Lecture 2. Viscosity solutions of the Hamilton-Jacobi equation on a non-compact manifold: Uniqueness

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We would like to prove uniqueness of viscosity solutions of the evolutionary Hamilton-Jacobi equation

$$\partial_t U(t,x) + H\big(x, \partial_x U(t,x)\big) = 0$$

for a given initial condition $u = U(0,\cdot)$.

Proofs usually go through the Maximum Principle, but the one we obtained.

Theorem 1 (Maximum Principle)

Assume that $u, v : [a,b] \times C \to \mathbb{R}$, *with* $C \subset M$ *compact, are continuous and respectively a viscosity subsolution and a viscosity supersolution of*

$$\partial_t u + H(x, \partial_x u) = c$$

on $]a,b[\times \mathring{C}$, *such that either* u *or* v *is locally Lipschitz on* $]a,b[\times \mathring{C}$. *Then the maximum of* $u - v$ *on* $[a,b] \times C$ *is achieved on* $[a,b] \times \partial C \cup \{a\} \times C$.

needs at least one of the functions to be locally Lipschitz.

As we already mentioned in the *stationary* case, by imposing coercivity on compact subsets fo the Hamiltonian H, all viscosity subsolutions are automatically locally Lipschitz.
In the evolutionary case this is not true.
In fact, as we observed in the first lecture, if U is a viscosity subsolution of

$$\frac{\partial U}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0,$$

and $\rho : [0,+\infty[\to \mathbb{R}$ which is continuous and non-increasing, then $V(x,s) = U(x,s) + \rho(s)$ is a viscosity subsolution of the same equation.
We can find such a function ρ which is nowhere locally Lipschitz and as (uniformly) small as we want. Hence non-locally Lipschitz viscosity subsolution are "dense".

Approximation by Lipschitz subsolutions

To circumvent the difficulty mentioned above, we will prove that viscosity subsolutions can be approximated by locally Lipschitz ones, at least in the case where the Hamiltonian $H(x,p)$ is coercive on compact subsets.
These results are well-known when M is the Euclidean space, see for example:
Hitoshi Ishii, *A Short Introduction to Viscosity Solutions and the Large Time Behavior of Solutions of Hamilton-Jacobi Equations* in Y. Achdou et al., Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications, Springer LNM **2074**, **(2013)**.

Our treatment follows closely this work of Hitoshi Ishii. Adaptation to the manifold setting is routine, as we will now see.

Sup-convolution in one variable

The usefulness of sup-convolution to improve regularity of viscosity subsolutions is already well established.
Assume $u : V \to \mathbb{R}$, is a continuous function, where V is an open subset of $\mathbb{R} \times M$.
Assume $K \subset V$ is compact subset. By continuity of u and compactness of K, we pick an open subset $O_1 \supset K$ whose closure $\bar{O}_1$ is compact and contained in V and we set

$$m = \sup_{\bar{O}_1} |u| < +\infty.$$

Again by compactness of K, we can find $\delta > 0$ and an open neighborhood $O_2 \subset O_1$ of K, with compact closure $\bar{O}_2 \subset O_1$ such that

$$[t - \delta, t + \delta] \times \{x\} \subset O_1, \text{ for every } (t, x) \in \bar{O}_2.$$

Since

$$[t-\delta, t+\delta] \times \{x\} \subset O_1, \text{ for every } (t,x) \in \bar{O}_2,$$

and u is defined on O_1, for $\epsilon > 0$, we can define $u_\epsilon : O_2 \to \mathbb{R}$ by

$$u_\epsilon(t,x) = \max_{s \in [-\delta, +\delta]} u(t+s, x) - \frac{s^2}{\epsilon} \tag{1}$$

Note that u_ϵ is continuous by continuity of u and compactness of $[-\delta, +\delta]$. We summarize the properties of u_ϵ in the following proposition.

Proposition 2

(1) *For every* $\epsilon > 0$*, we have* $u_\epsilon \geq u$*.*

(2) *For every* $0 < \epsilon < \epsilon'$*, we have* $u_\epsilon < u_{\epsilon'}$*.*

(3) *If* $(t, x) \in O_2$*, and* $s_\epsilon \in [-\delta, +\delta]$ *is such that* $u_\epsilon(t, x) = u(t + s_\epsilon, x) - (s_\epsilon)^2/\epsilon$*, then* $|s_\epsilon| \leq \sqrt{2\epsilon m}$*.*

(4) $u_\epsilon \to u$ *uniformly on* O_2*, when* $\epsilon \to 0$*.*

(5) *If* $(t, x), (t', x) \in O_2$*, with* $|t - t'| < \delta - \sqrt{2\epsilon m}$*, then*

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m} + |t - t'|}{\epsilon} |t - t'|.$$

In particular, we have

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m} + \delta}{\epsilon} |t - t'|.$$

Moreover, for every $x \in X$*, the map* $t \mapsto u_\epsilon(t, x)$ *is Lipschitz on every connected component of* $O_2 \cap \mathbb{R} \times \{x\}$ *with Lipschitz constant* $\leq 2\sqrt{2m/\epsilon}$*.*

Proof.

Part (1) $u_\epsilon \geq u$ and (2) $u_\epsilon < u_{\epsilon'}$, for $0 < \epsilon < \epsilon'$, are obvious from the definition $u_\epsilon(t,x) = \max_{s \in [-\delta, +\delta]} u(t+s,x) - s^2/\epsilon$.

To prove part (3), namely $|s_\epsilon| \leq \sqrt{2\epsilon m}$, if $u_\epsilon(t,x) = u(t+s_\epsilon,x) - (s_\epsilon)^2/\epsilon$, we notice that

$$u_\epsilon(t,x) = u(t+s_\epsilon,x) - \frac{(s_\epsilon)^2}{\epsilon} \geq u(t,x).$$

Therefore

$$\frac{(s_\epsilon)^2}{\epsilon} \leq u(t+s_\epsilon,x) - u(t,x) \leq 2 \sup_{O_1} |u| = 2m.$$

To prove part (4), namely $u_\epsilon \to u$ uniformly on O_2, when $\epsilon \to 0$, note that by part (3), we have

$$\sup_{(x,t)\in O_2} |u_\epsilon(t,x) - u(t,x)| \leq \sup_{(t,x)\in \bar{O}_2, |s| \leq \sqrt{2\epsilon m}} |u(t+s,x) - u(t,x)|.$$

By compactness of $\bar{O}_2$ and continuity of u, the right hand side of the inequality above tends to 0 as $\epsilon \to 0$.

To prove part (5), namely:
If $(t,x),(t',x) \in O_2$, with $|t-t'| < \delta - \sqrt{2\epsilon m}$, then

$$|u_\epsilon(t',x) - u_\epsilon(t,x)| \leq \frac{2\sqrt{2\epsilon m} + |t-t'|}{\epsilon}|t-t'|,$$

we choose s_ϵ such that $u_\epsilon(t,x) = u(t+s_\epsilon,x) - (s_\epsilon)^2/\epsilon$. By part (3), we have $|s_\epsilon| \leq \sqrt{2\epsilon m}$. Therefore, we get

$$|s_\epsilon + t - t'| \leq |s_\epsilon| + |t-t'| \leq \sqrt{2\epsilon m} + \delta - \sqrt{2\epsilon m} = \delta.$$

Hence, by the definition of u_ϵ, we obtain

$$\begin{aligned} u_\epsilon(t',x) &\geq u(t' + (s_\epsilon + t - t'), x) - \frac{(s_\epsilon + t - t')^2}{\epsilon} \\ &= u(t+s_\epsilon, x) - \frac{(s_\epsilon + t - t')^2}{\epsilon}. \end{aligned}$$

Subtracting this last inequality

$$u_\epsilon(t', x) \geq u(t + s_\epsilon, x) - \frac{(s_\epsilon + t - t')^2}{\epsilon}$$

from the equality $u_\epsilon(t, x) = u(t + s_\epsilon, x) - (s_\epsilon)^2/\epsilon$ yields

$$\begin{aligned} u_\epsilon(t, x) - u_\epsilon(t', x) &\leq \frac{(s_\epsilon + t - t')^2}{\epsilon} - \frac{(s_\epsilon)^2}{\epsilon} \\ &= \frac{(2s_\epsilon + t - t')(t - t')}{\epsilon} \\ &\leq \frac{2|s_\epsilon| + |t - t'|}{\epsilon}|t - t'| \\ &\leq \frac{2\sqrt{2\epsilon m} + |t - t'|}{\epsilon}|t - t'|, \end{aligned}$$

where we used $|s_\epsilon| \leq \sqrt{2\epsilon m}$, for the last inequality. By symmetry, we obtain

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m} + |t - t'|}{\epsilon}|t - t'|. \tag{2}$$

To finish the proof of part (5), we must show that,for t, t', x with $[t, t'] \times \{x\} \subset O_2$, we have

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m}}{\epsilon}|t - t'|.$$

For $\eta < \delta - \sqrt{2\epsilon m}$, pick $t = t_0 < t_1 < \cdots < t_n = t'$, with $|t_{i+1} - t_i| \leq \eta$. By applying (2) for t_i, t_{i+1} instead of t, t', we get

$$\begin{aligned} |u_\epsilon(t_{i+1}, x) - u_\epsilon(t_i, x)| &\leq \frac{2\sqrt{2\epsilon m} + |t_{i+1} - t_i|}{\epsilon}|t_{i+1} - t_i,| \\ &\leq \frac{2\sqrt{2\epsilon m} + \eta}{\epsilon}|t_{i+1} - t_i,|. \end{aligned}$$

Adding the inequalities, we obtain

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m} + \eta}{\epsilon}|t - t'|.$$

We can then let $\eta \to 0$, to conclude that

$$|u_\epsilon(t', x) - u_\epsilon(t, x)| \leq \frac{2\sqrt{2\epsilon m}}{\epsilon}|t - t'| = 2\sqrt{\frac{2m}{\epsilon}}|t - t'|.$$

This finishes the proof of (5), and also of the Proposition. □

Proposition 3

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian. If $u : V \to \mathbb{R}$ is a continuous function, defined on the open subset $V \subset \mathbb{R} \times M$, which is a viscosity subsolution on V of

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0. \tag{3}$$

Then for every compact subset $K \subset V$, we can find a sequence of continuous functions $u_n : K \to \mathbb{R}$, such that $u_n \to u$ uniformly on K, and for every n, the function u_n is a viscosity subsolution on the interior $\mathring{K}$ of K, not only of the same evolutionary Hamilton-Jacobi equation (3)*, but also of*

$$|\partial_t u_n(t,x)| + H(x, \partial_x u_n(t,x)) = C\sqrt{n},$$

for some $C < +\infty$.
In particular if H is coercive above each compact subset of M, then the u_n's are locally Lipschitz on $\mathring{K}$.

Proof. As was done above, we choose an open subset $O_1 \supset K$ whose closure $\bar{O}_1$ is compact and contained in V and we set

$$m = \sup_{\bar{O}_1} |u| < +\infty,$$

then we find $\delta > 0$ and an open neighborhood $O_2 \subset O_1$ of K, with compact closure $\bar{O}_2 \subset O_1$ such that

$$[t-\delta, t+\delta] \times \{x\} \subset O_1, \text{ for every } (t,x) \in \bar{O}_2.$$

For $1/n < \delta$, we then set $\hat{u}_n = u_{1/n} : O_2 \to \mathbb{R}$, where the function $u_{1/n}$ is defined as above by

$$\hat{u}_n(t,x) = \max_{s \in [-\delta, +\delta]} u(t+s, x) - ns^2.$$

By part (4) of Proposition 2, we indeed get the uniform convergence of $\hat{u}_n$ to u.
We will now check the fact that $\hat{u}_n$ is a viscosity subsolution of both Hamilton-Jacobi equations on O_2.

Assume $(t_0, x_0) \in O_2$, and that $\varphi : V \to \mathbb{R}$ is C^1 is such that $\hat{u}_n \le \varphi$ with equality at (t_0, x_0). By part (5) of Proposition 2, we know that $t \mapsto \hat{u}_n(t, x)$ is locally Lipschitz with local Lipschitz constant $\le 2\sqrt{2mn}$. This implies

$$|\partial_t \varphi(t_0, x_0)| \le 2\sqrt{2mn}. \qquad (4)$$

We now choose $s_n \in [-\delta, +\delta]$ such that

$$u(t_0 + s_n, x_0) - n s_n^2 = \hat{u}_n(t_0, x_0) = \varphi(t_0, x_0).$$

Since $(t_0, x_0) \in O_2$, we can find $\eta > 0$ and a neighborhood W of x_0 in M , such that $(t_0 + s, y) \in O_2$, for $|s| < \eta$ and all $y \in W$. By the definition of $\hat{u}_n$, for $(t, x) \in O_2$, we have

$$\hat{u}_n(t, x) = \max_{s \in [-\delta, +\delta]} u(t + s, x) - n s^2,$$

Therefore, for $|s| < \eta$ and $y \in W$, we get

$$u(t_0 + s + s_n, y) - n s_n^2 \le \hat{u}_n(t_0 + s, y) \le \varphi(t_0 + s, y).$$

Subtracting from this inequality the equality $u(t_0 + s_n, x_0) - n s_n^2 = \varphi(t_0, x_0)$, we obtain

$$u(t_0+s+s_n, y) - u(t_0+s_n, x_0) \le \varphi(t_0+s, y) - \varphi(t_0, x_0), \text{for } |s| < \eta, y \in W.$$

The last inequality, for $|s| < \eta, y \in W$,

$$u(t_0 + s + s_n, y) - u(t_0 + s_n, x_0) \leq \varphi(t_0 + s, y) - \varphi(t_0, x_0)$$

can be rewritten as

$$u(t_0 + s + s_n, y) \leq \varphi(t_0 + s, y) - \varphi(t_0, x_0) + u(t_0 + s_n, x_0).$$

Since this inequality is an equality at $s = 0$ and u is a viscosity subsolution of

$$\frac{\partial u}{\partial t}(t, x) + H\big(x, \frac{\partial u}{\partial x}(t, x)\big) = 0,$$

we must have

$$\partial_t \varphi(t_0, x_0) + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 0. \tag{5}$$

Therefore $\hat{u}_n$ is a viscosity subsolution of

$$\frac{\partial \hat{u}_n}{\partial t}(t, x) + H\big(x, \frac{\partial \hat{u}_n}{\partial x}(t, x)\big) = 0.$$

Using the already established inequalities (4)

$$|\partial_t \varphi(t_0, x_0)| \leq 2\sqrt{2mn}$$

and (5)

$$\partial_t \varphi(t_0, x_0) + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 0,$$

we also obtain

$$|\partial_t \varphi(t_0, x_0)| + H(x_0, \partial_x \varphi(t_0, x_0)) \leq 4\sqrt{2mn}.$$

Therefore $\hat{u}_n$ is a viscosity subsolution of

$$\left|\frac{\partial \hat{u}_n}{\partial t}(t,x)\right| + H\big(x, \frac{\partial \hat{u}_n}{\partial x}(t,x)\big) = C\sqrt{n},$$

with $C = 4\sqrt{2m}$. $\square$

We can now eliminate the need of a locally Lipschitz function in the Maximum Principle, by imposing coercivity above compact subsets.

Theorem 4 (Maximum Principle)

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive above every compact subset of M. Assume that $u, v : [a, b] \times C \to \mathbb{R}$, with $C \subset M$ compact, are continuous and respectively a viscosity subsolution and a viscosity supersolution of

$$\partial_t U + H(x, \partial_x U) = 0 \tag{6}$$

on $]a, b[\times \mathring{C}$. Then the maximum of $u - v$ on $[a, b] \times C$ is achieved on $[a, b] \times \partial C \cup \{a\} \times C$.

Proof. In fact, for every $\epsilon > 0$, with $a + \epsilon < b - \epsilon$ and every compact subset $\tilde{C} \subset \mathring{C}$, we can find a sequence of locally Lipschitz functions $u_n : V \to \mathbb{R}$, defined on the neighborhood V of $[a + \epsilon, b - \epsilon] \times \tilde{C}$ which are locally Lipschitz and viscosity subsolutions of (6) on V and converging uniformly to u on V.

Since u_n is locally Lipschitz, we can apply the already obtained maximum principle and get

$$\max_{[a+\epsilon,b-\epsilon]\times\tilde{C}} u_n - v = \max_{[a+\epsilon,b-\epsilon]\times\partial\tilde{C}\cup\{a+\epsilon\}\times\tilde{C}} u_n - v.$$

By uniform convergence of $u_n - V$ to $u - V$ on the compact set $[a+\epsilon, b-\epsilon] \times \tilde{C}$, we obtain

$$\max_{[a+\epsilon,b-\epsilon]\times\tilde{C}} u - v = \max_{[a+\epsilon,b-\epsilon]\times\partial\tilde{C}\cup\{a+\epsilon\}\times\tilde{C}} u - v.$$

For $\ell > 0$, define $C_\ell = \{x \in C \mid \bar{B}(x, 1/\ell) \subset C\}$. From what we just obtained

$$\max_{[a+\ell^{-1},b-\ell^{-1}]\times C_\ell} u - v = \max_{[a+\ell^{-1},b-\ell^{-1}]\times\partial C_\ell\cup\{a+\ell^{-1}\}\times C_\ell} u - v. \quad (7)$$

We obviously have $C_\ell \subset C_{\ell+1}$ and $\cup_\ell C_\ell = \mathring{C}$. Continuity of $u - v$ then implies $\max_{[a+\ell^{-1},b-\ell^{-1}]\times C_\ell} u - v \nearrow \max_{[a,b]\times C} u - v$. It can be checked that $\partial C_\ell \subset \bar{V}_{1/\ell}(\partial C) = \{x \in M \mid d(x, \partial C) \leq 1/\ell\}$, which by uniform continuity of $u - v$ implies that

$$\limsup_\ell \max_{[a+\ell^{-1},b-\ell^{-1}]\times\partial C_\ell\cup\{a+\ell^{-1}\}\times C_\ell} u - v \leq \max_{[a,b]\times\partial C\cup\{a\}\times C} u - v.$$

Passing to the limit in (7) yields

$$\max_{[a,b]\times C} u - v \leq \max_{[a,b]\times\partial C\cup\{a\}\times C} u - v,$$

which finishes to show that the general maximum principle holds. □

Corollary 5

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive above each compact subset of M and convex in the momentum p. Let $u : V \to \mathbb{R}$ be a continuous function defined on the open subset $V \subset \mathbb{R} \times M$ which is viscosity subsolution of the evolutionary Hamilton-Jacobi equation

$$\partial_t u(t,x) + H(x, \partial_x u(t,x)) = 0.$$

For every open set $V' \subset V$ whose closure $\bar{V}'$ is compact and contained in V, we can approximate uniformly u on V' by a C^∞ solution of the same evolutionary Hamilton-Jacobi equation.

Proof By Proposition 3 above, we can make a first approximation by a viscosity subsolution $u_1 : V' \to \mathbb{R}$ of

$$\partial_t u(t,x) + H(x, \partial_x u(t,x)) = 0,$$

which is locally Lipschitz on V'.

Therefore the function $u_2 : V' \to \mathbb{R}, (t, x) \to u_1(t, x) - \epsilon t$ is a locally Lipschitz viscosity subsolution of

$$\partial_t u + H(x, \partial_x u) = -\epsilon.$$

Note also that the variable t is bounded on the compact subset $\bar{V}'$ of $\mathbb{R} \times M$.

Hence by choosing appropriately ϵ, we can assume u_2 uniformly as close to u_1 as we wish. We can now consider the Hamiltonian $\hat{H} : T^*(\mathbb{R} \times M)$ defined by

$$\hat{H}(t, s, x, p) = s + H(x, p),$$

where we use the identification $T^*(\mathbb{R} \times M) = T^*\mathbb{R} \times T^*M = \mathbb{R} \times \mathbb{R} \times T^*M$.

With this identification, we get that the function u_2 is a locally Lipschitz viscosity subsolution of

$$\hat{H}(t, x, Du(t, x)) = -\epsilon.$$

Since the Hamiltonian $\hat{H}(t, s, x, p)$ is convex in the momentum (s, p), we can now invoque one of the properties of viscosity subsolutions we recalled in the first lecture, and approximate uniformly u_2 on V' by a C^∞ viscosity subsolution $u_3 : V' \to \mathbb{R}$ of

$$\hat{H}(t, x, Du(t, x)) = 0.$$

This means that u_3 is a uniform approximation of u, which is a viscosity subsolution of the evolutionary Hamilton-Jacobi equation

$$\partial_t u(t, x) + H(x, \partial_x u(t, x)) = 0. \quad \square$$

Domination and Subsolutions

From now on we will assume that the Hamiltonian $H : T^*M \to \mathbb{R}$ is Tonelli and $L : TM \to \mathbb{R}$ is its associated Lagrangian defined by

$$L(x, v) = \sup_{p \in T_x^* M} p(v) - H(x, p).$$

This definition implies the Fenchel inequality

$$p(v) \leq L(x, v) + H(x, p), \text{ for all } x \in M, v \in T_x M, p \in T_x^* M.$$

Recall now the definition of evolution dominated from the first lecture:

Definition 6 (Evolution dominated)

The function $U : [0, +\infty[\times M \to [-\infty, +\infty]$ is evolution dominated by the Lagrangian L if for every piecewise C^1 curve $\gamma : [a, b] \to M$, with $0 \leq a < b$, we have

$$U(b, \gamma(b)) \leq U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$$

or equivalently

$$U(t + s, x) \leq U(t, y) + h_s(y, x), \text{ for all } x, y \in M, t \geq 0, s > 0.$$

As we will presently explain for a function $U : S \to \mathbb{R}$, defined on a more general subset $S \subset \mathbb{R} \times M$, this last definition suggests two possible different definitions of domination.
For the definition of domination, we could impose the equality

$$U(b, \gamma(b)) \leq U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$$

for all curves $\gamma : [a, b] \to M$, such that $(a, \gamma(a))$ and $(b, \gamma(b))$ are both in S or only for curves whose graph $\mathrm{Graph}(\gamma)$ is contained in S, where as usual

$$\mathrm{Graph}(\gamma) = \{(t, \gamma(t)) \mid t \in [a, b]\}.$$

Obviously, this last possible definition is more natural, but the first one is useful because it is related to the minimal action and provide us with minimizers, which have good regularity properties.

Definition 7 (Evolution domination by a Lagrangian)

We will say that the function $U : S \to [-\infty, +\infty]$, where $S \subset \mathbb{R} \times M$ is evolution dominated by L on S, if for every piecewise C^1 curve $\gamma : [a, b] \to M$, with $a < b \in \mathbb{R}$ and $\mathrm{Graph}(\gamma) \subset S$, we have

$$U(b, \gamma(b)) \leq U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds. \tag{8}$$

Moreover, we will say that such a $U : S \to [-\infty, +\infty]$ is *strongly* evolution dominated by L on S, if for every $(t, x), (t', x') \in S$, with $t < t'$, it satisfies the stronger condition

$$U(t', x') \leq U(t, x) + h_{t'-t}(x, x').$$

Of course strong domination is equivalent to: (8) holds for all piecewise C^1 curves $\gamma : [a, b] \to M$ such that $(a, \gamma(a))$ and $(b, \gamma(b))$ are both in S.

Remarks 8

1) Note that, when $U(a, \gamma(a))$ is finite, the domination inequality

$$U(b, \gamma(b)) \leq U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds$$

is equivalent to

$$U(b, \gamma(b)) - U(a, \gamma(a)) \leq \mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, dt.$$

2) If $S \subset \mathbb{R} \times M$ is of the form $S = I \times M$, where I is an interval in $\mathbb{R}$, then $U : I \times M \to [-\infty, +\infty]$ is evolution dominated by L if and only if it is *strongly* evolution dominated by L.
3) In the next proposition we will show that an evolution dominated function defined on an open subset is locally strongly dominated.

Proposition 9 *Suppose $U : O \to \mathbb{R}$ is finite valued and evolution dominated by L on the open subset $O \subset \mathbb{R} \times M$. Then U is locally bounded on O. Moreover, the function U is locally strongly evolution dominated by L, i.e. for every $(t_0, x_0) \in O$ there exists a neighborhood $V \subset O$ of (t_0, x_0) such that the restriction $U|V$ is strongly evolution dominated by L on V.*

Proof. For $(t_0, x_0) \in O$, fix $[t_0 - 2\delta, t_0 + 2\delta] \times \bar{B}(x_0, 3r) \subset O$, with $\delta, r > 0$. For any $x \in \bar{B}(x_0, 2r)$ and any $t \in [t_0 - \delta, t_0 + \delta]$, the minimizing geodesic $\gamma_{x_0,x} : [t_0 - 2\delta, t] \to M$ joining x_0 to x is contained in $\bar{B}(x_0, 2r)$ and satisfies

$$\|\dot{\gamma}_{x_0,x}(s)\|_{\gamma_{x_0,x}(s)} = \frac{d(x_0, x)}{t - (t_0 - 2\delta)} \leq \frac{2r}{\delta},$$

since $t - (t_0 - 2\delta) \geq \delta$ and $d(x_0, x) \leq 2r$. Hence, using $t - (t_0 - 2\delta) \leq 3\delta$, the action $\mathbb{L}(\gamma_{x_0,x})$ satisfies

$$\mathbb{L}(\gamma_{x_0,x}) \leq [t - (t_0 - 2\delta)]A(2r/\delta) \leq 3\delta A(2r/\delta),$$

where $A(R) = \sup\{L(x, v) \mid \|v\|_x \leq R\}$, see Lecture 1.

Since, the function is evolution dominated by L on $O \supset [t_0 - 2\delta, t_0 + 2\delta] \times \bar{B}(x_0, 3r)$, we obtain

$$U(t,x) \leq U(t_0 - 2\delta, x_0) + \mathbb{L}(\gamma_{x_0,x})$$
$$\leq U(t_0 - 2\delta, x_0) + 3\delta A(2r/\delta).$$

This shows that U is bounded above on the compact neighborhood $[t_0 - \delta, t_0 + \delta] \times \bar{B}(x_0, 2r)$ of (t_0, x_0). In the same way the geodesic $\gamma_{x,x_0} : [t, t_0 + 2\delta] \to M$ joining x to x_0 is contained in $\bar{B}(x_0, 2r)$ satisfies

$$\mathbb{L}(\gamma_{x,x_0}) \leq (t_0 + 2\delta - t)A(d(x, x_0)/(t_0 + 2\delta - t)) \leq 3\delta A(2r/\delta).$$

Therefore $U(t_0 + 2\delta, x_0) \leq U(t,x) + 3\delta A(2r/\delta)$, which implies that U is bounded below on $[t_0 - \delta, t_0 + \delta] \times \bar{B}(x_0, 2r)$.
We then set

$$K = 2 \sup\{|U(t,x)| \mid (t,x) \in [t_0 - \delta, t_0 + \delta] \times \bar{B}(x_0, 2r)\} < +\infty.$$

Fix $(t,x), (t',x') \in [t_0 - \delta, t_0 + \delta] \times \bar{B}(x_0, 2r)$, with $t < t'$.
For $h_{t'-t}(x, x') \geq K$, we obviously get

$$U(t', x') - U(t,x) \leq K \leq h_{t'-t}(x, x').$$

It remains to show, for $t < t'$, that

$$U(t',x') - U(t,x) \leq h_{t'-t}(x,x'), \text{ for } h_{t'-t}(x,x') \leq K,$$

assuming (t,x) and (t',x') in a smaller neighborhood of (t_0,x_0). If $h_{t'-t}(x,x') \leq K$, pick a minimizer $\gamma : [t,t'] \to M$, with $\gamma(t) = x, \gamma(t') = x'$ and $h_{t'-t}(x,x') = \mathbb{L}(\gamma) \leq K$. By the superlinearity of L, for all $\tilde{K} \geq 0$, we have

$$K \geq h_{t'-t}(x,x') = \mathbb{L}(\gamma) \geq \tilde{K}\ell_g(\gamma) - C(\tilde{K})(t'-t),$$

or equivalently

$$\ell_g(\gamma) \leq \frac{K}{\tilde{K}} + \frac{C(\tilde{K})}{\tilde{K}}(t'-t).$$

Choose $\tilde{K} > 0$ such that $\frac{K}{\tilde{K}} \leq \frac{r}{2}$ and, once this $\tilde{K} > 0$ is fixed, choose $\epsilon > 0$, with $\epsilon \leq \delta$, such that $2\epsilon\frac{C(\tilde{K})}{\tilde{K}} \leq \frac{r}{2}$. Hence, if we further assume that

$$(t,x),(t',x') \in [t_0-\epsilon, t_0+\epsilon] \times \bar{B}(x_0,r),$$

we obtain $\ell_g(\gamma) \leq r$ and $\gamma([t,t']) \subset \bar{B}(x_0,2r)$.

Since $\mathrm{Graph}(\gamma) \subset [t_0 - \delta, t_0 + \delta] \times \bar{B}(x_0, 2r) \subset O$ and U is evolution dominated by L on O, we get

$$U(t', x') - U(t, x) \leq h_{t'-t}(x, x').$$

This finishes to prove that U is *strongly* evolution dominated by L on $[t_0 - \epsilon, t_0 + \epsilon] \times \bar{B}(x_0, r)$. □

We now show that evolution domination is the same as viscosity subsolution.

Proposition 10

Let H be a Tonelli Hamiltonian on the complete Riemannian manifold M. Suppose $U : O \to \mathbb{R}$ is a continuous function defined on the open subset $O \subset \mathbb{R} \times M$. Then U is a viscosity subsolution of

$$\partial_t U + H(x, \partial_x U) = 0,$$

on O if and only if it is evolution dominated by L on O.

Proof.

Assume that U is a viscosity subsolution of

$$\partial_t U + H(x, \partial_x U) = 0. \tag{9}$$

We prove that

$$U(b, \gamma(b)) - U(a, \gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s))\, dt,$$

holds for a piecewise C^1 curve $\gamma : [a, b] \to M$, with $\mathrm{Graph}(\gamma) \subset O$. If U is smooth, the Fenchel inequality

$$p(v) \leq L(x, v) + H(x, p), \text{ for all } x \in M, v \in T_x M, p \in T_x^* M$$

yields

$$\partial_x U(t, x)(v) \leq L(x, v) + H(x, \partial_x U(t, x)), \text{ for all } v \in T_x M.$$

Since the viscosity subsolution U of (9) is smooth on O, we have

$$\partial_t U(t, x) + H\left(x, \partial_x U(t, x)\right) \leq 0, \text{ everywhere on } O.$$

We combine the two inequalities

$$\partial_x U(t,x)(v) \leq L(x,v) + H(x, \partial_x U(t,x)), \text{ for all } v \in T_x M$$

and

$$\partial_t U(t,x) + H\left(x, \partial_x U(t,x)\right) \leq 0, \text{ everywhere on } O,$$

to obtain

$$\partial_t U(t,x) + \partial_x U(t,x)(v) \leq L(x,v),$$

for all (t,x,v) with $(t,x) \in O$ and $v \in T_x M$. Therefore, since $\mathrm{Graph}(\gamma) \subset O$ and γ is piecewise C^1, we obtain

$$\partial_t U(t,\gamma(t)) + \partial_x U(t,\gamma(t))(\dot\gamma(t)) \leq L(\gamma(t),\dot\gamma(t)), \text{ for all } t \in [a,b].$$

By integration, this proves the desired inequality.

$$U(b,\gamma(b)) - U(a,\gamma(a)) \leq \int_a^b L(\gamma(t),\dot\gamma(t))\, dt, \tag{10}$$

For U just continuous, since $\mathrm{Graph}(\gamma)$ is a compact subset of O, as we showed above, given a compact neighbourhood $V \subset O$ of $\mathrm{Graph}(\gamma)$, we can find a sequence U_n of smooth subsolutions converging to U on V. Since (10) holds for each U_n, it also holds for U.

Let us now assume that U satisfies

$$U(b,\gamma(b)) - U(a,\gamma(a)) \leq \int_a^b L(\gamma(s),\dot\gamma(s))\,ds, \qquad (11)$$

for every piecewise C^1 curve $\gamma : [a,b] \to M$, with $\mathrm{Graph}(\gamma) \subset O$.
What follows is a repeat of an argument in Lecture 1.
To prove that U is a viscosity subsolution of

$$\partial_t U + H(x, \partial_x U) = 0,$$

consider a C^1 function $\Phi : O \to \mathbb{R}$, with $\Phi \geq U$ and $\Phi(t,x) = U(t,x)$, for some $(t,x) \in O$. If $v \in T_x M$, let $\gamma : [t-1,t] \to M$ be a smooth curve with $\gamma(t) = x$ and $\dot\gamma(t) = v$. Since γ is continuous and O is open for $\epsilon > 0$ small enough, we have $\mathrm{Graph}(\gamma|[t-\epsilon,t]) \subset O$. Using $\Phi \geq U$ and inequality (11), we get

$$\begin{aligned}\Phi(t,\gamma(t)) - \Phi(t-\epsilon,\gamma(t-\epsilon)) &\leq U(t,\gamma(t)) - U(t-\epsilon,\gamma(t-\epsilon))\\ &\leq \int_{t-\epsilon}^t L(\gamma(s),\dot\gamma(s))\,ds.\end{aligned}$$

Dividing this just obtained inequality

$$\Phi(t, \gamma(t)) - \Phi(t - \epsilon, \gamma(t - \epsilon)) \le \int_{t-\epsilon}^{t} L(\gamma(s), \dot{\gamma}(s))\, ds$$

by ϵ, and letting $\epsilon \to 0$ yields

$$\partial_t \Phi(t, x) + \partial_x \Phi(t, x)(v) \le L(x, v),$$

or equivalently

$$\partial_t \Phi(t, x) + \partial_x \Phi(t, x)(v) - L(x, v) \le 0.$$

Taking the supremum over all $v \in T_x M$, we obtain

$$\partial_t \Phi(t, x) + H(x, \partial_x \Phi(t, x)) \le 0. \quad \square$$

Local construction of viscosity solutions

Before starting this local construction, we recall the well-known result:

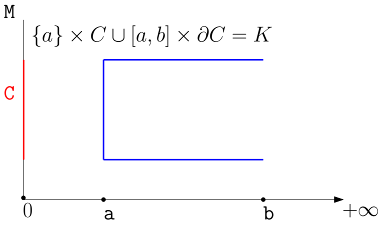
Proposition 11

The function $(t, x, y) \mapsto h_t(x, y)$ *is continuous on* $]0, +\infty[\times M \times M$.

In fact, this function is locally Lipschitz and even locally semi-concave.

Consider a compact subset $C \subset M$ and $a, b \in \mathbb{R}$, with $a < b$. Define the compact set $K = K(a, b, C)$ is defined by

$$K = [a, b] \times \partial C \cup \{a\} \times C.$$



Suppose that $U : K \to \mathbb{R}$ is a continuous function. We extend the function U defined on $K = [a,b] \times \partial C \cup \{a\} \times C$ to a function $\hat{U} : [a,b] \times C \to \mathbb{R}$ by defining it for $(t,x) \in]a,b] \times \mathring{C}$ as

$$\hat{U}(t,x) = \inf\{U(t',x') + h_{t-t'}(x',x) \mid t' < t, (t',x') \in K\}.$$

Proposition 12

The function $\hat{U}$ is continuous and strongly dominated by L on $]a,b] \times \mathring{C}$.

The strong domination results from the definition of $\hat{U}$ and the semi-group property of $h_t(x,y)$. The continuity of $\hat{U}$ follows from the continuity of $(t,x,y) \mapsto h_t(x,y)$ on $]0,+\infty[\times M \times M$ and the next Lemma.

Lemma 13

For every $(t_0,x_0) \in]a,b] \times \mathring{C}$, we can find a neighborhood V of (t_0,x_0) in $]a,b] \times \mathring{C}$ and $\epsilon > 0$, such that for all $(t,x) \in V$

$$\hat{U}(t,x) = \inf\{U(t',x') + h_{t-t'}(x',x) \mid t' \leq t - \epsilon, (t',x') \in K\}.$$

In particular, for $(t_0,x_0) \in]a,b] \times \mathring{C}$, we can find $(t',x') \in K$, with $t' < t$ such that $\hat{U}(t,x) = U(t',x') + h_{t_0-t'}(x',x_0)$.

Proof. For all $(t,x) \in]a,b] \times \mathring{C}$, we have

$$\hat{U}(t,x) \leq U(a,x) + h_{t-a}(x,x) \leq \sup_K U + (b-a)A(0) < +\infty,$$

where as before $A(0) = \sup_{x \in M} L(x,0)$. Set $\kappa = \sup_K U + (b-a)A(0) < +\infty$. We deduce that

$$\hat{U}(t,x) = \inf\{U(t',x') + h_{t-t'}(x',x) \mid t' < t, (t',x') \in K \text{ with } U(t',x') + h_{t-t'}(x',x) \leq \kappa\}.$$

Fix now $(t_0,x_0) \in]a,b] \times \mathring{C}$. Choose $r > 0$ and $\delta > 0$ such that $\bar{B}(x_0,2r) \subset \mathring{C}$ and $t_0 - \delta > a$.
For $(t,x) \in [t_0-\delta,b] \times \bar{B}(x_0,r)$, we have either

$$\hat{U}(t,x) = \inf\{U(a,x') + h_{t-a}(x',x) \mid x' \in C\},$$

in which case $t - a \geq t_0 - \delta - a > 0$ and the claim in the lemma is true for any $\epsilon \in]0, t_0 - \delta - a[$, or

$$\hat{U}(t,x) = \inf\{U(t',x') + h_{t-t'}(x',x) \mid t' < t, x' \in \partial C \text{ with } U(t',x') + h_{t-t'}(x',x) \leq \kappa\}.$$

Therefore to finish the proof of this Lemma 13, it suffices to show that

$$\inf\{t-t' \,|\, t' < t, \exists x \in \bar{B}(x_0, r), x' \in \partial C \text{ with } U(t', x') + h_{t-t'}(x', x) \leq \kappa\} > 0.$$

Suppose that we are given $x \in \bar{B}(x_0, r), x' \in \partial C$ such that

$$U(t', x') + h_{t-t'}(x', x) \leq \kappa.$$

Setting $m = \inf_{(t',x') \in K} U(t', x')$, we obtain

$$h_{t-t'}(x', x) \leq \kappa - m.$$

From the supelinearity of L, for every $\tilde{K} \geq 0$, we conclude

$$m - C(\tilde{K})(t - t') + \tilde{K} d(x, x') \leq \kappa.$$

We now note that $d(x, x') \geq r$, since
$x \in \bar{B}(x_0, r) \subset \bar{B}(x_0, 2r) \subset \mathring{C}$ and $x' \in \partial C$. Therefore

$$\tilde{K} r - \kappa + m \leq C(\tilde{K})(t - t').$$

We can choose $\tilde{K}$ such that $\tilde{K} r - \kappa + m \geq 1$. Hence

$$t - t' \geq \frac{1}{C(\tilde{K})}. \quad \square$$

Recall that the function $\hat{U} : [a, b] \times C \to \mathbb{R}$ is equal to the continuous function U on $K = [a, b] \times \partial C \cup \{a\} \times C$ and is defined by

$$\hat{U}(t, x) = \inf\{U(t', x') + h_{t-t'}(x', x) \mid t' < t, (t', x') \in K\},$$

for all $(t, x) \in]a, b] \times \mathring{C}$.

Proposition 14

If U is the restriction to $K = [a, b] \times \partial C \cup \{a\} \times C$ of a continuous function $V : [a, b] \times C \to \mathbb{R}$ which is strongly dominated by L on $[a, b] \times C$, then $\hat{U} \geq V$ everywhere. Moreover, $\hat{U}$ is continuous, strongly dominated by L on $[a, b] \times C$ and a viscosity solution of the Hamilton-Jacobi equation

$$\partial_t U + H(x, \partial_x U) = 0, \text{ on }]a, b[\times \mathring{C}.$$

Proof.

The inequality $\hat{U} \geq V$ is an easy consequence of the strong domination of V by L.

To prove the continuity of $\hat{U}$ on $[a,b]\times C$, since the previous proposition showed the continuity of $\hat{U}$ on $]a,b]\times \mathring{C}$, we have to prove continuity at points in $K=[a,b]\times \partial C\cup\{a\}\times C$.
We first prove it at a point in $\{a\}\times C$. Since $\hat{U}=U$ on K, it suffices to show that for a sequence $(t_n,x_n)\to(a,x)$, with $t_n>a$, we have $\hat{U}(t_n,x_n)\to U(a,x)$. This follows from

$$\begin{aligned} V(t_n,x_n)\leq \hat{U}(t_n,x_n) &\leq U(a,x_n)+h_{t_n-a}(x_n,x_n)\\ &\leq U(a,x_n)+A(0)(t_n-a), \end{aligned}$$

since $\lim_n V(t_n,x_n)=\lim_n U(a,x_n)=U(a,x)$ by continuity of V.
It remains to prove continuity of $\hat{U}$ at a point (t,x), with $t\in]a,b]$ and $x\in\partial C$. Suppose $(t_n,x_n)\to(t,x)$. Since U is continuous on $K=[a,b]\times\partial C\cup\{a\}\times C$, we can assume $x_n\in\mathring{C}$. Fix $t'\in[a,t[$, for n large enough $t'<t_n$, since $t_n\to t>t'$. For such large n, we have $V(t_n,x_n)\leq\hat{U}(t_n,x_n)\leq U(t',x)+h_{t_n-t'}(x_n,x)$. Passing to the limit, we get

$$\begin{aligned} U(t,x)\leq \liminf_n \hat{U}(t_n,x_n) &\leq \limsup_n \hat{U}(t_n,x_n)\\ &\leq U(t',x)+h_{t-t'}(x,x)\leq U(t',x)+A(0)(t-t'). \end{aligned}$$

If, in the just obtained inequality

$$U(t,x) \le \liminf_n \hat{U}(t_n,x_n) \le \limsup_n \hat{U}(t_n,x_n) \le U(t',x)+A(0)(t-t'),$$

we let $t' \to t$, we obtain

$$U(t,x) \le \liminf_n \hat{U}(t_n,x_n) \le \limsup_n \hat{U}(t_n,x_n) \le U(t,x).$$

This finishes the proof of the continuity of $\hat{U}$ on $[a,b] \times C$.
Since $\hat{U}$ is strongly dominated on $]a,b] \times \mathring{C}$ and continuous on its closure $[a,b] \times C$, by continuity of $h_t(x,y)$, it is also strongly dominated on $[a,b] \times C$.