pi} : G \to \mathrm{PGL}_n(\mathbb{C})$, any lift $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ is a projective representation with some associated multiplier α. Different lifts may give us projective representations with different multipliers, but they belong to the same cohomology class and all lifts are equivalent.

1.2. **Contents of the paper.** Fix a Schur multiplier α of G. Let (π, V, α) be a projective representation of G over a $\mathbb{C}$-vector space V with associated multiplier α. In Section 2, we show that (π, V, α) is a direct sum of irreducible projective representations (Theorem 2.3). Then to understand the set Rep_G^α of all projective representations of G with associated multiplier α, it suffices to understand the irreducible ones. We also prove that Schur's Lemma is true for projective representations (Theorem 2.12).

For any projective representation (π, V, α) and another multiplier α' with $[\alpha'] = [\alpha]$, we show that there exists a projective representation (π', V', α'), which is equivalent to (π, V, α). Therefore, to understand $\operatorname{Rep}_G^\alpha$, it suffices to understand $\operatorname{Rep}_G^{\alpha'}$ for some special α' with $[\alpha'] = [\alpha]$. In particular, we may choose α' to be unitary. This is the starting point of the character theory.

Fix a unitary Schur multiplier α. In Section 3, we construct a character theory for $\operatorname{Rep}_G^\alpha$ and prove some properties of the characters. More precisely, we show that the characters of distinct irreducible projective representations of G form an orthonormal basis of the space $\mathbb{H}_\alpha$, the space of α-class functions of G (Definition 3.13 and Theorem 3.15).

In Section 4, we define the inductions of projective representations and compute the characters of inductive projective representations.

Section 5 is the main part of this paper. We relate the projective representations of G to $\mathbb{C}[G]_\alpha$-modules, where $\mathbb{C}[G]_\alpha$ is the twisted group algebra. By studying the algebra $\mathbb{C}[G]_\alpha$, we prove that the degree of an irreducible projective representation of G divides the order of G (Theorem 5.6). As a byproduct, we obtain a sufficient condition for $H^2(G, \mathbb{C}^\times) = 0$ (Corollary 5.7). We also give another description of inductions of projective representations using tensor products and prove the Frobenius reciprocity (Propositions 5.8 and 5.11). Finally, we give a criteria for the irreducibility of an inductive projective representation (Proposition 5.13) and deduce a stronger result on the degrees of irreducible projective representations (Theorem 5.20). In the case where G is abelian, we can describe the degrees explicitly (Theorem 5.28).

In Section 6, we study the group of virtual projective characters $R_\alpha(G)$ and prove Artin's theorem.

In Section 7, we study the unitary projective representations of compact groups. As in the situation of linear representations, most properties of projective representations of finite groups carry over to unitary projective representations of compact groups. We also prove the Peter-Weyl Theorem (Theorem 7.10) for projective representations and deduce some consequences.

The theory of projective representations of finite groups has a long history ([9], [10]). Some of the results in these notes have been proven and published by others. For example, Subsection 4.3 on twists of projective representations and related topics are studied in [7], Proposition 5.24 is proved in [1, Theorem 1], a special case of Theorem 5.20 is proved in [8] (see also [6]), etc. The author claims no originality of these results. See the survey paper [3] for more discussion on the history and a more complete list of references.

Nevertheless, the treatment in these notes is different (e.g., the representation group G^* plays no role) and induces new results. The goal is to develop the theory of projective representations of finite groups (and compact groups) by exploiting the analogy with linear representations of finite groups (and compact groups). The readers will find out that the structure of these notes and some arguments are similar to those in the book [11] by J. P. Serre, which is the main source of motivations of these notes.

2. Basic properties

2.1. Complete reducibility.

Lemma 2.1. *Let $\pi : G \to \mathrm{GL}(V)$ be a projective representation of G in V with associated multiplier α. Let W be a G-stable subspace of V. Then there exists a complement W' of W in V, such that W' is also G-stable.*

Proof. Let W^0 be any complement of W in V and let p be the corresponding projection $V \to W$. Define

$$p' := \frac{1}{|G|} \sum_{g \in G} \pi(g) \circ p \circ \pi(g)^{-1}.$$

It is easy to see that $\mathrm{Im}(p') \subset W$. If $w \in W$, then $p'(w) = \frac{1}{|G|} \sum_{g \in G} \pi(g) \circ p \circ \pi(g)^{-1}(w) = \frac{1}{|G|} \sum_{g \in G} w = w$. Therefore, $\mathrm{Im}(p') = W$ and p' is also a projection from V to W. Let W' be the corresponding kernel of this projection. Note that

$$
\begin{aligned}
\pi(h) \circ p' \circ \pi(h)^{-1} &= \frac{1}{|G|} \sum_{g \in G} \pi(h)\pi(g) \circ p \circ \pi(g)^{-1}\pi(h)^{-1} \\
&= \frac{1}{|G|} \sum_{g \in G} \alpha(h,g)\pi(hg) \circ p \circ \alpha(h,g)^{-1}\pi(hg)^{-1} = p'.
\end{aligned}
$$

(2.1)

Thus $\pi(g) \circ p' = p' \circ \pi(g)$. If $w' \in W'$ and $g \in G$, we have $p'(w') = 0$, so $p'(\pi(g)w') = \pi(g)p'(w') = 0$. This shows that W' is G-stable. □

Remark 2.2. Let V be a projective representation of G. We see that V is irreducible if V is not a direct sum of two projective representations except for the trivial decomposition $V = 0 \oplus V$. A projective representation of degree 1 is evidently irreducible. On the other hand, if (π, V, α) is a projective representation of degree 1, it is easy to see that α is a 2-coboundary.

One may also prove the lemma by constructing a G-invariant inner product on V. An immediate consequence of the lemma is the following result.

Theorem 2.3. *Every projective representation of G is a direct sum of irreducible projective representations.*

Proof. Let V be a projective representation of G. We prove the theorem by induction on $\dim(V)$. If $\dim(V) \leq 1$, there is nothing to prove. Suppose that $\dim(V) \geq 2$. If V is irreducible, there is nothing to prove. Otherwise, by last proposition, $V = V_1 \oplus V_2$ with $\dim(V_i) < \dim(V)$ for $i = 1, 2$ and V_1 and V_2 are projective representations of G. By induction hypothesis, V_i $(i = 1, 2)$ are direct sums of irreducible projective representations, and so the same is true for V. □

Remark 2.4. Although we can decompose a projective representation into a direct sum of irreducible ones, the decomposition is not unique.

2.2. **Complete reducibility via twisted group algebra.** Let G be a finite group and let R be a commutative ring. Fix $\alpha \in Z^2(G, R^\times)$ a 2-cocycle. We denote by $R[G]_\alpha$ the α-twisted group algebra over R. This algebra has a basis (a_g) indexed by the elements of G. Each element f of $R[G]_\alpha$ can be uniquely written as

$$f = \sum_{g \in G} k_g a_g \text{ with } k_g \in R$$

and the multiplication in $R[G]_\alpha$ is given by

$$a_g a_h = \alpha(g,h)a_{gh}.$$

Let V be a free R-module and let $\pi : G \to \mathrm{GL}_R(V)$ be a projective representation of G in V. For each $g \in G$ and $v \in V$, set $a_g v = \pi(g)v$. By linearity this defines fx for any $f \in R[G]_\alpha$ and $v \in V$. Thus V is endowed with the structure of a left $R[G]_\alpha$-module. Conversely, such a structure defines a projective representation of G in V with associated multiplier α.

To say a ring or an algebra A is semisimple is equivalent to saying that each A-module M is semisimple, i.e., that each submodule M' of M is a direct summand in M as an A-module.

Proposition 2.5. *If R is a field of characteristic 0, then the algebran $R[G]_\alpha$ is semisimple.*

Proof. Let M be an $R[G]_\alpha$-module and $M' \subset M$ a submodule. Certainly, M' is a sub R-vector space of M and is a direct factor as an R-module. Let $p : M \to M'$ be an R-linear projection, define

$$P = \frac{1}{|G|} \sum_{g \in G} a_g p(a_g)^{-1}.$$

Then P is a projection and is $R[G]_\alpha$-linear, which implies that M' is a direct factor of M as an $R[G]_\alpha$-module. $\square$

Remark 2.6. This proposition corresponds to the complete reducibility of projective representations Theorem 2.3.

Corollary 2.7. *If R is a field of characteristic zero, the algebran $R[G]_\alpha$ is a product of matrix algebras over skew fields of finite degree over R.*

2.3. **Some lemmas.** In this subsection, we prove some basic lemmas that may help us simplify the study of projective representations of G.

Lemma 2.8. *Let (π, V, α) and (π', V', α') be two projective representations of G. If these two projective representations are equivalent, then α and α' have the same image in $H^2(G, \mathbb{C}^\times)$.*

Proof. By definition, there exist $\varphi : V \to V'$ and $\mu : G \to \mathbb{C}^\times$, such that

$$\mu(g)\pi'(g)(\varphi(v)) = \varphi(\pi(g)v) \text{ for all } g \in G.$$

Let $g, h \in G$, we have

$$\mu(gh)\pi'(gh)(\varphi(v)) = \varphi(\pi(gh)v).$$

On the other hand,

(2.2)
$$\varphi(\pi(gh)v) = \varphi(\alpha(g,h)^{-1}\pi(g)\pi(h)v) = \alpha(g,h)^{-1}\mu(g)\pi'(g)\mu(h)\pi'(h)\varphi(v)$$
$$= \mu(g)\mu(h)\alpha(g,h)^{-1}\alpha'(g,h)\pi'(gh)(\varphi(v)).$$

Comparing the above two equations, we have

$$\alpha(g,h)^{-1}\alpha'(g,h) = \mu(gh)\mu(g)^{-1}\mu(h)^{-1}.$$

Therefore, $[\alpha] = [\alpha']$. $\square$

Lemma 2.9. *Let (π, V, α) be a projective representation. Let α' be another multiplier such that $[\alpha] = [\alpha']$. Then there exists a projective representation (π', V') with multiplier α', such that (π, V, α) is equivalent to (π', V', α').*

Proof. By assumption, we may assume that

$$\alpha(g, h)^{-1} \alpha'(g, h) = \mu(gh)\mu(g)^{-1}\mu(h)^{-1}$$

for a function $\mu : G \to \mathbb{C}^\times$. Then we define $V' = V$ and $\pi'(g) = \mu'(g)\pi(g)$. The identity map $V \to V' = V$ gives us an equivalence of projective representations.   □

More generally, we have the following result.

Lemma 2.10. *Let α and α' be two multipliers. If $[\alpha] = [\alpha']$, then $\mathbb{C}[G]_\alpha \cong \mathbb{C}[G]_{\alpha'}$.*

Proof. This follows from the same argument of Lemma 2.9.   □

Lemma 2.11. *Let (π, V, α) and (π', V', α) be two projective representations of G. Let $\varphi : V \to V'$ be a G-morphism. Then the kernel of φ, the image of φ, and the cokernel of φ are either 0 or projective representations with associated multiplier α.*

Proof. The statement on the kernel and image of φ is clear. Assume that the cokernel of φ is nonzero. Let $\bar{v}'$ be an element of the cokernel of φ. Let $v' \in V'$ be a lift of $\bar{v}'$. Then we define $\bar{\pi}' : G \to \mathrm{GL}(\mathrm{Coker}(\varphi))$ by the equation

$$\bar{\pi}'(g)(\bar{v}') = \overline{\pi'(v')}.$$

It is easy to check that the definition is independent of the choice of v' and it defines a projective representation of G on $\mathrm{Coker}(\varphi)$ with associated multiplier α.   □

2.4. Schur's Lemma.

Theorem 2.12 (Schur's Lemma). *If V and W are irreducible projective representations of G with the same multiplier and $\varphi : V \to W$ is a map of projective representations, then*

    (1) *Either φ is an isomorphism or $\varphi = 0$.*
    (2) *If $V = W$, then $\varphi = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.*

Proof. The proof of this result is the same as the proof in the classical case. The first claim follows from the fact that $\mathrm{Ker}\,\varphi$ and $\mathrm{Im}\,\varphi$ are G-stable subspaces. For the second, since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue $\lambda \in \mathbb{C}$. Therefore, $\varphi - \lambda \cdot \mathrm{id}_V$ has a nonzero kernel. Since V is irreducible, we must have $\varphi - \lambda \cdot \mathrm{id}_V = 0$ and therefore $\varphi = \lambda \cdot \mathrm{id}_V$.   □

We deduce some corollaries of Schur's Lemma.

Corollary 2.13. *Let (π_1, V_1, α) and (π_2, V_2, α) be two irreducible projective representations of G with the same multiplier. Let $f : V_1 \to V_2$ be a linear map of vector spaces. Define*

$$f' = \frac{1}{|G|} \sum_{g \in G} \pi_2(g)^{-1} f \pi_1(g).$$

Then

    (1) *If $V_1 \not\cong V_2$, then $f' = 0$.*
    (2) *If $V_1 = V_2$ and $\pi_1 = \pi_2$, then f' is a homothety of ratio $\frac{1}{\dim(V_1)}\mathrm{Tr}(f)$.*

Proof. For any $h \in G$, we have

$$\pi_2(h)^{-1}f'\pi_1(h) = \frac{1}{|G|}\sum_{g \in G}\pi_2(h)^{-1}\pi_2(g)^{-1}f\pi_1(g)\pi_1(h)$$

(2.3)
$$= \frac{1}{|G|}\sum_{g \in G}(\pi_2(g)\pi_2(h))^{-1}f(\pi_1(g)\pi_1(h))$$

$$= \frac{1}{|G|}\sum_{g \in G}\pi_2(gh)^{-1}f\pi_1(gh) = f'.$$

Therefore, $f' : V_1 \to V_2$ is a map of projective representations. The first statement follows from Schur's Lemma. Assume now that $(\pi_1, V_1, \alpha) = (\pi_2, V_2, \alpha)$. By Schur's Lemma again, f' is a homothety. Let λ be the ratio of f'. Then

$$\dim V_1 \cdot \lambda = \mathrm{Tr}(f') = \mathrm{Tr}(\frac{1}{|G|}\sum_{g \in G}\pi_2(g)^{-1}f\pi_1(g)) = \mathrm{Tr}(f).$$

The claim follows. $\qquad\square$

We may write the above result in matrix form. Assume that

$$\pi_1(g) = (r_{i_1 j_1}(g)), \quad \pi_2(g) = (r_{i_2 j_2}(g)).$$

The linear maps f and f' are defined by matrices $(x_{i_2 i_1})$ and $(x'_{i_2 i_1})$ respectively.

Corollary 2.14. *With the notation as above, we have*

(1) *If $V_1 \not\cong V_2$, then*

$$\frac{1}{|G|}\sum_{g \in G}\alpha(g, g^{-1})^{-1}r_{i_2 j_2}(g^{-1})r_{j_1 i_1}(g) = 0$$

for any i_1, i_2, j_1, j_2.

(2) *If $V_1 = V_2$ and $\pi_1 = \pi_2$, then*

$$\frac{1}{|G|}\sum_{g \in G}\alpha(g, g^{-1})^{-1}r_{i_2 j_2}(g^{-1})r_{j_1 i_1}(g) = \frac{1}{\dim V_1}\delta_{i_2 i_1}\delta_{j_2 j_1}.$$

Here $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$

Proof. By definition,

$$x'_{i_2 i_1} = \frac{1}{|G|}\sum_{g, j_1, j_2}\alpha(g, g^{-1})^{-1}r_{i_2 j_2}(g^{-1})x_{j_2 j_1}r_{j_1 i_1}(g).$$

The right hand side is a linear form with respect to $x_{j_2 j_1}$. In case (1), this form vanishes for all systems of values of the $x_{j_2 j_1}$. Thus the coefficients are zero. The claim follows.

In case (2), we have $x'_{i_2 i_1} = \lambda\delta_{i_2 i_1}$, and

$$f' = \lambda = \frac{1}{\dim V_1}\mathrm{Tr}(f) = \frac{1}{\dim V_1}\sum_{j_2, j_1}\delta_{j_2 j_1}x_{j_2 j_1}.$$

Therefore, we have the following equation

$$\frac{1}{|G|} \sum_{g,j_1,j_2} \alpha(g,g^{-1})^{-1} r_{i_2 j_2}(g^{-1}) x_{j_2 j_1} r_{j_1 i_1}(g) = x'_{i_2 i_1} = \frac{1}{\dim V_1} \sum_{j_2,j_1} \delta_{i_2 i_1} \delta_{j_2 j_1} x_{j_2 j_1}.$$

Comparing the coefficients of $x_{j_2 j_1}$, the claim follows. $\square$

2.5. **Direct sums, tensor products, and dual projective representations.** Let (π, V, α) and (π', V', α') be two projective representations of G. The tensor product $V \otimes V'$ is a projective representation with associated multiplier $\alpha\alpha'$ via

$$g \cdot (v \otimes v') = \pi(g)(v) \otimes \pi'(g)(v').$$

If $\alpha = \alpha'$, the direct sum $V \oplus V'$ is a projective representation of G with associated multiplier α via

$$g \cdot (v \oplus v') = \pi(g)(v) \oplus \pi'(g)(v').$$

Let α be a multiplier of G. Define $\alpha^* : G \times G \to \mathbb{C}^\times$ by

$$\alpha^*(x,y) = \alpha(y^{-1}, x^{-1})^{-1}.$$

Lemma 2.15. *With the notation as above, we have*

    (1) α^* *is a multiplier.*
    (2) $[\alpha^*] = [\alpha]$ *as elements in* $H^2(G, \mathbb{C}^\times)$.

Proof. The first claim follows from the definition. We prove the second claim. By definition,

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z) \text{ for all } x,y,z \in G.$$

Let $y = x^{-1}$ we see that

$$\alpha(x,x^{-1}) = \alpha(x,x^{-1}z)\alpha(x^{-1},z) \text{ for all } x,z \in G.$$

Then let $z = x$, we see that

$$\alpha(x,x^{-1}) = \alpha(x^{-1},x) \text{ for all } x \in G.$$

For any $x,y \in G$, we have

$$\alpha(x,y) = \alpha(x,x^{-1})\alpha(x^{-1},xy)^{-1},$$

and

$$\alpha(y^{-1},x^{-1}) = \alpha(y^{-1},y)\alpha(y,y^{-1}x^{-1})^{-1}.$$

Combining the above two equations, we have

$$
\begin{aligned}
\alpha(x,y)\alpha(y^{-1},x^{-1}) &= \frac{\alpha(x,x^{-1})\alpha(y^{-1},y)}{\alpha(x^{-1},xy)\alpha(y,y^{-1}x^{-1})} \\
&= \alpha(x,x^{-1})\alpha(y^{-1},y)\alpha(xy,y^{-1}x^{-1})^{-1} \\
&= \mu(x)\mu(y)\mu(xy)^{-1}.
\end{aligned}
$$
(2.4)

Here $\mu : G \to \mathbb{C}^\times$ is defined by $\mu(x) = \alpha(x,x^{-1})$. We see that the difference of α and α^* is given by a 2-coboundary. The claim follows. $\square$

The dual $V^* = \mathrm{Hom}(V, \mathbb{C})$ of V is also a projective representation of G via

$$\pi^*(g) = \alpha(g, g^{-1})^{-1t}\pi(g^{-1}) : V^* \to V^*.$$

Let $\langle,\rangle$ be the natural pairing of V and V^*, then we have

$$\langle \pi^*(g)(v^*), \pi(g)(v) \rangle = \langle v^*, v \rangle.$$

By definition,

$$\pi^*(gh) = \alpha(gh, (gh)^{-1})^{-1t}\pi((gh)^{-1}) = \alpha(gh, (gh)^{-1})^{-1t}\pi(h^{-1}g^{-1})$$

(2.5)
$$= \alpha(gh, (gh)^{-1})^{-1}\alpha(h^{-1}, g^{-1})^{-1t}(\pi(h^{-1})\pi(g^{-1}))$$
$$= \alpha(gh, (gh)^{-1})^{-1}\alpha(h^{-1}, g^{-1})^{-1}\alpha(g, g^{-1})\alpha(h, h^{-1})\pi^*(g)\pi^*(h)$$
$$= \alpha(g, h)\pi^*(g)\pi^*(h).$$

Therefore, the multiplier attached to π^* is α^{-1}.

3. Character theory

In this section, we construct a character theory for unitary projective representations of finite groups.

3.1. Unitary projective representations.

Let (π, V, α) be a projective representation of G. By definition, we have

$$\pi(g)\pi(g^{-1}) = \alpha(g, g^{-1})\pi(1) = \alpha(g, g^{-1}).$$

We cannot expect that the eigenvalues of $\pi(g)$ are algebraic integers. Nevertheless, as we will show next, we can find another projective representation (π', V', α') such that it is equivalent to (π, V, α) and $\pi'(g)$ has roots of unity as eigenvalues.

Lemma 3.1. *Let α be a multiplier of G. Then there exists a unitary multiplier α' such that $[\alpha] = [\alpha']$.*

Proof. Denote by $|G|$ the order of G. Define $\beta : G \to \mathbb{C}^\times$ by $\beta(x) = \prod_{z \in G} \alpha(x, z)$. By definition, $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$. Thus

$$\prod_{z \in G} \alpha(x, y)\alpha(xy, z) = \prod_{z \in G} \alpha(x, yz)\alpha(y, z).$$

Therefore

$$\alpha(x, y)^{|G|} = \frac{\beta(x)\beta(y)}{\beta(xy)}.$$

Let $\beta' : G \to \mathbb{C}^\times$ be a map such that $(\beta')^{|G|} = \beta$ and $\beta'(1) = 1$. Define α' by

$$\alpha'(x, y) = \frac{\beta'(xy)}{\beta'(x)\beta'(y)}\alpha(x, y).$$

It is easy to see that α' is a unitary multiplier and $[\alpha'] = [\alpha]$. $\square$

Remark 3.2. From the proof of the above lemma, we see that the group $H^2(G, \mathbb{C}^\times)$ is annihilated by the order of G.

Assume that α is unitary and $[\alpha] \in H^2(G, \mathbb{C}^\times)$ has order A. Then α^A is a 2-coboundary, i.e., $\alpha^A(x, y) = \frac{\mu(xy)}{\mu(x)\mu(y)}$ for some function $\mu : G \to \mathbb{C}^\times$ with $\mu(1) = 1$. Note that

$\prod_{y \in G} \alpha^A(x, y) = \mu(x)^{-|G|}$, $\mu(x)$ is a root of unity. Let $\lambda : G \to \mathbb{C}^\times$ be another function such that $\lambda^A = \mu$ and $\lambda(1) = 1$. Let $\alpha''(x, y) = \alpha(x, y) \frac{\lambda(x)\lambda(y)}{\lambda(xy)}$. Then α'' is a unitary multiplier such that $[\alpha''] = [\alpha]$ and $(\alpha'')^A = 1$. In particular, $\alpha''(x, y)$ is an $|A|$-th root of unity.

From the above discussion, every element of $H^2(G, \mathbb{C}^\times)$ is represented by a function $\alpha : G \times G \to \{|G|$-th root of unity$\}$. Therefore $H^2(G, \mathbb{C}^\times)$ is a finite set.

By Lemma 2.9 and Lemma 3.1, every projective representation of G is equivalent to a unitary projective representation. Let $c \in H^2(G, \mathbb{C}^\times)$. In order to understand the projective representations belonging to c, it suffices to study the projective representations (π, V) with associated multiplier α, for a fixed unitary multiplier α with $[\alpha] = c$.

3.2. **Definition of characters of projective representations.** Let (π, V, α) be a unitary projective representation. Define $\chi_\pi : G \to \mathbb{C}$ by the equation

$$\chi_\pi(g) = \mathrm{Tr}(\pi(g)) \text{ for all } g \in G.$$

The function χ_π is called the *character* of the projective representation (π, V, α). Note that $\chi_\pi(g)$ is the sum of all eigenvalues of $\pi(g)$.

Remark 3.3. One can certainly define characters for all projective representations, but some of the following properties are only true for unitary ones.

Lemma 3.4. *If χ is the character of a unitary projective representation (π, V, α) of degree n, then*

(1) $\chi(1) = n$.
(2) $\chi(g^{-1}) = \alpha(g, g^{-1})\overline{\chi(g)}$. *Here $^-$ denotes the complex conjugation.*
(3) $\chi(hgh^{-1}) = \frac{\alpha(h,h^{-1})}{\alpha(h,gh^{-1})\alpha(g,h^{-1})}\chi(g) = \frac{\alpha(h,h^{-1})}{\alpha(h,g)\alpha(hg,h^{-1})}\chi(g)$ *for all $h, g \in G$.*

Proof. The first claim is clear. Note that G is a finite group, any element $g \in G$ has finite order. Let $|G|$ be the order of G, then

$$\begin{aligned}
I_{n \times n} = \pi(1) &= \pi(g^{|G|}) \\
&= \alpha(g, g^{|G|-1})^{-1}\pi(g)\pi(g^{|G|-1}) = \cdots \\
&= \alpha(g, g^{|G|-1})^{-1}\alpha(g, g^{|G|-2})^{-1}\cdots\alpha(g, g)^{-1}\pi(g)^{|G|}.
\end{aligned}$$

(3.1)

Since α is unitary, there exists a positive integer N, such that $\pi(g)^N = I_{n \times n}$. The eigenvalues of $\pi(g)$ are roots of unity. Assume that λ_i $(i = 1, \cdots, n)$ are eigenvalues of $\pi(g)$, then $\lambda_i^{-1} = \overline{\lambda_i}$ $(i = 1, \cdots, n)$ are eigenvalues of $\pi(g)^{-1}$. Therefore, we have

$$\chi(g^{-1}) = \mathrm{Tr}(\pi(g^{-1})) = \mathrm{Tr}(\alpha(g, g^{-1})\pi(g)^{-1}) = \alpha(g, g^{-1})\overline{\mathrm{Tr}\,\pi(g)} = \alpha(g, g^{-1})\overline{\chi(g)}.$$

The second claim follows. The third claim follows from the following equation

$$\pi(hgh^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})}\pi(h)\pi(g)\pi(h)^{-1} = \frac{\alpha(h, h^{-1})}{\alpha(h, g)\alpha(hg, h^{-1})}\pi(h)\pi(g)\pi(h)^{-1}$$

and the fact that similar matrices have the same eigenvalues. $\qquad\square$

Lemma 3.5. *Let V and W be two unitary projective representations of G with associated multipliers α and α' respectively. Then*

(1) *If $\alpha = \alpha'$, then $\chi_{V \oplus W} = \chi_V + \chi_W$.*
(2) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.
(3) $\chi_{V^*} = \overline{\chi_V}$.

Proof. The claims are clear. $\qquad\square$

3.3. **Orthogonality of characters.** If ϕ and ψ are two $\mathbb{C}$-valued functions on G, define

$$(\phi, \psi) = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

This is a scalar product, i.e., it is linear in ϕ, semilinear in ψ, and $(\phi, \phi) > 0$ for all $\phi \neq 0$.

Theorem 3.6. (1) *If χ is the character of a unitary irreducible projective represen-tation, then $(\chi, \chi) = 1$.*
(2) *If χ and χ' are the characters of two nonisomorphic unitary irreducible projective representations with the same associated multiplier α, then $(\chi, \chi') = 0$.*

Proof. We prove this result by using Corollary 2.14. Let (π, V) be an irreducible projective representation with character χ, given in matrix form $\pi(g) = (r_{ij}(g))$. Then $\chi(g) = \sum_i r_{ii}(g)$, hence

$$
\begin{aligned}
(\chi, \chi) &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \alpha(g, g^{-1})^{-1} \chi(g)\chi(g^{-1}) \\
\text{(3.2)} \qquad &= \frac{1}{|G|} \sum_{g \in G} \alpha(g, g^{-1})^{-1} \sum_i r_{ii}(g) \sum_j r_{jj}(g^{-1}) \\
&= \frac{1}{|G|} \sum_{i,j} \sum_{g \in G} \alpha(g, g^{-1})^{-1} r_{ii}(g) r_{jj}(g^{-1}) \\
&= \sum_{i,j} \frac{1}{\dim V} \delta_{ij} = 1
\end{aligned}
$$

The second claim follows by a similar argument. $\qquad\square$

A character of an irreducible projective representation is called an irreducible character. By the above theorem, the irreducible characters form an orthonormal system.

Corollary 3.7. *Let (V, α) be a unitary projective representation of G with character ϕ. Suppose that V decomposes into a direct sum of irreducible projective representations*

$$V = W_1 \oplus \cdots \oplus W_k.$$

Then, if (W, α) is an irreducible representation with character χ, the number of W_i iso-morphic to W is equal to the scalar product (ϕ, χ).

Proof. Let χ_i be the character of W_i, then $\phi = \sum_i \chi_i$. Thus $(\phi, \chi) = \sum_i (\chi_i, \chi)$. The theorem follows. $\qquad\square$

With the same notation as in last theorem, the number of W_i isomorphic to W does not depend on the chosen decomposition. This number is called the number of times that W occurs in V.

Corollary 3.8. *Two unitary projective representations with the same associated multiplier and the same character are isomorphic.*

Proof. The last corollary shows that they contain each given irreducible projective representation the same number of times. $\square$

The above results reduce the study of unitary projective representations with associated multiplier α to that of their characters. If $\chi_1, \ldots, \chi_l$ are the distinct irreducible characters of G, and if $W_1, \ldots, W_l$ denote the corresponding irreducible projective representations, each projective representation V with associated multiplier α is isomorphic to a direct sum

$$W = W_1^{\oplus m_1} \oplus \cdots \oplus W_l^{\oplus m_l}, \quad m_i \in \mathbb{Z}_{\geq 0}.$$

The character ϕ of W is equal to $\sum_i m_i \chi_i$, and we have $m_i = (\phi, \chi_i)$. The orthogonality relations imply

$$(\phi, \phi) = \sum_i m_i^2.$$

We have the following result.

Corollary 3.9. *If ϕ is the character of a unitary projective representation V, (ϕ, ϕ) is a positive integer and we have $(\phi, \phi) = 1$ if and only if V is irreducible.*

Proof. Indeed, $\sum_i m_i^2 = 1$ if and only if one of the m_i's is equal to 1 and the others to 0. $\square$

3.4. **The α-regular projective representation of G.** In this subsection, we study a special projective representation: the α-regular projective representation. Fix α a unitary multiplier of group G. The irreducible characters of G with associated multiplier α are denoted by $\chi_1, \ldots, \chi_l$, their degrees are $n_1, \ldots, n_l$.

Let R be the α-regular representation of G. It has a basis $(e_g)_{g \in G}$ such that $R(h)(e_g) = \alpha(h, g)e_{hg}$. If $h \neq 1$, we have $hg \neq g$ for all g, which shows that the diagonal entries of the matrix $R(h)$ are zero. In particular, $\text{Tr}(R(h)) = 0$. On the other hand, for $h = 1$, we have

$$\text{Tr}(R(1)) = \dim R = |G|.$$

Lemma 3.10. *The character r_G of the α-regular projective representation is given by*

$$r_G(1) = |G|, \quad r_G(h) = 0 \text{ if } h \neq 1.$$

Corollary 3.11.   (1) *Every irreducible projective representation W_i with associated multiplier α is contained in the α-regular projective representation with multiplicity equal to its degree n_i.*
    (2) *The degrees n_i satisfy the relation $\sum_i n_i^2 = |G|$.*
    (3) *If $g \in G$ is different from 1, we have $\sum_i n_i \chi_i(g) = 0$.*

Proof. For (1), the multiplicity is equal to (r_G, χ_i). Since

$$(r_G, \chi_i) = \frac{1}{|G|} \sum_{g \in G} r_G(g) \overline{\chi_i(g)} = \chi_i(1) = n_i,$$

the result holds. From (1), $r_G(g) = \sum_i n_i \chi_i(g)$ for all $g \in G$. The claims follow. $\quad\square$

The above result can be used in determining the irreducible projective representations of a group G with associated unitary multiplier α. Suppose we have constructed some mutually nonisomorphic irreducible projective representations of degrees $n_1, \ldots, n_k$ with associated multiplier α, in order that they be all the irreducible projective representations of G with associated multiplier α, it is necessary and sufficient that $\sum n_i^2 = |G|$. Furthermore, the above result has the following interesting corollary.

Corollary 3.12. *Let G be a group of order n, such that n is either a prime number or $n \in \{6, 10, 14, 15\}$. Then $H^2(G, \mathbb{C}^\times) = 0$.*

Proof. If n is a prime number, then G is a cyclic group and the corollary follows from the formula

$$H^2(G, \mathbb{C}^\times) = (\mathbb{C}^\times)^G / \operatorname{Norm}(\mathbb{C}^\times).$$

Assume now that $n \in \{6, 10, 14, 15\}$. Suppose that $H^2(G, \mathbb{C}^\times) \neq 0$. Then their exists a unitary multiplier α of G such that α is not a coboundary. By definition, there exists no projective representation of G with degree 1 and associated multiplier α. By the above results, we have

$$n = \sum_i n_i^2$$

with $n_i \geq 2$. This is impossible if $n \in \{6, 10, 14, 15\}$. The claim follows. $\quad\square$

See Corollary 5.7 for a stronger result.

3.5. The number of simple objects in $\operatorname{Rep}_G^\alpha$.

Definition 3.13. A function $f : G \to \mathbb{C}$ is called an α-*class function* if for all $g, h \in G$,

$$f(hgh^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})} f(g) = \frac{\alpha(h, h^{-1})}{\alpha(h, g)\alpha(hg, h^{-1})} f(g).$$

Let $\mathbb{H}_\alpha$ denote the space of α-class functions on G. The characters of projective representations belong to $\mathbb{H}_\alpha$.

Lemma 3.14. *Let α be a unitary multiplier. Let f be an α-class function on G. Let (π, V, α) be an irreducible projective representation of G. Let π_f be the linear map of V into itself defined by*

$$\pi_f = \sum_{g \in G} \overline{f(g)} \pi(g).$$

If V is irreducible with degree n and character χ, then π_f is a homothety of ratio λ given by

$$\lambda = \frac{1}{n} \sum_{g \in G} \overline{f(g)} \chi(g) = \frac{|G|}{n}(\chi, f).$$

Proof. For any $h \in G$, we have

$$\pi(h)\pi_f\pi(h)^{-1} = \sum_{g \in G} \overline{f(g)}\pi(h)\pi(g)\pi(h)^{-1}$$

(3.3)
$$= \sum_{g \in G} \overline{f(g)}\frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})}\pi(hgh^{-1})$$

$$= \sum_{g \in G} \overline{f(hgh^{-1})}\pi(hgh^{-1}) = \pi_f.$$

Therefore π_f is a map of projective representations. If V is irreducible, by Schur's Lemma, π_f is a homothety of ratio λ. Note that

$$n\lambda = \mathrm{Tr}(\pi_f) = \sum_{g \in G} \overline{f(g)}\,\mathrm{Tr}(\pi(g)) = \sum_{g \in G} \overline{f(g)}\chi(g) = |G|(\chi, f).$$

The claim follows. $\qquad\square$

Theorem 3.15. *Let α be a unitary multiplier. The characters (χ_i) of irreducible projective representations in Rep_G^α form an orthonormal basis of $\mathbb{H}_\alpha$. In particular, the number of irreducible projective representations with associated multiplier α (up to isomorphism) is equal to $\dim_{\mathbb{C}} \mathbb{H}_\alpha$.*

Proof. It suffices to show that the characters (χ_i) generate $\mathbb{H}_\alpha$. For this, it suffices to show that every element of $\mathbb{H}_\alpha$ orthogonal to all the χ_i is zero.

Let $f \in \mathbb{H}_\alpha$ and assume that f is orthogonal to all χ_i, i.e., $(\chi_i, f) = 0$ for all i. For each projective representation of G with multiplier α, put $\pi_f = \sum_{g \in G} \overline{f(g)}\pi(g)$. Since f is orthogonal to χ_i, the above lemma shows that π_f is zero as long as π is irreducible. From the direct sum decomposition, we see that π_f is always zero. Applying this to the α-regular projective representation R and computing the image of e_1 under π_f, we have

$$0 = \pi_f(e_1) = \sum_{g \in G} \overline{f(g)}R(g)(e_1) = \sum_{g \in G} \overline{f(g)}e_g.$$

Therefore $f(g) = 0$ for all g. The theorem follows. $\qquad\square$

The number $\dim_{\mathbb{C}} \mathbb{H}_\alpha$ is less or equal to the number of the conjugacy classes of G. Let $g \in G$. We say g is an α-*element* if $\frac{\alpha(h,h^{-1})}{\alpha(h,gh^{-1})\alpha(g,h^{-1})} = 1$ for all elements in $C_G(g) = \{h \in G \mid hg = gh\}$. If f is an α-class function on G, then we must have $f(g) = 0$ for g not an α-element.

Lemma 3.16.   (1) *Let $g \in G$. Then g is an α-element if and only if $\alpha(g, h) = \alpha(h, g)$ for all $h \in C_G(g)$.*
    (2) *If $g \in G$ is an α-element, then so are the conjugates of g.*

Proof. The first claim follows from the fact that $C_G(g)$ is a group and the following equation

$$\alpha(h, h^{-1}) = \alpha(h, gh^{-1})\alpha(h^{-1}, hgh^{-1}).$$

For the second claim, first note that $C_G(xgx^{-1}) = xC_G(g)x^{-1}$. By the following computation

$$\alpha(xgx^{-1}, xhx^{-1}) = \frac{\alpha(x, ghx^{-1})\alpha(gx^{-1}, xhx^{-1})}{\alpha(x, gx^{-1})}$$

(3.4)

$$= \frac{\alpha(x, ghx^{-1})}{\alpha(x, gx^{-1})} \frac{\alpha(gx^{-1}, x)\alpha(g, hx^{-1})}{\alpha(x, hx^{-1})}$$

$$= \frac{\alpha(x, ghx^{-1})\alpha(gx^{-1}, x)}{\alpha(x, gx^{-1})\alpha(x, hx^{-1})} \frac{\alpha(g, h)\alpha(gh, x^{-1})}{\alpha(h, x^{-1})}$$

$$= \frac{\alpha(x, ghx^{-1})\alpha(g, h)\alpha(gh, x^{-1})\alpha(x, x^{-1})}{\alpha(x, gx^{-1})\alpha(x, hx^{-1})\alpha(h, x^{-1})\alpha(g, x^{-1})}$$

we see that, if $hg = gh$ and $\alpha(g, h) = \alpha(h, g)$, then $\alpha(xgx^{-1}, xhx^{-1}) = \alpha(xhx^{-1}, xgx^{-1})$. The second claim follows from the first claim. $\qquad\square$

Corollary 3.17. *Let l_α be the number of the conjugacy classes of G which contain α-elements. Then $\dim_{\mathbb{C}} \mathbb{H}_\alpha = l_\alpha$.*

3.6. **Products of projective representations.** Let G_1 and G_2 be two finite groups. Let $\alpha_i \in Z^2(G_i, \mathbb{C}^\times)$ be a 2-cocycle of G_i ($i = 1, 2$). Then we define a map $\alpha_1 \times \alpha_2 : G_1 \times G_2 \to \mathbb{C}^\times$ by

$$(\alpha_1 \times \alpha_2)((g_1, g_2), (h_1, h_2)) = \alpha_1(g_1, h_1)\alpha_2(g_2, h_2) \text{ for all } g_1, h_1 \in G_1, \; g_2, h_2 \in G_2.$$

It is easy to check that $\alpha_1 \times \alpha_2 \in Z^2(G_1 \times G_2, \mathbb{C}^\times)$. Let (π_1, V_1, α_1) and (π_2, V_2, α_2) be projective representations of G_1 and G_2 respectively. We define a map $\pi_1 \times \pi_2 : G_1 \times G_2 \to \mathrm{GL}(V_1 \otimes V_2)$ by

$$\pi_1 \times \pi_2((g_1, g_2)) = \pi_1(g_1) \otimes \pi_2(g_2) \text{ for all } g_1 \in G_1, \; g_2 \in G_2.$$

Then $(\pi_1 \times \pi_2, V_1 \otimes V_2, \alpha_1 \times \alpha_2)$ is a projective representation of $G_1 \times G_2$. If moreover π_1 and π_2 are unitary projective representations, then so is $\pi_1 \times \pi_2$. Let χ_i be the character associated with the projective representation π_i of G_i ($i = 1, 2$), then the character χ of $\pi_1 \times \pi_2$ is given by

$$\chi((g_1, g_2)) = \chi_1(g_1) \cdot \chi_2(g_2) \text{ for all } g_1 \in G_1, \; g_2 \in G_2.$$

Proposition 3.18. *With the above notation,*

(1) *if π_i is irreducible ($i = 1, 2$), then $\pi_1 \times \pi_2$ is an irreducible projective representation of $G_1 \times G_2$;*

(2) *each irreducible projective representation of $G_1 \times G_2$ with multiplier $\alpha = \alpha_1 \times \alpha_2$ is isomorphic to a representation $\pi_1 \times \pi_2$, where π_i is an irreducible projective representation of G_i with multiplier α_i ($i = 1, 2$).*

Proof. We may assume that the projective representations are unitary. If π_i is irreducible, then

$$\frac{1}{|G_i|} \sum_{g_i \in G_i} |\chi_i(g_i)|^2 = 1 \text{ for } i = 1, 2.$$

Taking product, we have

$$\frac{1}{|G|} \sum_{(g_1, g_2) \in G_1 \times G_2} |\chi(g_1, g_2)|^2 = 1.$$

Thus $\pi_1 \times \pi_2$ is irreducible.

To prove (2), we show that each α-class function f on $G_1 \times G_2$, which is orthogonal to the characters of the form $\chi_1(g_1)\chi_2(g_2)$, is zero. Suppose f is such a function, we have

$$\sum_{g_1,g_2} f(g_1,g_2)\overline{\chi_1(g_1)\chi_2(g_2)} = 0.$$

Therefore,

$$\sum_{g_2}(\sum_{g_1} f(g_1,g_2)\overline{\chi_1(g_1)})\overline{\chi_2(g_2)} = 0.$$

This tells us

$$\sum_{g_1} f(g_1,g_2)\overline{\chi_1(g_1)} = 0.$$

So $f(g_1,g_2) = 0$. The claim follows. $\qquad\square$

By the above proposition, to understand the projective representations of $G_1 \times G_2$ with associated multiplier $\alpha_1 \times \alpha_2$, it suffices to understand the projective representations of G_1 with associated multiplier α_1 and the projective representations of G_2 with associated multiplier α_2. On the other hand, note that there are multipliers of $G_1 \times G_2$ which are not of the form $\alpha_1 \times \alpha_2$. We do not obtain all projective representations of $G_1 \times G_2$ by the construction in this subsection.

4. INDUCED PROJECTIVE REPRESENTATIONS

4.1. **Two descriptions of induced projective representations.** Fix a multiplier α of G. Let $H \subset G$ be a subgroup of G. We denote $\alpha_H : H \times H \to \mathbb{C}^\times$ the restriction of α. Thus $\alpha_H \in Z^2(H, \mathbb{C}^\times)$. Let $(\mathfrak{p}, W, \alpha_H)$ be a projective representation of H. Let V be the vector space

$$V = \{f : G \to W \mid f(hg) = \alpha(hg, g^{-1})\mathfrak{p}(h)f(g) \text{ for all } h \in H, g \in G\}.$$

We define a map $\pi : G \to \mathrm{GL}(V)$ by the equation $(\pi(g)f)(g') = \alpha(g', g)f(g'g)$.

Lemma 4.1. *With the above notation, the map π defines a projective representation of G with associated multiplier α. We write this projective representation as $\mathrm{Ind}_H^G(W)$.*

Proof. By definition, for any $f \in V$, $g, h \in G$,

$$(4.1)\qquad \begin{aligned} (\pi(g)(\pi(h)f))(g') &= \alpha(g', g)(\pi(h)f)(g'g) \\ &= \alpha(g', g)\alpha(g'g, h)f(g'gh) \\ &= \alpha(g', gh)\alpha(g, h)f(g'(gh)) \\ &= \alpha(g, h)(\pi(gh)f)(g'). \end{aligned}$$

Also $(\pi(1)f)(g') = f(g')$. The lemma follows. $\qquad\square$

Lemma 4.2. *For any $w \in W$, define $f_w : G \to W$ by*

$$f_w(g) = \begin{cases} \mathfrak{p}(g)w & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $f \in V$, we have

$$f = \sum_{Hx \in H\backslash G} \pi(x^{-1})f_{f(x)}.$$

Proof. Let $g \in G$. Assume that $g \in Hy$, then

$$(\sum_{Hx \in H\backslash G} \pi(x^{-1})f_{f(x)})(g) = \sum_{Hx \in H\backslash G} (\pi(x^{-1})f_{f(x)})(g)$$

(4.2)
$$= \sum_{Hx \in H\backslash G} \alpha(g, x^{-1})f_{f(x)}(gx^{-1})$$

$$= \alpha(g, y^{-1})f_{f(y)}(gy^{-1}) = f(g).$$

The lemma follows. $\qquad\square$

Define another vector space $V' = \oplus_{Hx \in H\backslash G}W_x$, where $W_x = W$ as vector spaces for all x. Fix a set $\{x_i\}$ of representatives of the right cosets $H\backslash G$. Define a map $\pi' : G \to \mathrm{GL}(V')$ by

(4.3)
$$\pi'(g)((w_i)_{w_i \in W_{x_i}}) = (\frac{\alpha(g, x_{\theta(i)}^{-1})}{\alpha(x_i^{-1}, x_igx_{\theta(i)}^{-1})}\mathfrak{p}(x_igx_{\theta(i)}^{-1})w_{\theta(i)}).$$

Here $\theta(i)$ is the index such that $x_ig \in Hx_{\theta(i)}$. We define a map $F : V' \to V$ by

$$F((w_i)_{w_i \in W_{x_i}}) = \sum_i \pi(x_i^{-1})f_{w_i}.$$

Lemma 4.3. *With the above notation, (π', V') is a projective representation of G with associated multiplier α. Moreover, the map F is an isomorphism of projective representations.*

Proof. It suffices to show that F is an isomorphism of vector spaces and $\pi(g) \circ F = F \circ \pi'(g)$. It is clear that F is injective. It is also surjective by last lemma. Write $\Delta_i = \frac{\alpha(g, x_{\theta(i)}^{-1})}{\alpha(x_i^{-1}, x_igx_{\theta(i)}^{-1})}\mathfrak{p}(x_igx_{\theta(i)}^{-1})$. To show that $\pi(g) \circ F = F \circ \pi'(g)$, it suffices to show that

$$\sum_i \alpha(g, x_i^{-1})\pi(gx_i^{-1})f_{w_i} = \sum_i \pi(x_i^{-1})f_{\Delta_i w_{\theta(i)}}.$$

For any $y \in G$, assume that $ygx_j^{-1} \in H$ and $yx_k^{-1} \in H$. Then $\theta(k) = j$. Note that

(4.4)
$$(\sum_i \alpha(g, x_i^{-1})\pi(gx_i^{-1})f_{w_i})(y) = \sum_i \alpha(g, x_i^{-1})\alpha(y, gx_i^{-1})f_{w_i}(ygx_i^{-1})$$

$$= \alpha(g, x_j^{-1})\alpha(y, gx_j^{-1})\mathfrak{p}(ygx_j^{-1})w_j.$$

On the other hand,

$$
\begin{aligned}
(\sum_i \pi(x_i^{-1}) f_{\Delta_i w_{\theta(i)}})(y) &= \sum_i \alpha(y, x_i^{-1}) f_{\Delta_i w_{\theta(i)}}(yx_i^{-1}) \\
&= \alpha(y, x_k^{-1}) \mathfrak{p}(yx_k^{-1}) \Delta_k w_{\theta(k)} \\
&= \alpha(y, x_k^{-1}) \mathfrak{p}(yx_k^{-1}) \frac{\alpha(g, x_{\theta(k)}^{-1})}{\alpha(x_k^{-1}, x_k g x_{\theta(k)}^{-1})} \mathfrak{p}(x_k g x_{\theta(k)}^{-1}) w_{\theta(k)} \\
&= \alpha(y, x_k^{-1}) \mathfrak{p}(yx_k^{-1}) \frac{\alpha(g, x_j^{-1})}{\alpha(x_k^{-1}, x_k g x_j^{-1})} \mathfrak{p}(x_k g x_j^{-1}) w_j \\
&= \alpha(y, x_k^{-1}) \frac{\alpha(g, x_j^{-1})}{\alpha(x_k^{-1}, x_k g x_j^{-1})} \alpha(yx_k^{-1}, x_k g x_j^{-1}) \mathfrak{p}(ygx_j^{-1}) w_j \\
&= \frac{\alpha(g, x_j^{-1})}{\alpha(x_k^{-1}, x_k g x_j^{-1})} \alpha(y, gx_j^{-1}) \alpha(x_k^{-1}, x_k g x_j^{-1}) \mathfrak{p}(ygx_j^{-1}) w_j \\
&= \alpha(g, x_j^{-1}) \alpha(y, gx_j^{-1}) \mathfrak{p}(ygx_j^{-1}) w_j.
\end{aligned}
$$

(4.5)

The lemma follows. $\qquad\square$

Corollary 4.4. $\dim_{\mathbb{C}} \operatorname{Ind}_H^G W = [G : H] \dim_{\mathbb{C}} W.$

4.2. The characters of induced projective representations. Let α be a unitary multiplier. The isomorphism F allows us to compute the characters of the induced projective representations. First, we prove the following lemma.

Lemma 4.5. *Let $\chi_{\mathfrak{p}}$ be the character of $(\mathfrak{p}, W, \alpha)$. Fix an element $g \in G$. Let $r \in G$ such that $rgr^{-1} \in H$. If $s \in Hr$, then*

$$
\frac{\alpha(g, r^{-1})}{\alpha(r^{-1}, rgr^{-1})} \chi_{\mathfrak{p}}(rgr^{-1}) = \frac{\alpha(g, s^{-1})}{\alpha(s^{-1}, sgs^{-1})} \chi_{\mathfrak{p}}(sgs^{-1}).
$$

Proof. Write $s = hr$ for $h \in H$. Then

$$
\chi_{\mathfrak{p}}(sgs^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, rgr^{-1}) \alpha(hrgr^{-1}, h^{-1})} \chi_{\mathfrak{p}}(rgr^{-1}).
$$

To prove the lemma, it suffices to prove that

(4.6)
$$
\begin{aligned}
&\alpha(g, r^{-1}) \alpha(r^{-1}h^{-1}, hrgr^{-1}h^{-1}) \alpha(h, rgr^{-1}) \alpha(hrgr^{-1}, h^{-1}) \\
&= \alpha(g, r^{-1}h^{-1}) \alpha(h, h^{-1}) \alpha(r^{-1}, rgr^{-1}).
\end{aligned}
$$

Since $\alpha(g, r^{-1}) \alpha(gr^{-1}, h^{-1}) = \alpha(g, r^{-1}h^{-1}) \alpha(r^{-1}, h^{-1})$, it suffices to prove that

(4.7)
$$
\begin{aligned}
&\alpha(r^{-1}, h^{-1}) \alpha(r^{-1}h^{-1}, hrgr^{-1}h^{-1}) \alpha(h, rgr^{-1}) \alpha(hrgr^{-1}, h^{-1}) \\
&= \alpha(gr^{-1}, h^{-1}) \alpha(h, h^{-1}) \alpha(r^{-1}, rgr^{-1}).
\end{aligned}
$$

This follows from the following computation.

$$LHS = \alpha(r^{-1}, h^{-1})\alpha(h, rgr^{-1})\alpha(r^{-1}h^{-1}, hrgr^{-1}h^{-1})\alpha(hrgr^{-1}, h^{-1})$$

$$= \alpha(r^{-1}, h^{-1})\alpha(h, rgr^{-1})\alpha(r^{-1}h^{-1}, hrgr^{-1})\alpha(gr^{-1}, h^{-1})$$

(4.8)

$$= \alpha(r^{-1}, h^{-1})\alpha(r^{-1}h^{-1}, h)\alpha(r^{-1}, rgr^{-1})\alpha(gr^{-1}, h^{-1})$$

$$= \alpha(r^{-1}, e)\alpha(h^{-1}, h)\alpha(r^{-1}, rgr^{-1})\alpha(gr^{-1}, h^{-1}) = RHS.$$

The lemma follows. $\qquad\square$

Theorem 4.6. *Let α be a unitary multiplier of G. Let $(\mathfrak{p}, W, \alpha_H)$ be a projective representation of H with character $\chi_{\mathfrak{p}}$. Let (π, V, α) be the projective representation of G induced from $(\mathfrak{p}, W)$. If χ_π is the character of G, then*

$$\chi_\pi(g) = \sum_{\substack{r \in H \backslash G \\ rgr^{-1} \in H}} \frac{\alpha(g, r^{-1})}{\alpha(r^{-1}, rgr^{-1})}\chi_{\mathfrak{p}}(rgr^{-1}) = \frac{1}{|H|} \sum_{\substack{s \in G \\ sgs^{-1} \in H}} \frac{\alpha(g, s^{-1})}{\alpha(s^{-1}, sgs^{-1})}\chi_{\mathfrak{p}}(sgs^{-1}).$$

Proof. The second equality follows from last lemma. It suffices to show the first equality. The space $V = \oplus_{x \in H \backslash G} W_x$ and $\pi(g)$ permutes the subspaces W_x. By definition, $\pi(g)$ sends W_{x_i} to $W_{x_{\theta^{-1}(i)}}$, where $\theta^{-1}(i)$ is the index such that $x_{\theta^{-1}(i)}gx_i^{-1} \in H$. If $\theta^{-1}(i) \neq i$, then $\pi(g)|_{W_{x_i}}$ dose not contribute to the trace. Therefore,

$$\chi_\pi(g) = \sum_{\substack{r \in H \backslash G \\ rgr^{-1} \in H}} \mathrm{Tr}_{W_r}(\pi(g)|_{W_r})$$

(4.9)

$$= \sum_{\substack{r \in H \backslash G \\ rgr^{-1} \in H}} \frac{\alpha(g, r^{-1})}{\alpha(r^{-1}, rgr^{-1})}\chi_{\mathfrak{p}}(rgr^{-1}).$$

The theorem follows. $\qquad\square$

4.3. **Twists of projective representations.** Assume that H is a normal subgroup of G, then for any $g \in G$, $gHg^{-1} = H$.

Proposition 4.7. *Let α be a multiplier of G. Let $(\mathfrak{p}, W, \alpha_H)$ be a projective representation of H. For any $g \in G$, define $\mathfrak{p}^g : H \to \mathrm{GL}(W)$ by*

$$\mathfrak{p}^g(h) := \frac{\alpha(h, g^{-1})}{\alpha(g^{-1}, ghg^{-1})}\mathfrak{p}(ghg^{-1}) \text{ for all } g \in G, h \in H.$$

Then $(\mathfrak{p}^g, W)$ is a projective representation of H with associated multiplier α_H.

Proof. It suffices to show that $\mathfrak{p}^g(h)\mathfrak{p}^g(h') = \alpha(h, h')\mathfrak{p}^g(hh')$. By definition, this is equivalent to

(4.10)
$$\frac{\alpha(h, g^{-1})}{\alpha(g^{-1}, ghg^{-1})}\mathfrak{p}(ghg^{-1})\alpha(h', g^{-1})\alpha(g^{-1}, gh'g^{-1})\mathfrak{p}(gh'g^{-1})$$

$$= \alpha(h, h')\alpha(hh', g^{-1})\alpha(g^{-1}, ghh'g^{-1})\mathfrak{p}(ghh'g^{-1}).$$

Thus it suffices to show that

$$(4.11) \quad \frac{\alpha(h, g^{-1})}{\alpha(g^{-1}, ghg^{-1})} \alpha(h', g^{-1}) \alpha(g^{-1}, gh'g^{-1}) \alpha(ghg^{-1}, gh'g^{-1})$$
$$= \alpha(h, h') \alpha(hh', g^{-1}) \alpha(g^{-1}, ghh'g^{-1}).$$

Note that $\alpha(ghg^{-1}, gh'g^{-1}) \alpha(g^{-1}, ghh'g^{-1}) = \alpha(g^{-1}, ghg^{-1}) \alpha(hg^{-1}, gh'g^{-1})$, it suffices to show that

$$\alpha(h, g^{-1}) \alpha(h', g^{-1}) \alpha(hg^{-1}, gh'g^{-1}) = \alpha(h, h') \alpha(hh', g^{-1}) \alpha(g^{-1}, gh'g^{-1}).$$

This follows from the following computation.

$$(4.12) \quad \begin{aligned} LHS &= \alpha(h', g^{-1}) \alpha(h, g^{-1}) \alpha(hg^{-1}, gh'g^{-1}) \\ &= \alpha(h', g^{-1}) \alpha(h, h'g^{-1}) \alpha(g^{-1}, gh'g^{-1}) \\ &= \alpha(h, h') \alpha(hh', g^{-1}) \alpha(g^{-1}, gh'g^{-1}) = RHS. \end{aligned}$$

The proposition follows. $\qquad \square$

Proposition 4.8. *With the notation as above,*
  (1) *if $\mathfrak{p}$ is irreducible, then $\mathfrak{p}^g$ is irreducible;*
  (2) *if (π, V, α) is the projective representation of G induced from $(\mathfrak{p}, W)$, then $\pi|_H \cong \oplus_{x \in H \backslash G} \mathfrak{p}^x$.*

Proof. For the first claim, we may assume that $\mathfrak{p}$ is unitary. Let χ^g be the character of $\mathfrak{p}^g$. Then $\chi^g(h) = c(h) \chi_{\mathfrak{p}}(ghg^{-1})$ with $c(h)$ a root of unity. Therefore

$$(\chi^g, \chi^g) = \sum_{h \in H} \chi^g(h) \overline{\chi^g(h)} = \sum_{h \in H} \chi_{\mathfrak{p}}(ghg^{-1}) \overline{\chi_{\mathfrak{p}}(ghg^{-1})} = (\chi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) = 1.$$

Thus $\mathfrak{p}^g$ is irreducible.

The second claim follows from the fact that $W_x \cong \mathfrak{p}^x$. $\qquad \square$

5. Twisted group algebras

5.1. **The structure of $\mathbb{C}[G]_\alpha$.** Since $\mathbb{C}$ is algebraically closed, each skew field or field of finite degree over $\mathbb{C}$ is equal to $\mathbb{C}$. Thus the twisted group algebra $\mathbb{C}[G]_\alpha$ is a product of matrix algebras $M_{n_i}(\mathbb{C})$. Let $\pi_i : G \to \mathrm{GL}(W_i)$ be the distinct irreducible projective representations of G with associated multiplier α ($i = 1, \ldots, l = l_\alpha$). Let $n_i = \dim W_i$. Then the ring $\mathrm{End}_{\mathbb{C}}(W_i)$ of endomorphisms of W_i is isomorphic to $M_{n_i}(\mathbb{C})$. The map $\pi_i : G \to \mathrm{GL}(W_i)$ extends by linearity to an algebra homomorphism $\Pi_i : \mathbb{C}[G]_\alpha \to \mathrm{End}(W_i)$. We thus obtain a homomorphism

$$\Pi : \mathbb{C}[G]_\alpha \to \prod_{i=1}^{l} \mathrm{End}(W_i) \cong \prod_{i=1}^{l} M_{n_i}(\mathbb{C}).$$

Proposition 5.1. *The homomorphism Π defined above is an isomorphism of $\mathbb{C}$-algebras.*

Proof. If $[\alpha] = [\beta]$, then $\mathbb{C}[G]_\alpha \cong \mathbb{C}[G]_\beta$ by Lemma 2.10. We may assume that α is unitary. Since both $\mathbb{C}[G]_\alpha$ and $\prod_{i=1}^{l} M_{n_i}(\mathbb{C})$ have the same dimension, it suffices to show that Π is surjective. Suppose otherwise, there would exist a nonzero linear form on $\prod_{i=1}^{l} M_{n_i}(\mathbb{C})$ vanishing on the image of Π. This would induce a nontrivial relation on the characters of

the projective representations π_i, which contradicts to the orthogonality formulas. Thus the claim follows. □

Let C be the set of conjugacy classes of α-elements of G. For each $c \in C$, fix an element $g_c \in c$. Set

$$e_c = \sum_{h \in G} a_h a_{g_c} a_h^{-1} = \sum_{h \in G} \frac{\alpha(h, g_c)\alpha(hg_c, h^{-1})}{\alpha(h, h^{-1})} a_{hg_c h^{-1}}.$$

It is easy to see that e_c is an element of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$, the center of the twisted group algebra. By the above proposition, $\dim_{\mathbb{C}}(\mathrm{Cent}\,.\mathbb{C}[G]_\alpha) = l$. Thus $(e_c)_{c \in C}$ form a basis of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$.

Remark 5.2. The definition of e_c depends on the choice of the fixed element $g_c \in c$. Let $g_c' = sg_c s^{-1} \in c$ be another element and define

$$e_c' = \sum_{h \in G} a_h a_{g_c'} a_h^{-1} = \sum_{h \in G} \frac{\alpha(h, g_c')\alpha(hg_c', h^{-1})}{\alpha(h, h^{-1})} a_{hg_c' h^{-1}}.$$

Then $e_c = \frac{\alpha(s, g_c s^{-1})}{\alpha(g_c s^{-1}, s)} e_c'$.

Indeed, let $g = g_c$, to see this, it suffices to prove

$$\frac{\alpha(hs, g)\alpha(hsg, h^{-1})\alpha(gs^{-1}, s)}{\alpha(hs, (hs)^{-1})} = \frac{\alpha(h, sgs^{-1})\alpha(hsgs^{-1}, h^{-1})\alpha(s, gs^{-1})}{\alpha(h, h^{-1})}.$$

This follows from

$$\begin{aligned}
&\alpha(h, sgs^{-1})\alpha(hsgs^{-1}, h^{-1})\alpha(s, gs^{-1})\alpha(hs, (hs)^{-1}) \\
=&\alpha(h, s)\alpha(hs, gs^{-1})\alpha(hsgs^{-1}, h^{-1})\alpha(hs, (hs)^{-1}) \\
=&\alpha(h, s)\alpha(hs, gs^{-1}h^{-1})\alpha(gs^{-1}, h^{-1})\alpha(hs, (hs)^{-1}) \\
=&\alpha(hs, gs^{-1}h^{-1})\alpha(gs^{-1}, h^{-1})\alpha(h, h^{-1})\alpha(s, s^{-1}h^{-1}) \\
=&\alpha(hs, gs^{-1}h^{-1})\alpha(h, h^{-1})\alpha(gs^{-1}, s)\alpha(g, s^{-1}h^{-1}).
\end{aligned}$$ (5.1)

In particular, in the case α is unitary, the difference between e_c and e_c' is given by a root of unity.

Proposition 5.3. *The homomorphism* Π_i *maps the* $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ *into the set of homotheties of* W_i *and defines an algebra homomorphism*

$$\omega_i : \mathrm{Cent}\,.\mathbb{C}[G]_\alpha \to \mathbb{C}.$$

If α *is unitary,* $f = \sum_{g \in G} k_g a_g$ *is an element of* $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$, *then*

$$\omega_i(f) = \frac{1}{n_i} \mathrm{Tr}_{W_i}(\Pi_i(f)) = \frac{1}{n_i} \sum_{g \in G} k_g \chi_i(g).$$

Moreover, the family $(\omega_i)_{1 \le i \le l}$ *defines an isomorphism of* $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$ *onto the algebra* $\mathbb{C}^l$.

Proof. The claims are clear. □

5.2. **Degrees of irreducible projective representations.** Let (π, V, α) be a unitary projective representation of G with character χ. Note that our α is unitary, therefore every eigenvalue of $\pi(g)$ is a root of unity. In particular, it is an algebraic integer. Thus the value $\chi(g)$, which is the sum of the eigenvalues of $\pi(g)$, is also an algebraic integer.

Lemma 5.4. *Let $f = \sum_{g \in G} k_g a_g$ be an element of $\mathrm{Cent}.\mathbb{C}[G]_\alpha$ such that k_g's are algebraic integers. Then f is integral over $\mathbb{Z}$. (Note that this makes sense since $\mathrm{Cent}.\mathbb{C}[G]_\alpha$ is a commutative ring.)*

Proof. By Remark 5.2, we may write $f = \sum_{c \in C} k_c e_c$ for some algebraic integers k_c. To show that f is integral over $\mathbb{Z}$, it suffices to show that each e_c is integral over $\mathbb{Z}$. Let $\mathcal{O} = \mathbb{Z}[\mathrm{Im}(\alpha)]$. It is contained in the ring of integers of the field $\mathbb{Q}(\mathrm{Im}(\alpha))$ and thus is finitely generated over $\mathbb{Z}$. Note that $e_c e_d$ is a linear combination with $\mathcal{O}$-coefficients of the e_c's, the subgroup $R = \oplus_{c \in C} \mathcal{O} \cdot e_c$ is a subring of $\mathrm{Cent}.\mathbb{C}[G]_\alpha$ and it is finitely generated over $\mathbb{Z}$. Every element in R is integral over $\mathbb{Z}$. The claim follows. $\qquad\square$

Lemma 5.5. *Let (π_i, W_i, α) be an irreducible unitary projective representation of G with degree n_i and character χ_i. Let $f = \sum_{g \in G} k_g a_g$ be an element of $\mathrm{Cent}.\mathbb{C}[G]_\alpha$ such that k_g's are algebraic integers. Then the number $\frac{1}{n_i} \sum_{g \in G} k_g \chi_i(g)$ is an algebraic integer.*

Proof. This number is the image of f under the homomorphism

$$\omega_i : \mathrm{Cent}.\mathbb{C}[G]_\alpha \to \mathbb{C}.$$

As f is integral over $\mathbb{Z}$, the same is true for its image under ω_i. $\qquad\square$

Theorem 5.6. *The degrees of the irreducible projective representations of G divide the order of G.*

Proof. It suffices to prove this for unitary irreducible projective representations. Let χ be the character of such a projective representation with multiplier α. First, we show that the element $\sum_{g \in G} \alpha(g, g^{-1})^{-1} \chi(g^{-1}) a_g$ is an element of $\mathrm{Cent}.\mathbb{C}[G]_\alpha$. It suffices to show that $a_h(\sum_{g \in G} \alpha(g, g^{-1})^{-1} \chi(g^{-1}) a_g) = (\sum_{g \in G} \alpha(g, g^{-1})^{-1} \chi(g^{-1}) a_g) a_h$ for any $h \in G$. This is equivalent to

$$\alpha(hgh^{-1}, hg^{-1}h^{-1})^{-1} \chi(hg^{-1}h^{-1}) \alpha(hgh^{-1}, h) = \alpha(g, g^{-1})^{-1} \chi(g^{-1}) \alpha(h, g)$$

$$(5.2) \qquad \Leftrightarrow \alpha(hgh^{-1}, hg^{-1}h^{-1}) = \frac{\alpha(hgh^{-1}, h) \alpha(h, h^{-1}) \alpha(g, g^{-1})}{\alpha(h, g^{-1}h^{-1}) \alpha(g^{-1}, h^{-1}) \alpha(h, g)}$$

$$\Leftrightarrow \alpha(hgh^{-1}, h) \alpha(h, gh^{-1}) \alpha(g, h^{-1}) = \alpha(h, h^{-1}) \alpha(h, g) \qquad \text{(equation (3.4))},$$

which is easy to see since α is a multiplier.

We may apply the above result to the element $\sum_{g \in G} \alpha(g, g^{-1})^{-1} \chi(g^{-1}) a_g$. The number

$$\frac{1}{n_i} \sum_{g \in G} k_g \chi_i(g) = \frac{1}{n_i} \sum_{g \in G} \alpha(g, g^{-1})^{-1} \chi(g^{-1}) \chi(g) = \frac{|G|}{n_i} (\chi, \chi) = \frac{|G|}{n_i}$$

is an algebraic integer. Therefore $n_i \mid |G|$. The claim follows. $\qquad\square$

Corollary 5.7. *Let G be a finite group of order N. Let l^c be the number of conjugacy classes of G. If the equation*

$$N = n_1^2 + \cdots + n_m^2$$

has no solution with $m \in \mathbb{Z}_{\geq 1}$, $m \leq l^c$, $n_i \in \mathbb{Z}_{\geq 2}$ *and* $n_i \mid N$ $(1 \leq i \leq m)$, *then* $H^2(G, \mathbb{C}^{\times}) = 0$.

Proof. This is a generalization of Corollary 3.12. The proof is similar. $\qquad \square$

If G is a group such that $|G| = pq$, where p, q are distinct prime numbers, it is easy to see that $H^2(G, \mathbb{C}^{\times}) = 0$ by the corollary.

5.3. Frobenius reciprocity.
Fix a unitary 2-cocycle $\alpha \in Z^2(G, \mathbb{C}^{\times})$. Let H be a subgroup of G and α_H the restriction of α on $H \times H$. If (W, α_H) is a projective representation of H, we may consider W as a $\mathbb{C}[H]_{\alpha_H}$-module. Let $W' = \mathbb{C}[G]_{\alpha} \otimes_{\mathbb{C}[H]_{\alpha_H}} W$ be the $\mathbb{C}[G]_{\alpha}$-module obtained by scalar extension from $\mathbb{C}[H]_{\alpha_H}$ to $\mathbb{C}[G]_{\alpha}$.

Proposition 5.8. *Let* $V = \mathrm{Ind}_H^G(W)$. *The injection* $W \to V$ *extends by linearity to a* $\mathbb{C}[G]_{\alpha}$-*homomorphism* $i : W' \to V$. *The homomorphism* i *is an isomorphism of* $\mathbb{C}[G]_{\alpha}$-*modules.*

Proof. This is a consequence of the fact that the elements $x \in H \backslash G$ form a basis of $\mathbb{C}[G]_{\alpha}$ as a $\mathbb{C}[H]_{\alpha_H}$-module and the decomposition $V = \oplus_x W_x$. $\qquad \square$

Corollary 5.9. (1) *If* V *is induced from* W *and if* E *is a* $\mathbb{C}[G]_{\alpha}$-*module, then we have a canonical isomorphism*
$$\mathrm{Hom}_H(W, E) = \mathrm{Hom}_G(V, E).$$

(2) *Induction is transitive in the following sense. If* G *is a subgroup of a finite group* L *and* α *is the restriction of a 2-cocycle in* $Z^2(L, \mathbb{C}^{\times})$, *then*
$$\mathrm{Ind}_G^L(\mathrm{Ind}_H^G(W)) \cong \mathrm{Ind}_H^L(W).$$

Proof. The claims follow from the properties of tensor products. $\qquad \square$

If f is an α_H-class function on H, consider the function f' on G defined by
$$f'(g) = \frac{1}{|H|} \sum_{\substack{s \in G \\ sgs^{-1} \in H}} \frac{\alpha(g, s^{-1})}{\alpha(s^{-1}, sgs^{-1})} f(sgs^{-1}).$$

We say that f' is induced from f and denote it by either $\mathrm{Ind}_H^G(f)$ or $\mathrm{Ind}(f)$.

Lemma 5.10. (1) *The function* $\mathrm{Ind}(f)$ *is an* α-*class function on* G.
(2) *If* f *is the character of a projective representation* W *of* H, *then* $\mathrm{Ind}(f)$ *is the character of the induced projective representation* $\mathrm{Ind}(W)$ *of* G.

Proof. The second claim is Theorem 4.6. The first claim follows from the fact that each α-class function is a linear combination of characters. $\qquad \square$

If V_1 and V_2 are two $\mathbb{C}[G]_{\alpha}$-modules, we set
$$(V_1, V_2)_G = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[G]_{\alpha}}(V_1, V_2)).$$

Proposition 5.11. (1) *If* V_i *is a unitary projective representation of* G *with character* $\chi_i (i = 1, 2)$, *then*
$$(\chi_1, \chi_2) = (V_1, V_2)_G.$$

(2) *If ψ is an α_H-class function on H and ϕ is an α-class function on G, then*

$$(\psi, \operatorname{Res} \phi)_H = (\operatorname{Ind} \psi, \phi)_G.$$

Proof. (1) If V_1 and V_2 are irreducible, the claim follows from the orthogonality formulas for characters. In the general case, we decompose V_1 and V_2 into direct sums of irreducible modules and it is easy to see that the claim follows.

(2) Since α is unitary, each α-class function is a linear combination of characters of projective representations, we may assume that ψ is the character of a $\mathbb{C}[H]_\alpha$-module W and ϕ is the character of a $\mathbb{C}[G]_\alpha$-module E. Then it suffices to show that

$$(W, \operatorname{Res} E)_H = (\operatorname{Ind} W, E)_G.$$

This is the same as

$$\dim(\operatorname{Hom}_H(W, \operatorname{Res} E)) = \dim(\operatorname{Hom}_G(\operatorname{Ind} W, E)),$$

which follows from Corollary 5.9. $\square$

5.4. A criteria for the irreducibility of the induced projective representations.
Fix a multiplier α of G. Let H and L be two subgroups of G. Let $\mathfrak{p} : H \to \operatorname{GL}(W)$ be a projective representation of H with multiplier α_H and $V = \operatorname{Ind}_H^G(W)$ be the corresponding induced projective representation of G. In the following, we determine the structure of the restriction $\operatorname{Res}_L^G(V)$ of V to L.

Let S be a set of representatives for the double cosets $L\backslash G/H$. For $s \in S$, let $H_s = s^{-1}Hs \cap L$, which is a subgroup of L. Set

$$\mathfrak{p}^s(x) = \frac{\alpha(x, s^{-1})}{\alpha(s^{-1}, sxs^{-1})}\mathfrak{p}(sxs^{-1}) \text{ for } x \in H_s,$$

we thus obtain a projective representation $\mathfrak{p}^s : H_s \to \operatorname{GL}(W)$. (See for example the proof of Proposition 4.7.) Denote this projective representation by $(\mathfrak{p}^s, W_s)$.

Proposition 5.12. *The representation $\operatorname{Res}_L^G(\operatorname{Ind}_H^G(W))$ is isomorphic to the direct sum of the projective representations $\operatorname{Ind}_{H_s}^L(W_s)$ for $s \in S$.*

Proof. Write $V = \oplus_{x \in H\backslash G}W_x$ with $W_x = W$. Let $V(s)$ be the subspace of V generated by $\pi(x)W$ with $x \in LsH$. Then $V = \oplus_{s \in S}V(s)$. It is easy to see that $V(s)$ is stable under L. It suffices to show that $V(s)$ is L-isomorphic to $\operatorname{Ind}_{H_s}^L(W_s)$. This follows from the identity $V(s) = \oplus_{x \in L/H_s}\pi(x)(W_s)$. The proposition follows. $\square$

We apply the above discussion to the special case $L = H$. For $g \in G$, we denote by H_g the subgroup $g^{-1}Hg \cap H$. The projective representation $\mathfrak{p}$ of H defines a projective representation $\operatorname{Res}_{H_g}^H \mathfrak{p}$ by restriction to H_g.

Proposition 5.13. *In order that the induced projective representation $V = \operatorname{Ind}_H^G W$ be irreducible, it is necessary and sufficient that the following two conditions be satisfied:*

(1) *W is irreducible.*
(2) *For each $s \in G - H$, the two representations $\mathfrak{p}^s$ and $\operatorname{Res}_{H_s}^H \mathfrak{p}$ of H_s are disjoint, i.e., $(\mathfrak{p}^s, \operatorname{Res}_{H_s}^H \mathfrak{p})_{H_s} = 0$.*

Proof. We may assume that α is unitary. In order that V be irreducible, it is the same that $(V, V)_G = 1$. We have

$$
\begin{aligned}
(V, V)_G &= (W, \operatorname{Res}_H^G V)_H \\
&= (W, \oplus_{s \in H \backslash G / H} \operatorname{Ind}_{H_s}^H (\mathfrak{p}^s))_H \\
&= \sum_{s \in H \backslash G / H} (W, \operatorname{Ind}_{H_s}^H (\mathfrak{p}^s))_H \\
&= \sum_{s \in H \backslash G / H} (\operatorname{Res}_{H_s}^H \mathfrak{p}, \mathfrak{p}^s)_{H_s}.
\end{aligned}
$$

(5.3)

For $s = 1$, we have $d_s := (\operatorname{Res}_{H_s}^H \mathfrak{p}, \mathfrak{p}^s)_{H_s} = (\mathfrak{p}, \mathfrak{p})_H \geq 1$. In order that $(V, V)_G = 1$, it is necessary and sufficient that $d_1 = 1$ and $d_s = 0$ for $s \neq 1$. These are exactly conditions (1) and (2). $\qquad\square$

Corollary 5.14. *Suppose that H is a normal subgroup of G. In order that $\operatorname{Ind}_H^G \mathfrak{p}$ be irreducible, it is necessary and sufficient that $\mathfrak{p}$ is irreducible and is not isomorphic to any of its twists $\mathfrak{p}^g$ for $g \notin H$.*

Proof. This is clear from the proposition since $H_s = H$ for all $s \in G$. $\qquad\square$

Proposition 5.15. *Let A be a normal subgroup of G and $\pi : G \to \operatorname{GL}(V)$ be an irreducible projective representation of G. Then*

(1) *either there exists a subgroup H of G, unequal to G and containing A, and an irreducible projective representation $\mathfrak{p}$ of H such that π is induced from $\mathfrak{p}$;*
(2) *or else the restriction $\operatorname{Res}_A^G \pi$ is isotypic, i.e., it is a direct sum of isomorphic projective representations of A.*

Proof. Let $V = \oplus V_i$ be the canonical decomposition of the representation $\pi|_A$ into a direct sum of isotypic representations. For $g \in G$, $\pi(g)$ permutes the V_i. Since V is irreducible, G permutes them transitively. Let V_{i_0} be one of these. If $V_{i_0} = V$, then we are in case (2). Otherwise, let H be the subgroup of G consisting of those $g \in G$ such that $\pi(g) V_{i_0} = V_{i_0}$. It is easy to see that $A \subset H$ and $H \neq G$. Moreover, V_{i_0} is an irreducible projective representation of H and $\operatorname{Ind}_H^G V_{i_0}$ is isomorphic to V. (Indeed, $\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_H^G V_{i_0}, V) = \dim_{\mathbb{C}} \operatorname{Hom}_H(V_{i_0}, V) > 0$. Since for any $s \in G - H$, $\operatorname{Res}_A^H V_{i_0}$ and $\operatorname{Res}_A^H V_{i_0}^s$ are disjoint, $\operatorname{Ind}_H^G V_{i_0}$ is irreducible.) This is case (1). The proposition follows. $\qquad\square$

5.5. Degrees of irreducible projective representations revisit.

Let α be a multiplier of G. Let A be a normal subgroup of G. Let $\mathfrak{p} : A \to \operatorname{GL}(W)$ be a projective representation of A with multiplier α. Define

$$
I_{\mathfrak{p}} = \{g \in G : \ \mathfrak{p}^g \cong \mathfrak{p}\}.
$$

It is easy to see that $I_{\mathfrak{p}}$ is a subgroup of G and A is a normal subgroup of $I_{\mathfrak{p}}$.

Lemma 5.16. *For any elements $g_1, g_2, h \in G$, we have*

(5.4)
$$
\begin{aligned}
&\alpha(h, (g_1 g_2)^{-1}) \alpha(g_2^{-1}, g_2 h g_2^{-1}) \alpha(g_1^{-1}, g_1 g_2 h (g_1 g_2)^{-1}) \\
&= \alpha(h, g_2^{-1}) \alpha(g_2 h g_2^{-1}, g_1^{-1}) \alpha((g_1 g_2)^{-1}, g_1 g_2 h (g_1 g_2)^{-1}).
\end{aligned}
$$

Proof. By definition,
$$\alpha(h, g_2^{-1}g_1^{-1})\alpha(g_2^{-1}, g_1^{-1}) = \alpha(h, g_2^{-1})\alpha(hg_2^{-1}, g_1^{-1}),$$
$$\alpha(g_2^{-1}, g_1^{-1})\alpha((g_1g_2)^{-1}, g_1g_2h(g_1g_2)^{-1}) = \alpha(g_2^{-1}, g_2hg_2^{-1}g_1^{-1})\alpha(g_1^{-1}, g_1g_2h(g_1g_2)^{-1}),$$
and
$$\alpha(g_2^{-1}, g_2hg_2^{-1})\alpha(hg_2^{-1}, g_1^{-1}) = \alpha(g_2hg_2^{-1}, g_1^{-1})\alpha(g_2^{-1}, g_2hg_2^{-1}g_1^{-1}),$$
the lemma follows.   □

Lemma 5.17. *Let* $(\mathfrak{p}, W, \alpha)$ *be an irreducible projective representation of* A. *One can extend* W *to a projective representation* $\mathfrak{p}'$ *of* $I_\mathfrak{p}$ *with some multiplier* β *such that*

(1) $\mathfrak{p}'(g)\mathfrak{p}(h)\mathfrak{p}'(g)^{-1} = \mathfrak{p}^g(h)$ *for all* $g \in I_p$ *and* $h \in A$.
(2) $\mathfrak{p}'(h) = \mathfrak{p}(h)$ *for all* $h \in A$.
(3) $\mathfrak{p}(h)\mathfrak{p}'(g) = \alpha(h, g)\mathfrak{p}'(hg)$.

Proof. For any $g \in I_\mathfrak{p}$, there exists a matrix $\rho(g)$ such that
$$\rho(g)\mathfrak{p}(h)\rho(g)^{-1} = \mathfrak{p}^g(h) \text{ for all } h \in A.$$
Note that $\rho(g_1g_2)\mathfrak{p}(h)\rho(g_1g_2)^{-1} = \rho(g_1)\rho(g_2)\mathfrak{p}(h)(\rho(g_1)\rho(g_2))^{-1}$ (which follows from last lemma) and $\mathfrak{p}$ is irreducible, by Schur's Lemma, there exists an element $\gamma(g_1, g_2) \in \mathbb{C}^\times$ such that $\rho(g_1)\rho(g_2) = \gamma(g_1, g_2)\rho(g_1g_2)$. Let $\{x_i\}$ be a set of right coset representatives of A in $I_\mathfrak{p}$. Define
$$\mathfrak{p}'(hx_i) = \alpha(h, x_i)^{-1}\mathfrak{p}(h)\rho(x_i), \quad \mathfrak{p}'(h) = \mathfrak{p}(h),$$
for all i and $h \in A$. It is easy to check that $(\mathfrak{p}', W)$ is a projective representation of $I_\mathfrak{p}$ that satisfies the properties.   □

Remark 5.18. With the notation as in the lemma, we have $\mathfrak{p}'(g)\mathfrak{p}(h) = \alpha(g, h)\mathfrak{p}'(gh)$. Indeed, since

(5.5)
$$\alpha(gh, g^{-1})\alpha(ghg^{-1}, g) = \alpha(g^{-1}, g)$$
$$= \alpha(g, g^{-1}) = \alpha(g, hg^{-1})\alpha(g^{-1}, ghg^{-1}),$$

we have
$$\alpha(h, g^{-1})\alpha(ghg^{-1}, g) = \alpha(g, h)\alpha(g^{-1}, ghg^{-1}).$$
Thus

(5.6)
$$\begin{aligned}
\mathfrak{p}'(g)\mathfrak{p}(h) &= \mathfrak{p}^g(h)\mathfrak{p}'(g) \\
&= \alpha(h, g^{-1})\alpha(g^{-1}, ghg^{-1})^{-1}\mathfrak{p}(ghg^{-1})\mathfrak{p}'(g) \\
&= \alpha(h, g^{-1})\alpha(g^{-1}, ghg^{-1})^{-1}\alpha(ghg^{-1}, g)\mathfrak{p}'(gh) \\
&= \alpha(g, h)\mathfrak{p}'(gh).
\end{aligned}$$

Therefore, the following equations hold:
$$\alpha(g, h) = \beta(g, h), \quad \alpha(h, g) = \beta(h, g),$$
where $g \in I_\mathfrak{p}$ and $h \in A$. Thus $\alpha\beta^{-1}$ is a well defined multiplier of the group $I_\mathfrak{p}/A$.

Let $Q_\mathfrak{p}$ be the quotient group $I_\mathfrak{p}/A$. Let $\mathfrak{q}$ be an irreducible projective representation of $Q_\mathfrak{p}$ with associated multiplier $\alpha\beta^{-1}$. We may also consider $\mathfrak{q}$ as a projective representation of $I_\mathfrak{p}$ via the natural map $I_\mathfrak{p} \to Q_\mathfrak{p}$.

Lemma 5.19. *The projective representation* $\mathrm{Ind}_{I_{\mathfrak{p}}}^{G}(\mathfrak{p}' \otimes \mathfrak{q})$ *is an irreducible projective representation of* G *with associated multiplier* α.

Proof. Note that the representation $\mathfrak{p}' \otimes \mathfrak{q}$ is an irreducible projective representation of $I_{\mathfrak{p}}$. Let $g \in G - I_{\mathfrak{p}}$. Then $(\mathfrak{p}' \otimes \mathfrak{q})^{g}|_{A}$ and $(\mathfrak{p}' \otimes \mathfrak{q})|_{A}$ are disjoint by the definition of $I_{\mathfrak{p}}$. By Proposition 5.13, the lemma follows. $\qquad\square$

Theorem 5.20. *Let A be a normal subgroup of G. Let d_A be the least common multiple of the degrees of the irreducible projective representations of A. (Note that $d_A \mid |A|$.) Then the degree of each irreducible projective representation π of G divides the number $d_A \cdot (G : A)$.*

Proof. We prove this theorem by induction on the order of G. In case (1) of Proposition 5.15, by induction, the degree of $\mathfrak{p}$ divides $d_A \cdot (H : A)$. Therefore, the degree of π divides $(G : H) d_A \cdot (H : A) = d_A \cdot (G : A)$.

In case (2) of Proposition 5.15, assume that $V|_A = W^{\oplus k}$ for an irreducible projective representation W of A. In the above discussion, take $\mathfrak{p} = W$. Then $I_{\mathfrak{p}} = G$, i.e., any twist of $\mathfrak{p}$ is isomorphic to $\mathfrak{p}$. We may extend W to a projective representation $\mathfrak{p}'$ of G with associated multiplier β. Define $W' = \mathrm{Hom}_A(W, V) = \{f : W \to V \mid f(\mathfrak{p}(a)w) = \pi(a)f(w)\}$. We define an action of G on W' via the equation

$$(\mathfrak{q}(g)f)(w) = \pi(g)f(\mathfrak{p}'(g)^{-1}w).$$

(1) By Lemma 5.17, $\mathfrak{p}'(g)^{-1}\mathfrak{p}(a) = \frac{\alpha(a,g)}{\alpha(g,g^{-1}ag)}\mathfrak{p}(g^{-1}ag)\mathfrak{p}'(g)^{-1}$. One has

$$\begin{aligned}
(\mathfrak{q}(g)f)(\mathfrak{p}(a)w) &= \pi(g)f(\mathfrak{p}'(g)^{-1}\mathfrak{p}(a)w) \\
&= \pi(g)\frac{\alpha(a,g)}{\alpha(g,g^{-1}ag)}f(\mathfrak{p}(g^{-1}ag)\mathfrak{p}'(g)^{-1}w) \\
&= \pi(g)\frac{\alpha(a,g)}{\alpha(g,g^{-1}ag)}\pi(g^{-1}ag)f(\mathfrak{p}'(g)^{-1}w) \\
&= \pi(a)\pi(g)f(\mathfrak{p}'(g)^{-1}w) = \pi(a)(\mathfrak{q}(g)f)(w).
\end{aligned}$$

(5.7)

Thus $\mathfrak{q}(g)f \in W'$.

(2) For any $g_1, g_2 \in G$,

$$\begin{aligned}
(\mathfrak{q}(g_1g_2)f)(w) &= \pi(g_1g_2)f(\mathfrak{p}'(g_1g_2)^{-1}w) \\
&= \alpha(g_1,g_2)^{-1}\beta(g_1,g_2)\pi(g_1)\pi(g_2)f(\mathfrak{p}'(g_2)^{-1}\mathfrak{p}'(g_1)^{-1}w) \\
&= \alpha(g_1,g_2)^{-1}\beta(g_1,g_2)(\mathfrak{q}(g_1)\mathfrak{q}(g_2)f)(w).
\end{aligned}$$

(5.8)

Thus $\mathfrak{q}$ is a projective representation of G with multiplier $\alpha\beta^{-1}$.

Consider the natural map

$$W \otimes_{\mathbb{C}} W' \to V,$$

it is easy to check that it is an isomorphism of projective G-representations. Furthermore, Since V is irreducible, W' is also irreducible as a projective representation of G. On the other hand, if $g \in A$, then $\mathfrak{q}(g)$ acts as scalar. Thus W' has a structure as an irreducible projective representation of G/A. Therefore, $\deg W' \mid (G : A)$ and $\deg V \mid d_A(G : A)$. The theorem follows. $\qquad\square$

The same argument proves the following result.

Theorem 5.21. *Let α be a multiplier of G. Let A be a normal subgroup of G. Let d_A^α be the least common multiple of the degrees of the irreducible projective representations of A with associated multiplier α. Then the degree of each irreducible projective representation π of G with multiplier α divides the number $d_A^\alpha \cdot (G : A)$.*

In particular, if $\alpha = 1$ and A is an abelian normal subgroup of G, then the degree of each irreducible representation of G divides the number $(G : A)$.

Corollary 5.22. *Let A be a cyclic normal subgroup of G. Then the degree of each irreducible projective representation π of G divides the number $(G : A)$.*

Proof. Since A is cyclic, $H^2(G, \mathbb{C}^\times) = 0$. Therefore every irreducible projective representation of A is equivalent to an linear irreducible representation, which has degree 1. Thus $d_A = 1$ and the claim follows. $\qquad\square$

This result gives us a stronger version of Corollary 5.7 and has useful applications. We explain the idea in the following simple but nontrivial example.

Example 5.23. Let $G = D_{2m}$ be the dihedral group of order $2m$. Let C_m be the normal subgroup of G generated by an element of order m. By the above corollary, the degree of each irreducible projective representation of G divides 2. By Corollary 5.7, we obtain the fact that $H^2(D_{2m}, \mathbb{C}^\times) = 0$ if m is odd.

Assume now that m is even. Let α be a multiplier of D_{2m} such that $[\alpha]$ is nontrivial. (For example, $m = 4$, $H^2(D_8, \mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$.) Then every irreducible projective representation of D_{2m} with multiplier α has degree 2 and there are $m/2$ of them up to isomorphism. (Note that in this case the number of conjugacy classes of G is $m/2 + 3$.) By the argument in Theorem 5.20, all these irreducible projective representations are induced from one-dimensional projective representations of C_m with multiplier $\alpha|_{C_m}$.

5.6. **On abelian groups.** In this subsection, we assume that G is abelian. In this case, one can say more about the degrees. First, we have the following result.

Proposition 5.24. *Let G be a finite abelian group and α a fixed multiplier of G. Then all the irreducible projective representations of G with multiplier α have the same degree. We denote this number d_G^α.*

Proof. One needs only the results in Section 3 to prove this. We may assume that α is unitary. Let π_i be an irreducible projective representation of G with multiplier α and character χ_i ($i = 1, 2$). We claim that there exists a one-dimensional linear representation* $\chi : G \to \mathbb{C}^\times$, such that $\pi_2 \cong \chi \otimes \pi_1$.

Indeed, let $\bar\pi_2$ be the projective representation of G defined by $\bar\pi_2(g) = \overline{\pi_2(g)}$. Here $\bar x$ is the complex conjugation of x. Then the associated multiplier of $\bar\pi_2$ is $\bar\alpha = \alpha^{-1}$ since α is unitary. The character of $\bar\pi_2$ is $\bar\chi_2$. Consider the projective representation $\pi_1 \otimes \bar\pi_2$. The associated multiplier is $\alpha \cdot \alpha^{-1} = 1$. Because G is abelian, there exists a one-dimensional linear representation $\tau : G \to \mathbb{C}^\times$, such that $\dim_\mathbb{C} \mathrm{Hom}_G(\tau, \pi_1 \otimes \bar\pi_2) \geq 1$, i.e., the number

$$\sum_{g \in G} \tau(g) \overline{\chi_1(g)} \overline{\bar\chi_2(g)}$$

---

*We do not use the word *character* to avoid the confusion with the characters of projective representations.

is a positive integer. Thus

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\pi_2, \bar{\tau} \otimes \pi_1) = \sum_{g \in G} \chi_2(g) \overline{\chi_1(g) \bar{\tau}(g)}$$

is a positive integer. Moreover, both π_2 and $\bar{\tau} \otimes \pi_1$ are irreducible and thus they are isomorphic. The claim follows. $\qquad \square$

Remark 5.25. From the proof of the proposition, one sees that each one-dimensional linear representation appears at most once in the space $\pi_1 \otimes \bar{\pi}_2$. The G-representation $\pi_1 \otimes \bar{\pi}_2$ is a direct sum of $(d_G^\alpha)^2$ distinct one-dimensional linear representations. In particular, $d_G^\alpha \le \sqrt{|G|}$, which is also a consequence of Corollary 3.11.

In the following, we describe the number d_G^α more precisely. Let α be a multiplier of group G. Let $A \subset G$ be a subgroup. We say that A is α-*symmetric* if $\alpha(a, b) = \alpha(b, a)$ for any $a, b \in A$.

Lemma 5.26. *If G is abelian and α-symmetric, then α is a coboundary.*

Proof. Let π be any irreducible projective representation of G with multiplier α. Then by assumption $\pi(a)\pi(b) = \pi(b)\pi(a)$ for any $a, b \in G$. Therefore, each $\pi(a)$ is an element of $\operatorname{Hom}_G(\pi, \pi)$. By Schur's Lemma, $\pi(a)$ is a scalar, say $\mu(a)$. Then $\alpha(a, b) = \frac{\mu(a)\mu(b)}{\mu(ab)}$ is a coboundary. $\qquad \square$

Lemma 5.27. *Let A be an α-symmetric subgroup of an abelian group G. Let $s \in G - A$. If $\alpha(a, s^i) = \alpha(s^i, a)$ for all $a \in A$ and $i \in \mathbb{Z}$, then the subgroup $B = \langle A, s \rangle$ is also α-symmetric.*

Proof. By definition, we have

$$\alpha(as^i, bs^j)\alpha(a, s^i) = \alpha(a, bs^{i+j})\alpha(s^i, bs^j),$$
$$\alpha(a, bs^{i+j})\alpha(b, s^{i+j}) = \alpha(a, b)\alpha(ab, s^{i+j}),$$
$$\alpha(s^i, s^j b)\alpha(s^j, b) = \alpha(s^i, s^j)\alpha(s^{i+j}, b).$$

Thus,

$$\alpha(as^i, bs^j) = \frac{\alpha(a, b)\alpha(ab, s^{i+j})\alpha(s^i, s^j)}{\alpha(a, s^i)\alpha(s^j, b)}.$$

Since $\alpha(s^i, s^j) = \alpha(s^j, s^i)$ for any $s \in G$, it is easy to see that $\alpha(as^i, bs^j) = \alpha(bs^j, as^i)$. The lemma follows. $\qquad \square$

Theorem 5.28. *Let G be an abelian group. Let α be a fixed multiplier of G. Let A be a maximal α-symmetric subgroup of G. Then $d_G^\alpha = (G : A)$.*

Proof. Let π be an irreducible projective representation of G with multiplier α. Consider the restriction $\pi|_A$, it is a projective representation of A with multiplier $\alpha|_A$, which is a coboundary. Thus $\pi|_A = \oplus_{i \in I} \chi_i$ is a finite direct sum of one-dimensional projective representations. Fix one $\chi \in \{\chi_i\}_{i \in I}$ and consider the projective representation $V' = \operatorname{Ind}_A^G \chi$. Note that here Ind_A^G is the one defined in Section 4 with respect to the multiplier α. First, we show that V' is irreducible. By Proposition 5.13, it suffices to show that χ is not isomorphic to χ^s for any $s \in G - A$. Suppose that there exists $s \in G - A$ such that $\chi \cong \chi^s$. From the definition of χ^s, we have $\alpha(a, s^{-1}) = \alpha(s^{-1}, a)$ for any $a \in A$.

Inductively, we see that $\alpha(a, s^i) = \alpha(s^i, a)$ for any $a \in A$ and $i \in \mathbb{Z}$. Therefore, $\langle A, s \rangle$ is an α-symmetric subgroup, which contradicts to the assumption on A. Thus V' is irreducible.

On the other hand,

$$\mathrm{Hom}_G(V', \pi) = \mathrm{Hom}_A(\chi, \pi|_A)$$

has a nontrivial element. So $V' \cong \pi$ and $\deg \pi = (G : A)$. The theorem follows. $\square$

Corollary 5.29. *Let α be a multiplier of an abelian group G. Then all the maximal α-symmetric subgroups of G have the same index in G, and this number is less or equal to $\sqrt{|G|}$.*

In particular, for any abelian group G and $\alpha \in Z^2(G, \mathbb{C}^\times)$, there exists a subgroup A of G with $|A| \geq \sqrt{|G|}$ and $\alpha|_A \in B^2(A, \mathbb{C}^\times)$.

6. The group of virtual projective characters

In this section, we study the group of virtual projective characters. The contents are similar to [11, Chapter 9], where the group of usual virtual characters is studied.

6.1. **Definitions and basic properties.** Let G be a finite group. Fix a unitary Schur multiplier α of G, such that $\alpha^N = 1$ where N is the order of $[\alpha]$ (see Remark 3.2). Let $\chi_1, \ldots, \chi_l$ be the set of irreducible projective characters with multiplier α. An α-class function on G is a character of a projective representation with multiplier α if and only if it is a linear combination of χ_i with non-negative coefficients. Define a group $R_\alpha(G)$ by

$$R_\alpha(G) = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_l.$$

An element of $R_\alpha(G)$ is called a *virtual projective character* attached to α. It is a subgroup of the vector space $\mathbb{H}_\alpha$.

Define a group

$$\mathcal{R}_\alpha(G) = R_\alpha(G) \oplus \cdots \oplus R_{\alpha^N}(G).$$

Then $\mathcal{R}_\alpha$ is a ring. The group $R(G) := R_{\alpha^N}(G)$ is the group of usual virtual characters and it is a subring of $\mathcal{R}_\alpha(G)$. Note that if χ (resp. τ) is an element in $R_\alpha(G)$ (resp. $R_\beta(G)$) and $\alpha \neq \beta$, $\chi + \tau$ has no meaning in terms of projective representations.

Let H be a subgroup of G. We also denote α the restriction of α to $H \times H$. The operations Ind and Res define two group homomorphisms

$$\mathrm{Ind} : R_\alpha(H) \to R_\alpha(G), \quad \mathrm{Res} : R_\alpha(G) \to R_\alpha(H).$$

They induce two natural maps

(6.1) $$\mathrm{Ind} : \mathcal{R}_\alpha(H) \to \mathcal{R}_\alpha(G), \quad \mathrm{Res} : \mathcal{R}_\alpha(G) \to \mathcal{R}_\alpha(H).$$

Lemma 6.1. *With the above notation,*

(1) *The two maps in (6.1) are group homomorphisms. Moreover, the map* Res *is a ring homomorphism.*

(2) *The image of* Ind $: \mathcal{R}_\alpha(H) \to \mathcal{R}_\alpha(G)$ *is an ideal of* $\mathcal{R}_\alpha(G)$.

Proof. The statement (1) follows easily from the definition. For (2), it suffices to show that

$$\mathrm{Ind}(\tau \cdot \mathrm{Res}(\chi)) = \mathrm{Ind}(\tau) \cdot \chi$$

for irreducible projective characters $\tau \in R_\beta(G)$ and $\chi \in R_\alpha(G)$. Let τ be the character of a projective representation W of H with multiplier β, χ be the character of a projective representation E of G with multiplier α. Then the identity above is equivalent to

$$\mathrm{Ind}(W \otimes \mathrm{Res}(E)) \cong \mathrm{Ind}(W) \otimes E.$$

This follows from the formula (4.3). $\qquad\square$

6.2. **Characters of symmetric and exterior powers.** Let $\pi : G \to \mathrm{GL}(V)$ be a projective representation of G with unitary multiplier α and character χ. The tensor product $W = V^{\otimes k}$ is a projective representation of G with multiplier α^k. It has two natural subprojective representations $\pi_S^k : G \to \mathrm{GL}(\mathrm{Sym}^k(V))$ and $\pi_A^k : G \to \mathrm{GL}(\mathrm{Alt}^k(V))$. Let χ_S^k and χ_A^k be the characters of π_S^k and π_A^k respectively. In the following, we compute the two characters explicitly. Define

$$S_T(\chi) = \sum_{k=0}^{\infty} \chi_S^k T^k, \quad A_T(\chi) = \sum_{k=0}^{\infty} \chi_A^k T^k,$$

where T is an indeterminate.

Lemma 6.2. *Let $g \in G$. Then*

$$S_T(\chi)(g) = \frac{1}{\det(1 - \pi(g)T)}, \quad A_T(\chi)(g) = \frac{1}{\det(1 + \pi(g)T)}.$$

Proof. Let (λ_i) be the eigenvalues of $\pi(g)$. Choose a basis (e_i) of V consisting eigenvectors of $\pi(g)$. Then $(e_{i_1} \otimes \cdots \otimes e_{i_k})_{i_1 \leq \cdots \leq i_k}$ (resp. $(e_{i_1} \otimes \cdots \otimes e_{i_k})_{i_1 < \cdots < i_k}$) is a basis of $\mathrm{Sym}^k V$ (resp. $\mathrm{Alt}^k V$). Thus $(\lambda_{i_1} \cdots \lambda_{i_k})_{i_1 \leq \cdots \leq i_k}$ (resp. $(\lambda_{i_1} \cdots \lambda_{i_k})_{i_1 < \cdots < i_k}$) are eigenvalues of $\mathrm{Sym}^k V$ (resp. $\mathrm{Alt}^k V$), i.e.,

(6.2)
$$\chi_S^k(g) = \sum_{i_1 \leq \cdots \leq i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$
$$\chi_A^k(g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

Therefore,

(6.3)
$$\frac{1}{\det(1 - \pi(g)T)} = \prod_i \frac{1}{1 - \lambda_i T}$$
$$= \prod_i \left(\sum_{k=0}^{\infty} \lambda_i^k T^k \right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{i_1 \leq \cdots \leq i_k} \lambda_{i_1} \cdots \lambda_{i_k} \right) T^k = S_T(\chi)(g).$$

The proof for $A_T(\chi)$ is the same. $\qquad\square$

Let f be a function on G. Define function $\Psi_\alpha^k(f)$ by

$$\Psi_\alpha^k(f)(g) = \alpha(g, g^{k-1})\alpha(g, g^{k-2}) \cdots \alpha(g, g)f(g^k).$$

Lemma 6.3. *With the above notation,*

(6.4)
$$S_T(\chi) = \exp(\sum_{k=1}^{\infty} \Psi_{\alpha}^k(\chi) T^k / k),$$

$$A_T(\chi) = \exp(\sum_{k=1}^{\infty} (-1)^{k-1} \Psi_{\alpha}^k(\chi) T^k / k).$$

Proof. We prove the lemma for $S_T(\chi)$. The other case is similar. It suffices to show that

$$\sum_{k=1}^{\infty} \Psi_{\alpha}^k(\chi)(g) T^k / k = -\log \det(1 - \pi(g)T).$$

Since $\log \det(1 - \pi(g)T) = \sum_i \log(1 - \lambda_i T)$, it suffices to show that

$$\Psi_{\alpha}^k(\chi)(g) = \sum_i \lambda_i^k.$$

This follows from the definition of $\Psi_{\alpha}^k(\chi)$ and the fact

$$\pi(g)^k = \alpha(g, g^{k-1})\alpha(g, g^{k-2}) \cdots \alpha(g, g)\pi(g^k).$$

$\square$

Proposition 6.4. *With the above notation,*

(6.5)
$$n\chi_S^n = \sum_{k=1}^{n} \Psi_a^k(\chi)\chi_S^{n-k},$$

$$n\chi_A^n = \sum_{k=1}^{n} (-1)^{k-1} \Psi_a^k(\chi)\chi_A^{n-k}.$$

Proof. We only prove the first equality. The other case is similar. By the above computation, we have

$$\sum_{k=0}^{\infty} \chi_S^k T^k = \exp(\sum_{k=1}^{\infty} \Psi_{\alpha}^k(\chi) T^k / k).$$

Taking derivative with respect to T on both sides, we obtain

(6.6)
$$\sum_{k=1}^{\infty} k\chi_S^k T^{k-1} = \exp(\sum_{k=1}^{\infty} \Psi_{\alpha}^k(\chi) T^k / k) \sum_{k=1}^{\infty} \Psi_{\alpha}^k(\chi) T^{k-1}$$

$$= \sum_{k=0}^{\infty} \chi_S^k T^k \sum_{k=1}^{\infty} \Psi_{\alpha}^k(\chi) T^{k-1}.$$

Comparing the coefficients of T^{n-1}, the proposition follows. $\square$

Corollary 6.5. *With the above notation,*
   (1) Ψ_{α}^n *sends* $R_{\alpha}(G)$ *to* $R_{\alpha^n}(G)$.
   (2) *Let* χ *be an irreducible projective character in* $R_{\alpha}(G)$. *If* $(n, |G|) = 1$, *then* $\Psi_{\alpha}^n(\chi)$
       is an irreducible projective character in $R_{\alpha^n}(G)$.

Proof. Using induction on n, the first claim follows easily from Proposition 6.4. For (2), we see that $\Psi_\alpha^n(\chi)$ is an element in $R_{\alpha^n}(G)$ by (1). Therefore, to show it is an irreducible projective character, it suffices to show that $\Psi_\alpha^n(\chi)(1) \geq 0$ and $(\Psi_\alpha^n(\chi), \Psi_\alpha^n(\chi)) = 1$. By the assumption, these two conditions hold. The claim follows. $\square$

6.3. **Artin's theorem.** For any subset H of G, let H_α be the subset of H consisting of α-elements. We have the following result, which corresponds to [11, Theorem 17].

Theorem 6.6. *Let X be a family of subgroups of G. Let $\mathrm{Ind}_X : \oplus_{H \in X} R_\alpha(H) \to R_\alpha(G)$ be the homomorphism induced from Ind_H^G, $H \in X$. Then the following conditions are equivalent.*

(1) *G_α is the union of the conjugates of H_α, $H \in X$.*
(2) *The cokernel of $\mathrm{Ind}_X : \oplus_{H \in X} R_\alpha(H) \to R_\alpha(G)$ is finite.*
(3) *For each projective character of G in $R_\alpha(G)$, there exist virtual projective characters $\chi_H \in R_\alpha(H)$, $H \in X$, and a positive integer d, such that*

$$d\chi = \sum_{H \in X} \mathrm{Ind}_H^G(\chi_H).$$

Proof. The proof is the same as the proof of [11, Theorem 17]. We give details here for completeness. Since $R_\alpha(G)$ is a finitely generated group, it is clear that (2)$\Leftrightarrow$(3). First, we show that (2)$\Rightarrow$(1). Let $S = \cup_{H \in X, g \in G}(gH_\alpha g^{-1}) \subset G_\alpha$. Every function of G with the form $\sum_{H \in X} \mathrm{Ind}_H^G(f_H)$ $(f_H \in R_\alpha(H))$ vanishes outside S. If (2) is true, then each α-class function on G vanishes outside S, which shows that $S = G_\alpha$, i.e., (1) is true.

Conversely, suppose (1) is satisfied. To prove (2), it suffices to show that the $\mathbb{Q}$-linear map

$$\mathbb{Q} \otimes \mathrm{Ind}_X : \oplus_{H \in X} \mathbb{Q} \otimes R_\alpha(H) \to \mathbb{Q} \otimes R_\alpha(G)$$

is surjective. Then it suffices to show the surjectivity of the $\mathbb{C}$-linear map

$$\mathbb{C} \otimes \mathrm{Ind}_X : \oplus_{H \in X} \mathbb{C} \otimes R_\alpha(H) \to \mathbb{C} \otimes R_\alpha(G).$$

This is equivalent to the injectivity of the adjoint map

$$\mathbb{C} \otimes \mathrm{Res}_X : \mathbb{C} \otimes R_\alpha(G) \to \oplus_{H \in X} \mathbb{C} \otimes R_\alpha(H),$$

which is obvious because G_α is covered by the conjugates of H_α ($H \in X$). The theorem follows. $\square$

Remark 6.7. Let A be the subring of $\mathbb{C}$ generated by $|G|$-th root of unity. If $\alpha = 1$, then $\mathrm{Spec}(A \otimes R(G))$ is connected in the Zariski topology (see [11, Proposition 31]). In general, what can we say about the map $\mathrm{Spec}(A \otimes \mathcal{R}_\alpha(G)) \to \mathrm{Spec}(A \otimes R(G))$? We hope to come back to this question in a future work.

7. On compact groups

In this section, we study the (unitary) projective representations of compact groups. The set up is similar as for finite groups, but there are some subtle differences and we need to introduce some new definitions. In this section, G is a compact topological group with identity element 1. Fix a Haar measure $\int_G \cdot \, dg$ on G.

7.1. **The set up.**

Definition 7.1. Let S^1 be the unit circle in $\mathbb{C}$. A continuous map $\alpha : G \times G \to S^1$ is called a *multiplier* (or a *factor set* or a *2-cocycle*) on G if

(1) $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$.
(2) $\alpha(x, 1) = \alpha(1, x) = 1$ for all $x \in G$.

Remark 7.2. Note that for finite groups, the multipliers are defined using $\mathbb{C}^\times$, not S^1. But each element $c \in H^2(G, \mathbb{C}^\times)$ is represented by an element in $Z^2(G, S^1)$ (Lemma 3.1). Therefore, the definition here does not lose the generality. The same statement is true for pro-finite groups by [12, Chap. 1, Proposition 8].

Definition 7.3. Let V be an n-dimensional Hilbert vector space over $\mathbb{C}$. (n is not necessarily finite). A *projective representation* of G over V is a continuous map $\pi : G \to \mathrm{U}(V)$ such that $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$ for all $x, y \in G$, where α is the associated multiplier, $\mathrm{U}(V)$ is the set of bounded invertible linear operators from V to V. Here *continuous* means that the map $(g, v) \mapsto \pi(g)v$ is a continuous map from $G \times V$ to V. We denote this projective representation by (π, V, α) or (π, V).

The other notions are defined in the same way as in finite group case. From the following lemma, every projective representation with multiplier $\alpha \in Z^2(G, S^1)$ is unitary and we may and will take $\mathrm{U}(V)$ to be the set of unitary operators from V to V.

Lemma 7.4. *Let (π, V, α) be a projective representation of G. Then there exists a G-invariant Hermitian inner product $\langle, \rangle$ on V, i.e., $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ for any $g \in G$ and $v, w \in V$.*

In particular, $\pi(g)$ is unitary and the eigenvalues of $\pi(g)$ have absolute value 1.

Proof. Let $(,)$ be any Hermitian inner product on V. Given $v, w \in V$, the function $f : g \mapsto (\pi(g)v, \pi(g)w)$ is a continuous function. Hence f is integrable. Define $\langle v, w \rangle = \int_G (\pi(g)v, \pi(g)w) \, dg$. Since α is unitary, it is easy to check that this defines a G-invariant Hermitian inner product. $\qquad\square$

From now on, we assume that all projective representations are unitary.

Corollary 7.5. *Every projective representation of G is completely reducible. i.e., it is a direct sum of irreducible projective representations.*

7.2. **Schur's Lemma and finite dimensional projective representations.** We prove Schur's Lemma for projective representations of compact groups and study finite dimensional projective representations. The situation is very similar to the case of linear representations.

Lemma 7.6. *Let (π_1, V_1) and (π_2, V_2) be two projective representations of G with multiplier α. If $A \in \mathrm{Hom}_G(V_1, V_2)$ is a bounded linear operator, then $A^*A\pi_1(g) = \pi_1(g)A^*A$ for all $g \in G$. Here A^* is the adjoint of A.*

Proof. This follows from

$$\langle A^*A\pi_1(g)v, w \rangle = \langle A\pi_1(g)v, Aw \rangle = \langle \pi_2(g)Av, Aw \rangle$$

(7.1)
$$= \langle \alpha(g, g^{-1})Av, \pi_2(g^{-1})Aw \rangle = \langle \alpha(g, g^{-1})Av, A\pi_1(g^{-1})w \rangle$$

$$= \langle \alpha(g, g^{-1})\pi_1(g)A^*Av, \alpha(g, g^{-1})w \rangle = \langle \pi_1(g)A^*Av, w \rangle,$$

where $g \in G$, $v, w \in V_1$. □

Lemma 7.7 (Schur's Lemma)**.** *Let (π_1, V_1) and (π_2, V_2) be two projective representations of G with multiplier α. Assume that π_1 is irreducible and $A \in \mathrm{Hom}_G(V_1, V_2)$ is a nonzero bounded linear operator. Then $A(V_1) \subset V_2$ is a closed subspace for π_2 and $\pi_1 \cong \pi_2|_{A(V_1)}$.*

Proof. By Lemma 7.6, $A^*A \in \mathrm{Hom}_G(V_1, V_1)$. Because π_1 is irreducible, we must have $A^*A = \lambda \cdot \mathrm{id}$ for some λ. Indeed, for any bounded normal operator $O = O^*$ in $\mathrm{Hom}_G(V_1, V_1)$, the norm closed unital $*$-algebra $C^*(O)$ generated by O is contained in $\mathrm{Hom}_G(V_1, V_1)$. Since O is normal, $C^*(O)$ is commutative and is isomorphic to $\mathbb{C}(\sigma(O))$ by the spectral theorem (see for example [2, II.2.3.1 Corollary].) If $\sigma(O) \neq \{\mathrm{point}\}$, then we can find nonzero self-adjoint operators B_1 and B_2 in $C^*(O)$ such that $B_1 B_2 = B_2 B_1 = 0$. Thus $\langle B_1 v, B_2 w \rangle = 0$ for all $v, w \in V_1$. In particular, the closed subspaces $\overline{B_1(V_1)}$ and $\overline{B_2(V_1)}$ are nonzero, orthogonal, G-stable. This contradicts to the irreducibility of V_1. Therefore, $\sigma(O)$ is a point and we must have $O = \lambda \cdot \mathrm{id}$.

Note that $B = \lambda^{1/2} A$ is an isometry, hence a unitary operator in $\mathrm{Hom}_G(V_1, A(V_1))$. Since A is a multiple of an isometry, $A(V_1)$ is closed. By Lemma 7.6 again, $BB^* \in \mathrm{Hom}_G(V_2, V_2)$ and it is the orthogonal projection onto $B(V_1) = A(V_1)$. It follows that $A(V_1)$ is G-stable and the lemma follows. □

Remark 7.8. Let (π, V, α) be a projective representation of G. Then π is irreducible if and only if $\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot \mathrm{id}_V$. Indeed, it suffices to verify the *only if* part. Let $A \in \mathrm{Hom}_G(V, V)$. If A is normal, i.e., $A^* = A$, the argument in Lemma 7.7 shows that $A = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$. For general nonzero $A \in \mathrm{Hom}_G(V, V)$, applying the argument to AA^* and A^*A, we have $AA^* = \xi \cdot \mathrm{id}_V$ and $A^*A = \eta \cdot \mathrm{id}_V$ with $\xi, \eta \in \mathbb{R}_{>0}$. Then $\xi = \eta$ and A must be normal. The claim follows. (See [13, Theorem 12] for the statement for linear representations of compact groups.)

Most of the properties of projective representations of finite groups carry over to finite dimensional projective representations of compact groups. As for the proof, one only needs to replace $\frac{1}{|G|} \sum_{g \in G}$ by $\int_G \mathrm{d}\, g$. First, Corollaries 2.13 and 2.14 are true for finite dimensional projective representations of compact groups.

Let (π, V, α) be a finite dimensional projective representation of G. Define the *character* of π $\chi_\pi : G \to \mathbb{C}$ by the equation

$$\chi_\pi(g) = \mathrm{Tr}(\pi(g)) \text{ for all } g \in G.$$

Then Lemmas 3.4, 3.5 and Theorem 3.6 carry over. We collect some other properties in the following proposition.

Proposition 7.9. *Let (π_1, V_1) and (π_2, V_2) be finite dimensional projective representations of G with the same multiplier. Then*

(1) *If $\chi_{\pi_1} = \chi_{\pi_2}$, then $\pi_1 \cong \pi_2$.*
(2) *$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle_2 = \dim_\mathbb{C} \mathrm{Hom}_G(\pi_1, \pi_2)$. Here $\langle, \rangle_2$ is the scalar product defined by equation (7.2).*
(3) *If G is abelian, then all finite dimensional irreducible projective representations of G with the same multiplier have the same degree.*

7.3. **The Peter-Weyl theorem.** From now on, we fix a multiplier α. Let $L^2(G)$ be the space of measurable functions on G for which $\int_G |f(g)|^2 \, dg < \infty$. If $f \in L^2(G)$, define $||f||_2 = (\int_G |f(g)|^2 \, dg)^{1/2}$. Let $L^1(G)$ be the space of measurable functions on G for which $\int_G |f(g)| \, dg < \infty$. If $f \in L^1(G)$, define $||f||_1 = \int_G |f(g)| \, dg$. Given $f, f' \in L^2(G)$, define an inner product by

$$(7.2) \qquad \langle f, f' \rangle_2 = \int_G f(g) \overline{f'(g)} \, dg.$$

With this inner product, $L^2(G)$ is a Hilbert space. Furthermore, $ff' \in L^1(G)$ and we have the following inequalities.

$$(7.3) \qquad \begin{aligned} &||ff'||_1 \leq ||f||_2 ||f'||_2, \\ &|\langle f, f' \rangle_2| \leq ||f||_2 ||f'||_2 \text{ (Schwarz inequality)}. \end{aligned}$$

If $f : G \to \mathbb{C}$ and $g \in G$, define $r(g)f := r_\alpha(g)f : G \to \mathbb{C}$ by

$$(r(g)f)(g_0) = \alpha(g_0, g) f(g_0 g)$$

for all $g_0 \in G$. It is easy to check that $r(g)f \in L^2(G)$ if $f \in L^2(G)$ and $r(g)$ is an element in $U(L^2(G))$. Then $r : G \to U(L^2(G))$ defines a unitary projective representation of G with associated multiplier α. We call it the *right translation* or *right regular projective representation* of G on $L^2(G)$ with respect to α. It is also easy to check that $\langle, \rangle_2$ is G-invariant, i.e.,

$$\langle r(g)f, r(g)f' \rangle_2 = \langle f, f' \rangle_2.$$

Thus $(r, L^2(G), \alpha)$ decomposes as a direct sum of irreducible unitary projective representations by Corollary 7.5.

Let (π, V, α) be a finite dimensional projective representation of G. Let $\langle, \rangle$ be a G-invariant Hermitian inner product on V. Given $v, w \in V$, the function $f : g \mapsto \langle \pi(g)v, w \rangle$ is a *matrix coefficient* of π. Let $\mathcal{A}_\alpha(G)$ be the space spanned by all matrix coefficients of finite dimensional irreducible projective representations of G with multiplier α. The main result in this subsection is the following theorem.

Theorem 7.10 (Peter-Weyl Theorem). *$\mathcal{A}_\alpha(G)$ is dense in $L^2(G)$.*

The strategy of the proof is similar as for linear representations (see for example [5]). First, we prove some lemmas.

Lemma 7.11. *With the above notation, the functions $g \mapsto \alpha(g, g^{-1})\overline{f(g^{-1})}$, $g \mapsto \alpha(g, h)f(gh)$, $g \mapsto \alpha(h, g)\alpha(h^{-1}, h)^{-1}f(hg)$ are matrix coefficients of π. We call them the adjoint of f, the right translation of f, the left translation of f, respectively.*

Proof. Note that

$$(7.4) \qquad \begin{aligned} \overline{f(g^{-1})} &= \overline{\langle \pi(g^{-1})v, w \rangle} = \langle w, \pi(g^{-1})v \rangle \\ &= \langle \pi(g)w, \pi(g)\pi(g^{-1})v \rangle = \alpha(g, g^{-1})^{-1} \langle \pi(g)w, v \rangle. \end{aligned}$$

This shows that $g \mapsto \alpha(g, g^{-1})\overline{f(g^{-1})}$ is a matrix coefficient. Similarly, it is easy to see that

$$(7.5) \qquad \begin{aligned} f(gh) &= \alpha(g, h)^{-1} \langle \pi(g)(\pi(h)v), w \rangle, \\ f(hg) &= \alpha(h, g)^{-1}\alpha(h^{-1}, h) \langle \pi(g)v, \pi(h^{-1})w \rangle. \end{aligned}$$

The claims follow easily. $\qquad\square$

Denote by $\mathcal{C}(G)$ the space of all continuous functions from G to $\mathbb{C}$. It is dense in $L^2(G)$.

Lemma 7.12. *Let $f \in L^2(G)$. Then the map $g \mapsto r(g)f$ is a continuous map from G to $L^2(G)$.*

Proof. Let $\epsilon > 0$. Choose $\phi \in \mathcal{C}(G)$ such that $||f - \phi||_2 < \epsilon/3$. Note that G is compact, each continuous function on G is uniformly continuous. In particular, for the function $\alpha(g, -)\phi(g-)$, there exists an open neighborhood U of $1 \in G$ such that if $h^{-1}h' \in U$, then $|\alpha(g, h)\phi(gh) - \alpha(g, h')\phi(gh')| < \epsilon/3$ for all $g \in G$. Note that

$$||r(h)f - r(h')f||_2 \le ||r(h)f - r(h)\phi||_2 + ||r(h)\phi - r(h')\phi||_2 + ||r(h')\phi - r(h')f||_2$$

(7.6)
$$= 2||f - \phi||_2 + ||r(h)\phi - r(h')\phi||_2$$

$$< \epsilon.$$

The continuity follows. $\qquad\square$

Lemma 7.13. *Let $f \in L^2(G)$. For every $\epsilon > 0$, there exist finitely many $g_i \in G$ and Borel sets $B_i \subset G$ such that G is the disjoint union of the B_i's and $||r(g)f - r(g_i)f||_2 < \epsilon$ for all i and $g_i \in B_i$.*

Proof. By Lemma 7.12, there exists an open neighborhood U of 1 such that $||r(g)f - f||_2 < \epsilon$ for all $g \in U$. Note that $\{hU \mid h \in G\}$ is an open cover of G and G is compact, there exist finitely many $g_1, \dots, g_n$ such that $G = \cup_{i=1}^n g_i U$. Let $B_i = g_i U - \cup_{j=1}^{i-1} g_j U$. It is easy to check that these objects satisfy the property in the statement. $\qquad\square$

Lemma 7.14. *Let $f \in L^2(G)$ and $f_1 \in L^1(G)$. Define $F : G \to \mathbb{C}$ by*

$$F(g') = \int_G \alpha(g', g)f(g'g)f_1(g)\, \mathrm{d}\, g.$$

Then F is an element in $L^2(G)$ and it is a limit of a sequence of functions, each of which is a finite linear combination of right translates of f.

Proof. Let $\epsilon > 0$. Choose g_i and B_i as in Lemma 7.13. Set $e_i = \int_{B_i} f_1(g)\, \mathrm{d}\, g$. Then

$$||F - \sum_{i=1}^n e_i r(g_i)f||_2 \le \sum_{i=1}^n \int_{B_i} |f_1(g)| \cdot ||r(g)f - r(g_i)f||_2\, \mathrm{d}\, g$$

(7.7)
$$\le \sum_{i=1}^n \int_{B_i} |f_1(g)|\epsilon\, \mathrm{d}\, g = \epsilon||f_1||_1.$$

The lemma follows. $\qquad\square$

For compact group G, a function $f : G \to \mathbb{C}$ is called an *α-class function* if it satisfies the equation in Definition 3.13.

Lemma 7.15. *Let f be any integrable function on G. Set*

$$f'(g) = \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f(hgh^{-1})\, \mathrm{d}\, h.$$

Then f' is an α-class function on G.

Proof. Note that

$$f'(i^{-1}gi) = \int_G \frac{\alpha(h, i^{-1}gih^{-1})\alpha(i^{-1}gi, h^{-1})}{\alpha(h, h^{-1})} f(hi^{-1}gih^{-1}) \, \mathrm{d}\, h$$

$$= \int_G \frac{\alpha(h'i, i^{-1}gii^{-1}(h')^{-1})\alpha(i^{-1}gi, i^{-1}(h')^{-1})}{\alpha(h'i, i^{-1}(h')^{-1})} f(h'g(h')^{-1}) \, \mathrm{d}\, h' \ (h' = hi^{-1}).$$

Then to show that f' is an α-class function, it suffices to show that

(7.8)
$$\begin{aligned}&\alpha(i^{-1}, i)\alpha(h, gh^{-1})\alpha(g, h^{-1})\alpha(hi, i^{-1}h^{-1})\\ =&\alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, h^{-1})\alpha(hi, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}).\end{aligned}$$

Since $\alpha(h, i)\alpha(hi, i^{-1}h^{-1}) = \alpha(h, h^{-1})\alpha(i, i^{-1}h^{-1})$, it suffices to show that

(7.9)
$$\begin{aligned}&\alpha(i^{-1}, i)\alpha(h, gh^{-1})\alpha(g, h^{-1})\alpha(i, i^{-1}h^{-1})\\ =&\alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, i)\alpha(hi, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}).\end{aligned}$$

This follows from the following computation.

(7.10)
$$\begin{aligned}RHS &= \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, gh^{-1})\alpha(i, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1})\\ &= \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, gh^{-1})\alpha(i, i^{-1}gi)\alpha(gi, i^{-1}h^{-1})\\ &= \alpha(h, gh^{-1})[\alpha(i^{-1}, gi)\alpha(i, i^{-1}gi)][\alpha(g, i)\alpha(gi, i^{-1}h^{-1})]\\ &= \alpha(h, gh^{-1})\alpha(i, i^{-1})\alpha(g, h^{-1})\alpha(i, i^{-1}h^{-1}) = LHS.\end{aligned}$$

The lemma follows. $\qquad\square$

Lemma 7.16. *Let $f : G \to \mathbb{C}$ be an α-class function. Then $f'(g) = \alpha(g, g^{-1})\overline{f(g^{-1})}$ is also an α-class function.*

Proof. One needs to show that

$$f'(hgh^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})} f'(g).$$

This is equivalent to

$$\frac{\alpha(hgh^{-1}, hg^{-1}h^{-1})\alpha(h, g^{-1}h^{-1})\alpha(g^{-1}, h^{-1})}{\alpha(h, h^{-1})} = \frac{\alpha(h, h^{-1})\alpha(g, g^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})}.$$

Note that

(7.11)
$$\begin{aligned}&\alpha(hgh^{-1}, hg^{-1}h^{-1})\alpha(h, g^{-1}h^{-1})\alpha(g^{-1}, h^{-1})\alpha(h, gh^{-1})\alpha(g, h^{-1})\\ =&\alpha(hgh^{-1}, h)\alpha(hg, g^{-1}h^{-1})\alpha(g^{-1}, h^{-1})\alpha(h, gh^{-1})\alpha(g, h^{-1})\\ =&\alpha(hgh^{-1}, h)\alpha(hg, g^{-1})\alpha(h, h^{-1})\alpha(h, g)\alpha(hg, h^{-1})\\ =&\alpha(h, h^{-1})[\alpha(h, g)\alpha(hg, g^{-1})][\alpha(hg, h^{-1})\alpha(hgh^{-1}, h)]\\ =&\alpha(h, h^{-1})\alpha(g, g^{-1})\alpha(h, h^{-1}).\end{aligned}$$

The lemma follows. $\qquad\square$

Lemma 7.17. *Let $f : G \to \mathbb{C}$ be an α-class function. Then*

$$\frac{f(h^{-1}g)}{\alpha(h, h^{-1}g)} = \frac{f(gh^{-1})}{\alpha(gh^{-1}, h)}.$$

Proof. Since $gh^{-1} = h(h^{-1}g)h^{-1}$, it suffices to prove

$$\frac{\alpha(h, h^{-1})}{\alpha(h, h^{-1}gh^{-1})\alpha(h^{-1}g, h^{-1})} = \frac{\alpha(gh^{-1}, h)}{\alpha(h, h^{-1}g)}.$$

Note that

(7.12)
$$\begin{aligned}
&\alpha(gh^{-1}, h)\alpha(h, h^{-1}gh^{-1})\alpha(h^{-1}g, h^{-1}) \\
&=\alpha(h, h^{-1}g)\alpha(h^{-1}gh^{-1}, h)\alpha(h^{-1}g, h^{-1}) = \alpha(h, h^{-1}g)\alpha(h^{-1}, h).
\end{aligned}$$

The lemma follows. $\qquad\square$

With the above preparation, now we can prove Theorem 7.10.

Proof of Theorem 7.10. Let $\overline{\mathcal{A}_\alpha(G)}$ be the closure of $\mathcal{A}_\alpha(G)$ in $L^2(G)$. Since $\mathcal{A}_\alpha(G)$ is stable under the operations in Lemma 7.11, $\overline{\mathcal{A}_\alpha(G)}$ is also stable under those operations. Suppose that $\overline{\mathcal{A}_\alpha(G)} \neq L^2(G)$. Then $\overline{\mathcal{A}_\alpha(G)}^\perp \neq \{0\}$ and it is stable under the operations in Lemma 7.11. Let $f_0 \in \overline{\mathcal{A}_\alpha(G)}^\perp$ and $f_0 \neq 0$. Fix U an open neighborhood of 1. Let $\mathbb{I}_U$ be the characteristic function on U, $|U|$ the Haar measure of U, and

$$f_U(g) = |U|^{-1} \int_G \alpha(g, g_0)\mathbb{I}_U(g_0)f_0(gg_0)\,\mathrm{d}\,g_0.$$

Since $\mathbb{I}_U, f_0 \in L^2(G)$, by Schwarz inequality, we see that f_U is continuous. Furthermore, $f_0 = \lim_{U \to \{1\}} f_U$ in $L^2(G)$. Because $f_0 \neq 0$, there exist U such that $f_U \neq 0$. Since $\overline{\mathcal{A}_\alpha(G)}$ is G-stable by right translation and the right translation of G on $L^2(G)$ is unitary, $\overline{\mathcal{A}_\alpha(G)}^\perp$ is also G-stable. Hence linear combinations of right translates of f_0 belong to $\overline{\mathcal{A}_\alpha(G)}^\perp$. By Lemma 7.14, $f_U \in \overline{\mathcal{A}_\alpha(G)}^\perp$. In particular, $\overline{\mathcal{A}_\alpha(G)}^\perp$ contains a nonzero continuous function. Let f_1 be such a function. We may assume that $f_1(1) \in \mathbb{R} - \{0\}$. Define

$$f_2(g) = \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f_1(hgh^{-1})\,\mathrm{d}\,h.$$

By Lemma 7.15, f_2 is an α-class function. It is easy to see that f_2 is continuous and $f_2(1) \in \mathbb{R} - \{0\}$. Moreover, for any $f' \in \overline{\mathcal{A}_\alpha(G)}$, $f''(g) = \alpha(h^{-1}, g)\alpha(h, h^{-1})^{-1}\alpha(h^{-1}g, h)f'(h^{-1}gh)$ is also an element in $\overline{\mathcal{A}_\alpha(G)}$ by Lemma 7.11. Note that

(7.13)
$$\begin{aligned}
\langle f_2, f'\rangle_2 &= \int_G f_2(g)\overline{f'(g)}\,\mathrm{d}\,g \\
&= \int_G \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f_1(hgh^{-1})\overline{f'(g)}\,\mathrm{d}\,h\,\mathrm{d}\,g \\
&= \int_G \int_G \frac{\alpha(h, h^{-1}g)\alpha(h^{-1}gh, h^{-1})}{\alpha(h, h^{-1})} f_1(g)\overline{f'(h^{-1}gh)}\,\mathrm{d}\,h\,\mathrm{d}\,g \\
&= \int_G \int_G f_1(g)\overline{f''(g)}\,\mathrm{d}\,g\,\mathrm{d}\,h = 0.
\end{aligned}$$

Thus $f_2 \in \overline{\mathcal{A}_\alpha(G)}^\perp$. Define $f_3(g) = f_2(g) + \alpha(g, g^{-1})\overline{f_2(g^{-1})}$. Then f_3 is in $\overline{\mathcal{A}_\alpha(G)}^\perp$ and is an α-class function by Lemma 7.16. Moreover, it is easy to check that $f_3(g) =$

$\alpha(g, g^{-1})\overline{f_3(g^{-1})}$. Define

$$K(g, h) = f_3(gh^{-1})\alpha(gh^{-1}, h)^{-1}.$$

Since

$$\alpha(hg^{-1}, g)\alpha(gh^{-1}, h) = \alpha(hg^{-1}, gh^{-1})\alpha(1, h) = \alpha(hg^{-1}, gh^{-1}),$$

one gets $K(g, h) = \overline{K(h, g)}$. Define

$$(Tf)(g) = \int_G K(g, h)f(h)\,\mathrm{d}\,h.$$

Then T is a nonzero self-adjoint Hilbert-Schmidt operator on $L^2(G)$. Hence T has a nonzero real eigenvalue γ and the eigenspace $V_\gamma \subset L^2(G)$ is finite dimensional (see for example [2, I.8.4.1 and I.8.5.5].) Let $f \in V_\gamma$. Then

$$
\begin{aligned}
(7.14) \quad (T(r(g_0)f))(g) &= \int_G K(g, g_1)\alpha(g_1, g_0)f(g_1g_0)\,\mathrm{d}\,g_1 \\
&= \int_G K(g, g_1g_0^{-1})\alpha(g_1g_0^{-1}, g_0)f(g_1)\,\mathrm{d}\,g_1 \\
&= \int_G f_3(gg_0g_1^{-1})\frac{\alpha(g_1g_0^{-1}, g_0)}{\alpha(gg_0g_1^{-1}, g_1g_0^{-1})}f(g_1)\,\mathrm{d}\,g_1 \\
&= \int_G f_3(gg_0g_1^{-1})\frac{\alpha(g, g_0)}{\alpha(gg_0g_1^{-1}, g_1)}f(g_1)\,\mathrm{d}\,g_1 \\
&= \int_G K(gg_0, g_1)\alpha(g, g_0)f(g_1)\,\mathrm{d}\,g_1 \\
&= \alpha(g, g_0)(Tf)(gg_0) = \gamma(r(g_0)f)(g).
\end{aligned}
$$

The eigenspace V_γ is stable under right translation. Now $r : G \to \mathrm{U}(V_\gamma)$ is a finite dimensional unitary projective representation of G with multiplier α. Let $W \subset V_\gamma$ be an irreducible sub projective representation and $\{e_1, \ldots, e_n\}$ an orthonormal basis of W with respect to r. Then $g \mapsto \langle r(g)e_i, e_j\rangle_2 = \int_G \alpha(g_0, g)e_i(g_0g)\overline{e_j(g_0)}\,\mathrm{d}\,g_0$ is a matrix coefficient in $\mathcal{A}_\alpha(G)$. Since $f_3 \in \overline{\mathcal{A}_\alpha(G)}^\perp$, we have

$$
\begin{aligned}
(7.15) \quad 0 &= \int_G f_3(g)\left(\int_G \overline{\alpha(g_0, g)e_j(g_0g)}e_j(g_0)\,\mathrm{d}\,g_0\right)\mathrm{d}\,g \\
&= \int_G \left(\int_G f_3(g)\overline{\alpha(g_0, g)e_j(g_0g)}\,\mathrm{d}\,g\right)e_j(g_0)\,\mathrm{d}\,g_0 \\
&= \int_G \left(\int_G f_3(g_0^{-1}g)\overline{\alpha(g_0, g_0^{-1}g)}\,\overline{e_j(g)}\,\mathrm{d}\,g\right)e_j(g_0)\,\mathrm{d}\,g_0 \\
&= \int_G \left(\int_G f_3(g_0^{-1}g)\overline{\alpha(g_0, g_0^{-1}g)}e_j(g_0)\,\mathrm{d}\,g_0\right)\overline{e_j(g)}\,\mathrm{d}\,g \\
&= \int_G \left(\int_G f_3(gg_0^{-1})\overline{\alpha(gg_0^{-1}, g_0)}e_j(g_0)\,\mathrm{d}\,g_0\right)\overline{e_j(g)}\,\mathrm{d}\,g \quad \text{(Lemma 7.17)} \\
&= \int_G (Te_j)(g)\overline{e_j(g)}\,\mathrm{d}\,g = \gamma\langle e_j, e_j\rangle_2.
\end{aligned}
$$

Hence $\gamma = 0$, which is a contradiction. Therefore, we must have $\overline{\mathcal{A}_\alpha(G)} = L^2(G)$. □

7.4. **Some corollaries.** In the following we deduce some consequences of Theorem 7.10.

7.4.1. *Unitary irreducible projective representations are finite dimensional.*

Lemma 7.18. *Let (π_1, V_1) and (π_2, V_2) be two irreducible projective representations of G with multiplier α. Assume that V_1 and V_2 are separable Hilbert spaces (not necessarily finite dimensional). Fix orthonormal bases $\{e_k^i\}_{k=1}^{d_{\pi_i}}$ for V_i and let $r_{kl}^i = \langle \pi_i(g)e_l^i, e_k^i \rangle$ be the matrix coefficients. If $\pi_1 \not\cong \pi_2$, then*

$$\int_G \alpha(g, g^{-1})^{-1} r_{ij}^1(g) r_{kl}^2(g^{-1}) \, \mathrm{d}\, g = 0,$$

for all $1 \leq i, j \leq d_{\pi_1}$ and $1 \leq k, l \leq d_{\pi_2}$.

Proof. The proof is similar to the proof of Corollary 2.13. For any bounded linear operator $B : V_1 \to V_2$, define $A = \int_G \pi_2(g) B \pi_1(g)^{-1} \, \mathrm{d}\, g$. Then $A \in \mathrm{Hom}_G(V_1, V_2)$. If $\pi_1 \not\cong \pi_2$, then $A = 0$ by Lemma 7.7. Take $B = B_{ij}$ such that $B_{ij}(v) = \langle v, e_j^1 \rangle e_k^2$. Then

$$\begin{aligned}
&\int_G \alpha(g, g^{-1})^{-1} r_{ij}^1(g) r_{kl}^2(g^{-1}) \, \mathrm{d}\, g \\
(7.16) \qquad &= \int_G \alpha(g, g^{-1})^{-1} \langle \pi_1(g)e_j^1, e_i^1 \rangle \langle \pi_2(g^{-1})e_l^2, e_k^2 \rangle \, \mathrm{d}\, g \\
&= \int_G \langle e_j^1, \pi_1(g^{-1})e_i^1 \rangle \langle \pi_2(g^{-1})e_l^2, e_k^2 \rangle \, \mathrm{d}\, g \\
&= \int_G \langle \pi_2(g^{-1})e_l^2, \overline{\langle e_j^1, \pi_1(g^{-1})e_i^1 \rangle} e_k^2 \rangle \, \mathrm{d}\, g \\
&= \int_G \langle e_l^2, \pi_2(g) \overline{\langle e_j^1, \pi_1(g)^{-1}e_i^1 \rangle} e_k^2 \rangle \, \mathrm{d}\, g \\
&= \int_G \langle e_l^2, \pi_2(g) B_{ij}(\pi_1(g)^{-1}e_i^1) e_k^2 \rangle \, \mathrm{d}\, g = \langle e_l^2, A e_i^1 \rangle = 0.
\end{aligned}$$

The proposition follows. □

Proposition 7.19. *Every irreducible projective representation of G is finite dimensional.*

Proof. Let π be an irreducible projective representation of G with orthonormal basis $\{e_i^\pi\}_{i=1}^{d_\pi}$. For each ρ finite dimensional irreducible unitary projective representation of G, fix an orthonormal basis $\{e_i^\rho\}_{i=1}^{d_\rho}$. Suppose that $d_\pi = \infty$. From Lemma 7.18, we have

$$\begin{aligned}
0 &= \int_G \langle \rho(g)e_j^\rho, e_i^\rho \rangle \langle \alpha(g, g^{-1})^{-1}\pi(g)e_l^\pi, e_k^\pi \rangle \, \mathrm{d}\, g \\
(7.17) \qquad &= \int_G \langle \rho(g)e_j^\rho, e_i^\rho \rangle \overline{\langle e_k^\pi, \alpha(g, g^{-1})^{-1}\pi(g)e_l^\pi \rangle} \, \mathrm{d}\, g
\end{aligned}$$

for all finite dimensional ρ and $1 \leq i, j \leq d_\rho$. Since the functions $\langle \rho(g)e_j^\rho, e_i^\rho \rangle$ are dense in $L^2(G)$ by Theorem 7.10, we must have $\langle e_k^\pi, \alpha(g, g^{-1})^{-1}\pi(g)e_l^\pi \rangle = 0$ for all l, k. Therefore $\pi = 0$, which is a contradiction. The proposition follows. □

Denote by $\widehat{G}_\alpha$ the set of isomorphism classes of finite dimensional irreducible projective representations of G with multiplier α. Fix an element (ρ, V_ρ) for each class and denote

by d_ρ the degree of ρ. Then every projective representation π of G with multiplier α decomposes as $\pi \cong \oplus_{\rho \in \widehat{G}_\alpha} m_\rho \cdot \rho$ for some $m_\rho \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

7.4.2. *Trace formula twisted by α and decomposition of $L^2(G)$.* Let G be a unimodular group (not necessarily compact) and $\Gamma \subset G$ a discrete normal subgroup such that $\Gamma \backslash G$ is compact. Let $\alpha \in Z^2(\Gamma \backslash G, S^1)$ be a unitary multiplier. We may view α as an element in $Z^2(G, S^1)$ via the natural projection $G \times G \to \Gamma \backslash G \times \Gamma \backslash G$. The right regular representation r_α of G with respect to α over $L^2(\Gamma \backslash G)$ is defined by

$$(r_\alpha(h)(f))(g) = \alpha(g, h) f(gh).$$

Let $\phi \in \mathcal{C}(G)$ with compact support. Define $R(\phi) : L^2(\Gamma \backslash G) \to L^2(\Gamma \backslash G)$ by

$$(R(\phi)f)(x) = \int_G \phi(g) \alpha(x, g) f(xg) \, \mathrm{d}\, g = \int_G \phi(x^{-1}g) \alpha(x, x^{-1}g) f(g) \, \mathrm{d}\, g.$$

It is easy to check that this is well defined. Note that we may write $R(\phi) = \int_G \phi(g) r_\alpha(g) \, \mathrm{d}\, g$. Thus $R(\phi)$ sends each irreducible component of $(r_\alpha, L^2(\Gamma \backslash G), \alpha)$ to itself. Moreover,

$$
\begin{aligned}
(R(\phi)f)(x) &= \int_G \phi(x^{-1}g) \alpha(x, x^{-1}g) f(g) \, \mathrm{d}\, g \\
(7.18) \qquad &= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \phi(x^{-1}\gamma g) \alpha(x, x^{-1}\gamma g) f(g) \, \mathrm{d}\, g = \int_{\Gamma \backslash G} K_\phi(x, g) f(g) \, \mathrm{d}\, g,
\end{aligned}
$$

where $K_\phi(x, g) = \sum_{\gamma \in \Gamma} \phi(x^{-1}\gamma g) \alpha(x, x^{-1}\gamma g)$. Then $R(\phi)$ is of trace class and

$$\mathrm{Tr}(R(\phi)) = \int_{\Gamma \backslash G} K_\phi(x, x) \, \mathrm{d}\, x = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \phi(x^{-1}\gamma x) \alpha(x, x^{-1}\gamma x) \, \mathrm{d}\, x.$$

(See for example [4, Lemma 4.1].) Let $\mathfrak{o}$ be the set of conjugacy classes of Γ. For each class in $\mathfrak{o}$, fix an element γ and denote this conjugacy class by $\mathfrak{o}_\gamma$. If γ is an element of a group H, denote by H^γ the centralizer of γ in H. With the above notation, $\mathfrak{o}_\gamma = \{\delta^{-1}\gamma\delta \mid \delta \in \Gamma^\gamma \backslash \Gamma\}$. Therefore

$$
\begin{aligned}
\mathrm{Tr}(R(\phi)) &= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \phi(x^{-1}\gamma x) \alpha(x, x^{-1}\gamma x) \, \mathrm{d}\, x \\
&= \sum_{\mathfrak{o}_\gamma} \sum_{\delta \in \Gamma^\gamma \backslash \Gamma} \int_{\Gamma \backslash G} \phi(x^{-1}\delta^{-1}\gamma\delta x) \alpha(x, x^{-1}\delta^{-1}\gamma\delta x) \, \mathrm{d}\, x \\
(7.19) \qquad &= \sum_{\mathfrak{o}_\gamma} \int_{\Gamma^\gamma \backslash G} \phi(x^{-1}\gamma x) \alpha(x, x^{-1}\gamma x) \, \mathrm{d}\, x \\
&= \sum_{\mathfrak{o}_\gamma} \int_{G^\gamma \backslash G} \left(\int_{\Gamma^\gamma \backslash G^\gamma} \phi(x^{-1}y^{-1}\gamma y x) \alpha(x, x^{-1}y^{-1}\gamma y x) \, \mathrm{d}\, y \right) \mathrm{d}\, x \\
&= \sum_{\mathfrak{o}_\gamma} \mathrm{vol}(\Gamma^\gamma \backslash G^\gamma) \int_{G^\gamma \backslash G} \phi(x^{-1}\gamma x) \alpha(x, x^{-1}\gamma x) \, \mathrm{d}\, x.
\end{aligned}
$$

The following result corresponds to Corollary 3.11 and Proposition 5.1 in finite group case.

Proposition 7.20. *Let G be a compact group. Consider $L^2(G)$ as a unitary projective representation of G via r. Then*

$$(r, L^2(G), \alpha) \cong \bigoplus_{\rho \in \widehat{G}_\alpha} \rho^{\oplus d_\rho}.$$

Proof. For G compact, by Proposition 7.19, we know that

$$(r, L^2(G), \alpha) \cong \oplus_{\rho \in \widehat{G}_\alpha} m_\rho \cdot \rho,$$

for some $m_\rho \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Applying the above discussion to $\Gamma = \{1\}$, we obtain

$$\phi(1) = \sum_{\rho \in \widehat{G}_\alpha} m_\rho \cdot \mathrm{Tr}(\rho(\phi)),$$

for $\phi \in \mathcal{C}(G)$. Here $\rho(\phi) = R(\phi)|_\rho$. In particular, if we take $\phi = \overline{\chi_\pi}$, where χ_π is the character of $\pi \in \widehat{G}_\alpha$, then we have

$$d_\pi = \phi(1) = \sum_{\rho \in \widehat{G}_\alpha} m_\rho \cdot \int_G \chi_\rho(g)\overline{\chi_\pi(g)}\,\mathrm{d}\,g = m_\pi.$$

The proposition follows. $\qquad\square$

Remark 7.21. One may construct explicit projections for this decomposition as in [11, Chap. 2, Prop. 8].

7.4.3. *On non-abelian Fourier analysis.* Let $\psi \in L^2(G)$. Since $\rho \in \widehat{G}_\alpha$ is unitary, we have $\overline{r_{ij}^\rho(g)} = \alpha(g, g^{-1})^{-1} r_{ji}^\rho(g^{-1})$. By Corollary 2.14 (for compact groups) and Theorem 7.10, the family $\{d_\rho^{1/2} r_{ij}^\rho\}_{\rho \in \widehat{G}_\alpha}$ is an orthonormal basis for $L^2(G)$. Thus we may write

$$\psi = \sum_{\rho;i,j} c_{\rho;ij} d_\rho^{1/2} r_{ij}^\rho.$$

Then

$$\|\psi\|_2^2 = \sum_{\rho \in \widehat{G}_\alpha} \sum_{i,j=1}^{d_\rho} |c_{\rho;ij}|^2.$$

Proposition 7.22. *If $\psi \in L^2(G)$, then*

$$\|\psi\|_2^2 = \sum_{\rho \in \widehat{G}_\alpha} d_\rho \cdot \mathrm{Tr}(\rho_\psi \rho_\psi^*) = \sum_{\rho \in \widehat{G}_\alpha} d_\rho \cdot \|\rho_\psi\|_{HS}.$$

Here $\|M\|_{HS} = \sum_{i,j} m_{ij}^2$ for a matrix $M = (m_{ij})$ of finite rank; $\rho_\psi = \int_G \psi(g)\rho(g)^{-1}\,\mathrm{d}\,g$.

Proof. With the above notation, the claim follows from

$$\begin{aligned}
(7.20) \qquad c_{\rho;ij} &= \int_G \psi(g)\overline{d_\rho^{1/2} r_{ij}^\rho(g)}\,\mathrm{d}\,g \\
&= d_\rho^{1/2} \int_G \psi(g)\langle \rho(g)^{-1} e_i^\rho, e_j^\rho \rangle \,\mathrm{d}\,g = d_\rho^{1/2}\langle \rho_\psi e_i^\rho, e_j^\rho \rangle = d_\rho^{1/2}[\rho_\psi]_{ji}.
\end{aligned}$$

$\qquad\square$

7.4.4. *The space of α-class functions.* Let $\mathbb{H}_\alpha$ be the closed subspace of $L^2(G)$ consisting of square-integrable α-class functions. Let χ_ρ be the character of the projective representation $\rho \in \widehat{G}_\alpha$. We have the following result corresponds to Theorem 3.15 in finite group case.

Proposition 7.23. *The characters $(\chi_\rho)_{\rho \in \widehat{G}_\alpha}$ form an orthonormal basis of $\mathbb{H}_\alpha$.*

Proof. Let f be an α-class function on G. Let (π, V, α) be an irreducible projective representation of G. Let π_f be the linear map of V into itself defined by $\pi_f = \int_G \overline{f(g)} \pi(g) \, \mathrm{d}\, g$. Argue as in Lemma 3.14, π_f is a map of projective representations. By Schur's Lemma, π_f is a homothety of ratio λ given by

$$\lambda = \int_G \overline{f(g)} \chi_\pi(g) \, \mathrm{d}\, g = \langle \chi_\pi, f \rangle_2.$$

Now to prove the proposition, it suffices to show that every element of $\mathbb{H}_\alpha$ orthogonal to all the χ_ρ is zero. Let $f \in \mathbb{H}_\alpha$ with $\langle \chi_\rho, f \rangle_2 = 0$ for all $\rho \in \widehat{G}_\alpha$. The above discussion shows that π_f is zero as long as π is irreducible. From the direct sum decomposition, we see that π_f is always zero. Applying this to the α-regular projective representation $(r, L^2(G))$ we obtain

$$(7.21) \qquad \int_G \overline{f(g)} \alpha(x, g) \psi(xg) \, \mathrm{d}\, g = \int_G \overline{f(x^{-1}g)} \alpha(x, x^{-1}g) \psi(g) \, \mathrm{d}\, g = 0$$

for all $\psi \in L^2(G)$ and $x \in G$. Then it is easy to see that f is the zero function. The proposition follows. $\square$

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