The Ringel–Hall Lie algebra of a spherical object

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Abstract

For an integer $w$, let $S_w$ be the algebraic triangulated category generated by a $w$-spherical object. We determine the Picard group of $S_w$ and show that each orbit category of $S_w$ is triangulated and is triangle equivalent to a certain orbit category of the bounded derived category of a standard tube. Moreover, we describe the Ringel–Hall Lie algebra associated to $S_w$ in the sense of Peng and Xiao.

1. Introduction

The Ringel–Hall approach to constructing Lie algebras from triangulated categories was introduced by Peng and Xiao [12–14]. For example, any Kac–Moody Lie algebra $g$ can be constructed via this approach from the bounded derived category $D_Q$ of finite-dimensional representations of a quiver $Q$ whose underlying graph is the Dynkin diagram of $g$. This improves and generalizes previous work by Ringel [15] on a categorical explanation of Gabriel’s theorem [4] on the relation between indecomposable quiver representations and root systems. See [3, 18] for further developments in this direction. Inspired by Peng–Xiao’s work, Toën introduced in [16] the notion of a Hall algebra for algebraic triangulated categories, with certain finiteness conditions (cf. also [17, 19]), with the expectation to realize the whole quantized/universal enveloping algebra of the Kac–Moody Lie algebra $g$ from the derived category $D_Q$.

For an integer $w$ let $S_w$ be the algebraic triangulated category over a field $k$ generated by a $w$-spherical object. In [10], the authors computed the Hall algebra of $S_w$. The next step is to describe the associated Ringel–Hall Lie algebra.

In order to apply Peng–Xiao’s construction, we need to first prove that the ‘root category’ $S_w/\Sigma^2$, that is, the orbit category of $S_w$ with respect to the square of the shift functor $\Sigma$, is triangulated. In fact, we will prove that each reasonable orbit category of $S_w$ carries a nice triangle structure. Moreover, we shall provide a characterization of these categories in terms of orbit categories of bounded derived categories of standard tubes. More precisely, we have the following theorem.

**Theorem 1.1.** Let $n$ be a positive integer.

(a) There are group isomorphisms $f : \text{Aut}(S_w) \to k^\times \times \mathbb{Z}$ and $g : \text{Aut}(D^b(T_n)) \to k^\times \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$. Here $T_n$ is the standard tube of rank $n$ and $D^b(T_n)$ is the bounded derived category. In particular, $f(\Sigma) = (1, 1)$, $g(\tau) = (1, 1, 0)$ and $g(\Sigma) = (1, 0, 1)$, where $\tau$ is the Auslander–Reiten translation of $T_n$.

(b) For any $a \in k^\times$, the orbit category $S_w/f^{-1}(a, n)$ carries a canonical triangle structure such that the projection functor $S_w \to S_w/f^{-1}(a, n)$ is a triangle functor. Moreover, it is
triangle equivalent to the perfect derived category of the dg algebra \( \Lambda = k \langle s, r, r^{-1} \rangle / (s^2, sr = (-1)^{w} arcs) \) with trivial differential, where deg\((s) = w \) and deg\((r) = n \).

(c) Let \( m \) be the greatest common divisor of \( n \) and \( d = 1 - w \), \( n' = n/m \), \( d' = d/m \), let \( c \) be an inverse of \( d' \) modulo \( n' \) and let \( a, b \in k^* \) be such that \( a = ((-1)^{d}b)^{n'} \). Then the orbit category \( \mathcal{S}_{w}/f^{-1}(a, n) \) is triangle equivalent to the orbit category \( D^b(T_{nm})/g^{-1}(b, c, m) \) (which admits a canonical triangle structure by Keller \([7, \text{ Theorem 9.9}]\)).

In particular, the orbit category \( \mathcal{S}_{w}/\Sigma^2 \) is triangle equivalent to the root category of the standard homogeneous tube (that is, the standard tube of rank 1) if \( w \) is odd, and to the cluster tube of rank 2 if \( w \) is even. This characterization helps us to obtain the following description of the associated Ringel–Hall Lie algebra.

**Theorem 1.2.** Let \( \mathfrak{g} \) denote the Ringel–Hall Lie algebra of \( \mathcal{S}_{w}/\Sigma^2 \) with scalar extended to \( \mathbb{Q} \).

(a) If \( w \) is odd, then \( \mathfrak{g} \) is isomorphic to the infinite-dimensional Heisenberg Lie algebra.

(b) If \( w \) is even, then the centre of \( \mathfrak{g} \) is infinite-dimensional. The quotient of \( \mathfrak{g} \) by its centre has a basis \( \{a_x \mid x \in \mathbb{N} \cup \{0\}\} \cup \{b_y, c_y \mid y \in \mathbb{N} - \frac{1}{2}\} \) and the structure constants are

1. \( [a_x, a_{x'}] = 0 \), \( [b_y, b_{y'}] = 0 \), \( [c_y, c_{y'}] = 0 \);
2. \( [a_x, b_y] = b_{y+x} + \text{sgn}(y-x)b_{|y-x|} \);
3. \( [a_x, c_y] = -c_{y+x} - \text{sgn}(y-x)c_{|y-x|} \);
4. \( [b_y, c_{y'}] = 2a_{y+y'} - 2a_{|y-y'|} \),

where for an integer \( r \), \( \text{sgn}(r) = 1 \) if \( r \) is positive and \( \text{sgn}(r) = -1 \) if \( r \) is negative.

We remark that the cluster tube of rank 2 is not proper, and we refer to [18] for Lie algebras constructed from non-proper 2-periodic triangulated categories (cf. also Section 4). The Lie algebra obtained in Theorem 1.2(b) seems likely to be the first Lie algebra computed from a non-proper 2-periodic triangulated category.

This paper is organized as follows. In Section 2, we give some preliminary results, including results on derived categories of dg categories, triangulated orbit categories, the algebraic triangulated category \( \mathcal{S}_{w} \) generated by a \( w \)-spherical object and the bounded derived category of a standard tube. In particular, we prove part (a) of Theorem 1.1. In Section 3, we prove Theorem 1.1(b) and (c). The proof of the first part of (b) is a variant of Keller’s proof of Theorem 4 \([7]\). In the proof of the second part of (b) and the part (c), we compute the dg algebras for both orbit categories and compare them. Section 4 is devoted to the description (Theorem 1.2) of the Ringel–Hall Lie algebra associated to \( \mathcal{S}_{w}/\Sigma^2 \). In an appendix, by using covering and the universal property of orbit categories, we construct an explicit triangle equivalence between the two orbit categories in Theorem 1.1(c) in the case when \( n \) is even and \( a = b = 1 \).

2. Preliminaries

Let \( k \) be a field.

2.1. The derived category of a dg category

We follow [6]. Let \( \mathcal{A} \) be a differential graded (=dg) \( k \)-category (we identify a dg \( k \)-algebra with a dg \( k \)-category with one object). Let \( \text{Diff}\mathcal{A} \) be the dg category of (right) dg modules over \( \mathcal{A} \). For two dg modules \( M \) and \( N \) over \( \mathcal{A} \) and for an integer \( n \), the degree \( n \) component of the
morphism complex $\text{Hom}_{\text{Dif}_A}(M, N)$ consists of the homogeneous morphisms from $M$ to $N$ of degree $n$; here $M$ and $N$ are considered as graded modules over the underlying graded category of $A$. The differential of $\text{Hom}_{\text{Dif}_A}(M, N)$ is induced from the differentials of $M$ and $N$. The shift of complexes is a dg functor $\Sigma : \text{Dif}_A \to \text{Dif}_A$, which takes a homogeneous morphism $f$ of degree $n$ to $(-1)^n f$.

The derived category $\mathcal{D}_A$ of $A$ has the same objects as $\text{Dif}_A$ and its morphisms are obtained from the closed morphisms in $\text{Dif}_A$ of degree 0 by formally inverting all quasi-isomorphisms. It is triangulated with suspension functor $\Sigma$ the shift functor. Let $\text{per}_A$ denote the smallest triangulated subcategory of $\mathcal{D}_A$ containing all free dg $A$-modules and closed under taking direct summands. Let $\mathcal{D}_{fd}_A$ be the full subcategory of $\mathcal{D}_A$ consisting of those dg modules that have finite-dimensional total cohomology. It is a triangulated subcategory of $\mathcal{D}_A$.

Let $A$ be a dg $k$-category. We define $H^0 A$ to be the $k$-category that has the same objects as $A$ and whose morphism space $\text{Hom}_{H^0 A}(X, Y)$ between two objects $X$ and $Y$ is the zeroth cohomology of the complex $\text{Hom}_A(X, Y)$. The Yoneda embedding $A \hookrightarrow \text{Dif}_A$ of dg categories induces an embedding $H^0 A \hookrightarrow \mathcal{D}_A$ of $k$-categories. In particular, we have, for $A \in A$ and the corresponding free module $A^\wedge = \text{Hom}_A(?, A)$,

$$H^* \text{Hom}_A(A, A) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}_A}(A^\wedge, \Sigma^p A^\wedge).$$

Let $A$ and $B$ be two dg $k$-categories. A dg $A$-$B$-bimodule is by definition a dg module over the tensor product $B \otimes A^{op}$. Given a dg $A$-$B$-bimodule $X$, one can define a pair of adjoint standard triangle functors

$$\mathcal{D}_A \xrightarrow{T_X} \mathcal{D}_B.$$

2.2. Orbit categories

We follow [7].

Let $C$ be a $k$-category, and $F$ be an auto-equivalence of $C$. The orbit category $C/F$ is defined as the category whose objects are the same as those of $C$ and the morphism space $\text{Hom}_{C/F}(X, Y)$ between two objects $X$ and $Y$ is

$$\text{Hom}_{C/F}(X, Y) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_C(X, F^p Y).$$

The following remarkable result is due to Keller.

**Theorem 2.1** [7, Theorem 9.9]. Let $\mathcal{H}$ be a small hereditary abelian $k$-category with the Krull–Schmidt property where all morphism and extension spaces are finite-dimensional. Let $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H})$ be a standard equivalence with a dg lift. Suppose

1. for each indecomposable $U$ of $\mathcal{H}$, only finitely many objects $F^i U, i \in \mathbb{Z}$, lie in $\mathcal{H}$;
2. there is an integer $N \geq 0$ such that the $F$-orbit of each indecomposable of $C$ contains an object $\Sigma^n U$ for some $0 \leq n \leq N$ and some indecomposable $U$ of $\mathcal{H}$.

Then the orbit category $C/F$ admits a natural triangle structure such that the projection functor $C \to C/F$ is a triangle functor.

In Keller’s proof, a dg orbit category is defined and the triangle structure of $C/F$ come from the (nice) dg structure of the dg orbit category. In this case, we say that the orbit category $C/F$ admits a canonical triangle structure. In particular, the projection functor $\pi_C : C \to C/F$ is a triangle functor given by a tensor functor.
2.3. The algebraic triangulated category generated by a spherical object

Let \( C \) be a triangulated \( k \)-category. For an integer \( \omega \), an object \( S \) of \( C \) is called a \( \omega \)-spherical object if the graded endomorphism algebra

\[
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_C(S, \Sigma^p S)
\]

is isomorphic to \( \Lambda = k[s]/(s^2) \), where \( s \) is of degree \( \omega \). Let \( \mathcal{S}_w \) be the algebraic triangulated category over \( k \) generated by a \( \omega \)-spherical object \( S \). The following result was proved in [10].

**Theorem 2.2.** The category \( \mathcal{S}_w \) is triangle equivalent to \( \text{per}(\Lambda) \) and to \( \mathcal{D}_{fd}(\Gamma) \), where \( \Gamma = k[t] \) if \( w \neq 1 \) and \( \Gamma = k[[t]] \) if \( w = 1 \) with \( \deg t = 1 - w \). Here both \( \Lambda \) and \( \Gamma \) are viewed as dg algebras with trivial differentials. In particular, \( \Sigma^n S, n \in \mathbb{Z} \), are precisely the \( \omega \)-spherical objects in \( \mathcal{S}_w \).

Let \( \text{Aut}(\mathcal{S}_w) \) be the group of triangle automorphisms of \( \mathcal{S}_w \) that admit dg lifts. The suspension functor \( \Sigma \) belongs to \( \text{Aut}(\mathcal{S}_w) \) and is central. Let \( a \) be a non-zero element of \( k \). We define \( \varphi_a \) to be the automorphism of \( \Lambda \) taking \( s \) to \( as \). The induced push-out functor \( \varphi_{a,s} : \mathcal{S}_w \to \mathcal{S}_w \) also belongs to \( \text{Aut}(\mathcal{S}_w) \). In fact, \( \text{Aut}(\mathcal{S}_w) \) is generated by \( \Sigma \) and \( \varphi_{a,s}, a \in k^\times \).

**Lemma 2.3.** The group \( \text{Aut}(\mathcal{S}_w) \) is isomorphic to \( k^\times \times \mathbb{Z} \).

**Proof.** Let \( F \) be an element of \( \text{Aut}(\mathcal{S}_w) \). Then \( FS \) is also a \( \omega \)-spherical object, so it follows from Theorem 2.2 that \( FS \cong \Sigma^n(F) S \) for some \( n(F) \in \mathbb{Z} \). Moreover, there is a non-zero element \( a(F) \) of \( k \) such that \( Fs = a(F) \Sigma^n(F)s \). Sending \( F \) to \( (a(F), n(F)) \) defines a group homomorphism \( f : \text{Aut}(\mathcal{S}_w) \to k^\times \times \mathbb{Z} \). The map \( f \) is surjective because, for any \( a \in k^\times \) and any \( n \in \mathbb{Z} \), we have \( f(\varphi_{a,s}(\Sigma^n)) = (a,n) \). It is injective because any element in the kernel is isomorphic to the identity functor on the objects \( \Sigma^p S, p \in \mathbb{Z} \) and on all the morphism spaces \( \text{Hom}_{\mathcal{S}_w}(S, \Sigma^p S) \) and, since \( S \) generates \( \mathcal{S}_w \), such a functor must be isomorphic to the identity functor. \( \square \)

2.4. Standard tubes

Let \( n \) be a positive integer. Let \( T_n \) be the standard tube of rank \( n \), that is, the hereditary abelian category of finite-dimensional nilpotent representations of a cyclic quiver with \( n \) vertices.

**Lemma 2.4.** The derived category \( \mathcal{D}^b(T_n) \) is triangle equivalent to \( \text{per}(\Lambda') \), where \( \Lambda' \) is the quotient of the path algebra of the graded cyclic quiver with each arrow in degree 1 modulo all paths of length 2.

**Proof.** It is easy to check that \( \Lambda' \) is the graded endomorphism algebra of the simple objects of \( T_n \). For degree reasons, there is no non-trivial \( A_{\infty} \)-structure on \( \Lambda' \) such that the identities of the simple objects are strict units (cf. the proof of Keller, Yang and Zhou [10, Theorem 2.1]). So, by Lefèvre-Hasegawa [11, Theorem 7.6.0.6], we obtain the desired result. \( \square \)

Let us denote by \( \alpha_1 : 1 \to 2, \ldots, \alpha_n : n \to 1 \) the arrows in the quiver of \( \Lambda' \). For \( a \in k^\times \) we define \( \psi_a \) as the unique automorphism of \( \Lambda' \) taking \( \alpha_1 \) to \( a \alpha_1 \), and \( \alpha_i \) to \( \alpha_i \), \( i = 2, \ldots, n \). We define \( c \) as the unique automorphism of \( \Lambda' \) taking the vertex \( i \) to \( i + 1 \), \( i = 1, \ldots, n \). The push-out \( c \) is exactly the Auslander–Reiten translation \( \tau \).
Lemma 2.5. The group $\text{Aut}(D^b(T_n))$ is isomorphic to $k^\times \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$.

Proof. The proof is similar to that for Lemma 2.3. Let $S_i$ denote the simple object of $T_n$ corresponding to the vertex $i$, $i = 1, \ldots, n$. Let $F$ be an element of $\text{Aut}(D^b(T_n))$. By the shape of the Auslander–Reiten quiver of $D^b(T_n)$, we have $F(S_1) = \Sigma^{n(F)} S_{i(F)} = \Sigma^{n(F)} \tau^{-i(F)} S_1$ for some $n(F) \in \mathbb{Z}$ and $i(F) = 1, \ldots, n$. Moreover, there are non-zero elements $a_1(F), \ldots, a_n(F)$ of $k$ such that $F(a_i) = a_i(F) \tau^{-i(F)} \Sigma^{n(F)} a_i$. Sending $F$ to $\prod_{i=1}^n a_i(F)$, $i(F) - 1, n(F)$ defines a group homomorphism $: \text{Aut}(D^b(T_n)) \to k^\times \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$. It is surjective because, for any $a \in k^\times$, any $i \in \mathbb{Z}/n\mathbb{Z}$ and any $m \in \mathbb{Z}$, we have $f(\psi_n, x \tau ^m) = (a, i, m)$. Let $F$ be in the kernel of $f$. Then $\prod_{i=1}^n a_i(F) = 1$. So the maps $a_1(F)^{-1} \ldots a_{i-1}(F)^{-1} : F(S_i) = S_i \to S_i$ define a natural isomorphism on the generators $S_1, \ldots, S_n$, and hence $F$ is isomorphic to the identity functor.

3. The triangle structure

Let $k$ be a field and $w$ be an integer. Let $S_w$ be the algebraic triangulated category generated by a $w$-spherical object in this section, we will prove that, for any triangle auto-equivalence $F$ of $S_w$ that is not identical on isoclasses of objects, the orbit category $S_w/F$ admits a canonical triangle structure.

3.1. The triangle structure

Let $\Gamma = \Gamma_d$ be the graded algebra $k[t]$ with $\text{deg}(t) = d = 1 - w$ when $w \neq 1$ or the ring of power series $k[[t]]$ when $w = 1$, viewed as a dg algebra with trivial differential. Then $D_{fd}(\Gamma) \cong S_w$; see Theorem 2.2. When $w = 1$, $D_{fd}(\Gamma)$ is exactly the bounded derived category of a homogeneous tube and the orbit category $S_1/F$ is triangulated by Theorem 2.1. So, in the rest of this subsection we assume that $w \neq 1$, that is, $d \neq 0$.

Let $n \in \mathbb{N}$. The following lemma on graded modules over $\Gamma$ is well known.

Lemma 3.1. (a) Up to degree shifting, an indecomposable graded $\Gamma$-module is isomorphic to one of the following modules: $\Gamma/(tp)$ $(p \in \mathbb{Z})$, $\Gamma$, $k[t^{-1}]$ (the graded dual of $\Gamma$), $M = k[t, t^{-1}]$.

(b) Let $X$ be a finite-dimensional graded $\Gamma$-module. Then

$$\text{Hom}_{\text{Grmod}(\Gamma)} \left( M, \bigoplus_{p \in \mathbb{Z}} X(\langle np \rangle) \right) = 0 = \text{Hom}_{\text{Grmod}(\Gamma)} \left( \bigoplus_{p \in \mathbb{Z}} X(\langle np \rangle), M \right).$$

Recall that the automorphism group of $D_{fd}(\Gamma)$ consists of the functors $\varphi \circ \Sigma^n$, $a \in k^\times$, $n \in \mathbb{Z}$. Here, by abuse of notation, $\varphi_a$ denotes the automorphism $t \mapsto at$ of $\Gamma$. It is useful to observe that $\varphi_{a, x}$ is isomorphic to the identity on objects.

Theorem 3.2. The orbit category $D_{fd}(\Gamma)/\varphi_{a, x} \Sigma^n$ admits a canonical triangle structure.

Proof. Recall that $D_{fd}(\Gamma)$ denotes the dg category of dg $\Gamma$-modules. Let $A$ be the dg subcategory of $D_{fd}(\Gamma)$ consisting of strictly perfect dg $\Gamma$-modules with finite-dimensional total cohomology (a dg $\Gamma$-module is strictly perfect if, as a graded module, it is the direct sum of finite copies of shifts of $\Gamma$). Then we have an equivalence of triangulated categories

$$H^0 A \cong D_{fd}(\Gamma).$$
The functor \( \varphi_{a,s} \Sigma^n \) is a dg auto-equivalence of the dg category \( \mathcal{A} \). For any \( X,Y \in \mathcal{A} \), \( \text{Hom}_{H^0 \mathcal{A}}(X,(\varphi_{a,s} \Sigma^n)pY) \) vanishes for all but finitely many \( p \in \mathbb{Z} \). Let \( \mathcal{B} = \mathcal{A}/\varphi_{a,s} \Sigma^n \) be the dg orbit category of \( \mathcal{A} \). We have an equivalence of categories

\[
H^0 \mathcal{A}/\Sigma^n \simto H^0 \mathcal{B}.
\]

The canonical dg functor \( \pi : \mathcal{A} \to \mathcal{B} \) yields an \( \mathcal{A}\mathcal{B} \)-bimodule

\[
(B,A) \to \text{Hom}_{\mathcal{B}}(B,\pi A),
\]

which induces the standard functors (cf. Subsection 2.1)

\[
\mathcal{D}A \xrightarrow{\pi} \mathcal{D}B \quad \text{and} \quad \mathcal{D}B \xrightarrow{\pi} \mathcal{D}A.
\]

Note that we also have a natural embedding

\[
i : \mathcal{D}A \to \mathcal{D} \Gamma
\]

given by the \( \mathcal{A} \mathcal{\Gamma} \)-bimodule

\[
(\Gamma, A) \to \text{Hom}_{\mathcal{D} \mathcal{\Gamma}}(\Gamma, A).
\]

This embedding identifies \( H^0 \mathcal{A} \) with \( D_{fd}(\Gamma) \).

Let \( \mathcal{M} = \text{per } \mathcal{B} \) be the triangulated subcategory of \( \mathcal{D} \mathcal{B} \) generated by the representable functors. By abuse of notation, we still denote the representable functor \( X^\wedge \) by \( X \) for any \( X \in \mathcal{B} \). To prove that \( D_{fd}(\Gamma)/\varphi_{a,s} \Sigma^n \) admits a canonical triangle structure, it suffices to show that \( H^0 \mathcal{B} \) is extension closed in \( \mathcal{M} \), that is, for any \( X,Y \in H^0 \mathcal{B} \) and \( f \in \text{Hom}_{H^0 \mathcal{B}}(X,Y) \), the third term of the following triangle in \( \mathcal{M} \) is isomorphic to an object in \( H^0 \mathcal{B} \):

\[
X \xrightarrow{f} Y \to E \to \Sigma X.
\]

We apply the right adjoint \( \pi_p \) of \( \pi_* \) to the triangle above; we get a triangle in \( \mathcal{D}A \)

\[
\pi_p X \to \pi_p Y \to \pi_p E \to \Sigma \pi_p X.
\]

Applying the functor \( i \), we have a triangle in \( \mathcal{D} \Gamma \)

\[
i \pi_p X \to i \pi_p Y \to i \pi_p E \to \Sigma i \pi_p X.
\]

It suffices to show that

\[
i \pi_p E \cong \bigoplus_{p \in \mathbb{Z}} (\varphi_{a,s} \Sigma^n)^p Z \cong \bigoplus_{p \in \mathbb{Z}} \Sigma^{np} Z, \quad \text{for some } Z \in \mathcal{A}.
\]

Below we consider the functor \( \varphi_{a,s} \Sigma^n \) only on objects, so we drop \( \varphi_{a,s} \). Note that \( i \pi_p X \) and \( i \pi_p Y \) are direct sums of \( \Sigma^n \)-orbits of objects in \( D_{fd}(\Gamma) \). We have

\[
\dim H^m(i \pi_p X) < \infty \quad \text{and} \quad \dim H^m(i \pi_p Y) < \infty, \quad \forall m \in \mathbb{Z}.
\]

Let us rewrite the triangle as follows

\[
\bigoplus_{p \in \mathbb{Z}} \Sigma^{np} X \to \bigoplus_{p \in \mathbb{Z}} \Sigma^{np} Y \to N \to \bigoplus_{p \in \mathbb{Z}} \Sigma^{np+1} X.
\]

Applying the cohomological functor \( H^* \) to the above triangle gives a long exact sequence of graded \( \Gamma \)-modules

\[
\bigoplus_{p \in \mathbb{Z}} H^*(X)\langle np \rangle \to \bigoplus_{p \in \mathbb{Z}} H^*(Y)\langle np \rangle \to H^*(N) \to \bigoplus_{p \in \mathbb{Z}} H^*(X)\langle np + 1 \rangle.
\]

Then it follows, by Lemma 3.1, that any degree shifting of \( M = H^*(M) \) is not a direct summand of \( H^*(N) \). By Lemma A.3, we know that any shift of \( M \) is not a direct summand of \( N \). As a consequence, \( N \) must be the direct sum of a \( \Sigma^n \)-orbit. As \( \dim H^m(N) < \infty \) for each \( m \in \mathbb{Z} \), \( N \) only has objects in \( D_{fd}(\Gamma) \) as its direct summands. Again, by \( \dim H^m(N) < \infty \) for each
\( m \in \mathbb{Z} \), we have
\[
N \cong \left( \bigoplus_{p \in \mathbb{Z}} \Sigma^{np}Z_1 \right) \oplus \ldots \oplus \left( \bigoplus_{p \in \mathbb{Z}} \Sigma^{np}Z_r \right),
\]
for some indecomposable objects \( Z_1, \ldots, Z_r \) in \( D_{fd}(\Gamma) \); namely, \( N \) is the direct sum of a \( \varphi_{a, s} \Sigma^n \)-orbit of objects in \( D_{fd}(\Gamma) \). This completes the proof.

**Remark 3.3.** Recall that \( \Lambda = k[s]/s^2 \) is the graded algebra with \( \deg(s) = w \). In [7, Section 3], it is shown that \( D^b(\Lambda)/\Sigma^2 \cong \text{per}(\Gamma)/\Sigma^2 \) is not triangulated for \( w = 0 \) (that is, \( d = 1 \)). Similarly, \( \text{per}(\Gamma)/\Sigma^n \) is not triangulated for all \( w \in \mathbb{Z} \) and all \( n \in \mathbb{N} \). Indeed, we can use the argument in [7, Section 3]. The endomorphism algebra of \( \Gamma \) in the orbit category is a polynomial ring \( k[u] \) with \( u \in \text{Hom}_{\text{per}(\Gamma)}(\Gamma, \Sigma \Gamma) \), where \( |l| \) is the least common multiple of \( n \) and \( |d| \) and \( l \) has the same sign as \( d \). The endomorphism \( 1 + u \) is monomorphic but does not admit a left inverse. By copying the proof of Theorem 3.2, we can also obtain some evidence (and some clue about the missing cone of \( 1 + u \)). The morphism \( 1 + u \) induces a triangle in \( D(\Gamma) \)
\[
\bigoplus_{p \in \mathbb{Z}} \Sigma^{np} \Gamma \xrightarrow{f} \bigoplus_{p \in \mathbb{Z}} \Sigma^{np} \Gamma \rightarrow N \rightarrow \Sigma \bigoplus_{p \in \mathbb{Z}} \Sigma^{np} \Gamma,
\]
where \( f \) is the morphism with components
\[
\Sigma^{np} \Gamma \xrightarrow{(1, n)} \Sigma^{np} \Gamma \oplus \Sigma^{np+1} \Gamma \leftarrow \bigoplus_{p \in \mathbb{Z}} \Sigma^{np} \Gamma;
\]
namely, \( N \) is the Milnor colimit of the sequence
\[
\cdots \rightarrow \Sigma^p \bigoplus_{i=0}^{m-1} \Sigma^i \Gamma \xrightarrow{\Sigma^p v} \Sigma^{p+1} \bigoplus_{i=0}^{m-1} \Sigma^i \Gamma \rightarrow \cdots,
\]
where \( m \) is the greatest common divisor of \( n \) and \( |d| \), and \( v \) is the diagonal matrix
\[
v = \text{diag}(u, \Sigma u, \ldots, \Sigma^{m-1} u).
\]
Thus, \( N \) is isomorphic in \( D(\Gamma) \) to \( \bigoplus_{i=0}^{m-1} \Sigma^i k[t, t^{-1}] \), which is not the direct sum of the \( \Sigma^n \)-orbit of any object in \( \text{per}(\Gamma) \).

### 3.2. The dg algebra for \( S_w/\varphi_{a, s} \Sigma^n \)

We have a nice by-product of the proof of Theorem 3.2.

**Proposition 3.4.** Let \( w \in \mathbb{Z}\setminus\{1\} \), \( a \in k^\times \) and \( n \in \mathbb{N} \). The orbit category \( S_w/\varphi_{a, s} \Sigma^n \) is triangle equivalent to \( \text{per} \Lambda \), where
\[
\hat{\Lambda} = \hat{\Lambda}_{w, a, n} = k(s, r, r^{-1})/(s^2, sr = (-1)^nwars)
\]
is the graded algebra with \( \deg(s) = w \) and \( \deg(r) = n \), viewed as a dg algebra with trivial differential.

**Proof.** Let \( A \) and \( B \) be dg categories as defined in the proof of Theorem 3.2. Let \( S = \Gamma/\Gamma' \) be the 1-dimensional simple dg \( \Gamma \)-module concentrated in degree 0. Recall from Theorem 2.2 that the dg endomorphism algebra of (a strictly perfect resolution of) \( S \) in \( A \) is related by a zigzag of quasi-isomorphisms to \( \Lambda = k[s]/s^2 \) with \( \deg(s) = w \). Thus, the dg endomorphism
algebra
\[ \bigoplus_{p \in \mathbb{Z}} \text{Hom}_A(\tilde{S}, (\varphi_{a,*} \Sigma^n)^p \tilde{S}) = \bigoplus_{p \in \mathbb{Z}} \Sigma^m \text{Hom}_A(\tilde{S}, \tilde{S}) \]
of the image \( \tilde{S} \) of \( S \) in \( B \) is related by a zigzag of quasi-isomorphisms to \( \tilde{A} \) (note that the composition in the orbit category is twisted by \( \varphi_{a,*} \Sigma^n \)). It follows from the construction that every object in \( B \) is an iterated cone of closed morphisms of degree 0 between shifts of copies of \( \tilde{S} \), since every object in \( A \) is an iterated cone of closed morphisms of degree 0 between shifts of copies of \( S \). Thus, the restriction from \( B \) to the one-object dg subcategory \( \{ \tilde{S} \} \) induces a triangle equivalence \( \text{per} \tilde{A} \cong \text{per} B \). The latter category, by the proof of Theorem 3.2, is triangle equivalent to \( H^0 \mathcal{B} = D_{fd}(\Gamma)/\varphi_{a,*} \Sigma^n \), and the desired result follows.

3.3. The Auslander–Reiten quiver

Let \( a \in k^\times \), \( n, w \in \mathbb{Z} \) and \( d = 1 - w \).

If \( w = 1 \), then the Auslander–Reiten quiver of \( S_w \) consists of \( \mathbb{Z} \) copies of homogeneous tubes, and \( \Sigma \) acts transitively on them. Thus, the Auslander–Reiten quiver of the orbit category \( S_1/\varphi_{a,*} \Sigma^n \) consists of \( n \) homogeneous tubes.

If \( w \neq 1 \), then the Auslander–Reiten quiver of \( S_w \) consists of \( |d| \) copies of \( \mathbb{Z} A_\infty \). An object \( M \) and \( \Sigma^p M \) are in the same component if and only if \( p \) is a multiple of \( d \). Thus, the Auslander–Reiten quiver of \( S_w/\varphi_{a,*} \Sigma^n \) consists of \( m \) copies of tubes of rank \( n' \), where \( m \) is the greatest common divisor of \( n \) and \( d \), and \( n' = n/m \).

3.4. Orbit categories of the bounded derived category of a standard tube

Let \( n, c \in \mathbb{Z}/n'\mathbb{Z} \), \( b \in k^\times \) and \( m \in \mathbb{N} \).

Proposition 3.5. The orbit category \( D^b(T_{n'})/\psi_{b,*} \tau^c \Sigma^m \) is triangle equivalent to \( \text{per} \tilde{\Lambda}' \), where
\[ \tilde{\Lambda}' = \tilde{\Lambda}_{n',b,c,m} = k(s, r, r^{-1})/(s^2, sr = (-1)^{n'm}bn'r) \]
is the graded algebra with \( \text{deg}(s) = w \) and \( \text{deg}(r) = n \), viewed as a dg algebra with trivial differential. Here \( n = n'm \), and \( w = 1 - md' \) for \( d' \) with \( cd' \equiv 1 \pmod{n'} \).

Proof. The proof is similar to that for Proposition 3.4. Let \( \Lambda' \) be the dg category of strictly perfect dg \( \Lambda' \)-modules, where \( \Lambda' \) was defined in Subsection 2.4, and let \( B' \) be the dg orbit category with respect to the dg automorphism \( \psi_{b,*} \tau^c \Sigma^m \). Then \( H^0 \Lambda' = D^b(T_{n'}) \) is triangulated and, by Theorem 2.1, \( H^0 B' = D^b(T_{n'})/\psi_{b,*} \tau^c \Sigma^m \) is also triangulated. The dg endomorphism algebra
\[ \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\Lambda'}(\tilde{S}_1, (\psi_{b,*} \tau^c \Sigma^m)^p \tilde{S}_1) = \bigoplus_{p \in \mathbb{Z}} \Sigma^m \text{Hom}_A(\tilde{S}_1, \tilde{S}_{1+c}) \]
of the image \( \tilde{S}_1 \) of \( S_1 \) in \( B' \) is \( \tilde{\Lambda}' \). The triangulated orbit category \( H^0 B' = D^b(T_{n'})/\psi_{b,*} \tau^c \Sigma^m \) is generated by \( \tilde{S}_1 \). Thus, the restriction from \( B' \) to the one-object dg subcategory \( \tilde{S}_1 \) induces a triangle equivalence \( \text{per} \tilde{\Lambda}' \cong \text{per} B' = H^0 B' = D^b(T_{n'})/\psi_{b,*} \tau^c \Sigma^m \).

3.5. An equivalence

Let \( n, w \in \mathbb{Z} \) and \( d = 1 - w \). Let \( m \) be the greatest common divisor of \( n \) and \( d \), and let \( d' = d/m \) and \( n' = n/m \). Let \( c \) be an inverse of \( d' \) modulo \( n' \). Let \( a, b \in k^\times \).

Combining Propositions 3.4 and 3.5, we obtain the following theorem.
Theorem 3.6. The two orbit categories $S_w/\varphi_{a,n} \Sigma^n$ and $D^b(T_n)/\psi_{b,s} \tau^n \Sigma^m$ are triangle equivalent if and only if $a = ((-1)^d b)^n$.

Proof. Note first that Proposition 3.4 is also valid for the case $w = 1$, owing to Proposition 3.5.

The ‘if’ part: If $a = ((-1)^d b)^n$, then $\Lambda'$ and $\Lambda$ are the same dg algebra, in particular, $\text{per}(\Lambda') = \text{per}(\Lambda)$, implying that $S_w/\varphi_{a,n} \Sigma^n$ and $D^b(T_n)/\psi_{b,s} \tau^n \Sigma^m$ are triangle equivalent.

The ‘only if’ part: Let $F : D^b(T_n)/\psi_{b,s} \tau^n \Sigma^m \to S_w/\varphi_{a,n} \Sigma^n$ be a triangle equivalence. Then due to the shape of the Auslander–Reiten quiver, $F(\Omega_1) \cong \Sigma^p \Omega$ for some integer $p$. Therefore, the graded endomorphism algebras $\Lambda'$ of $\Omega_1$ and $\Lambda$ of $\Omega$ are isomorphic, which implies that $a = ((-1)^d b)^n$.

In the appendix, we construct an explicit equivalence for the case $a = b = 1$ and $n = 2$ using covering and the universal property of orbit categories.

Example 3.7. Let $w = 2$ and $n \in \mathbb{N}$. Then $S_2 = C_{\overline{A}_\infty}$ is known as the cluster category of $\overline{A}_\infty$ (see [5, 9]). By Theorem 3.6, when $n$ is even, the orbit category $C_{\overline{A}_\infty}/\Sigma^n$ and the cluster tube $C_n = D^b(T_n)/\tau^{-1} \circ \Sigma$ of rank $n$ are triangle equivalent, whereas when $n$ is odd, they are not triangle equivalent.

4. Ringel–Hall Lie algebras associated to spherical objects

Let $k$ be a finite field with $|k| = q$ and $w$ be an integer. Let $S_w$ be the triangulated category over $k$ generated by a $w$-spherical object. As shown by Theorem 3.2, the orbit category $S_w/\Sigma^2$ admits a canonical triangle structure. It is 2-periodic (cf. [8]), so we can associate a Lie algebra to it via the Ringel–Hall approach in the sense of Peng–Xiao. In this section, we determine this Lie algebra.

4.1. The Ringel–Hall Lie algebra

We recall the definition of the Ringel–Hall Lie algebra of a 2-periodic triangulated category following [14]. Let $\mathcal{R}$ be a Hom-finite $k$-linear triangulated category with suspension functor $\Sigma$. By ind $\mathcal{R}$ we denote a set of representatives of the isoclasses of all indecomposable objects in $\mathcal{R}$.

Given any objects $X, Y, L$ in $\mathcal{R}$, we define

$$W(X, Y; L) = \{(f, g, h) \in \text{Hom}_\mathcal{R}(X, L) \times \text{Hom}_\mathcal{R}(L, Y) \times \text{Hom}_\mathcal{R}(Y, \Sigma X)| X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} \Sigma X \text{ is a triangle}\}.$$ 

The action of $\text{Aut}(X) \times \text{Aut}(Y)$ on $W(X, Y; L)$ induces the orbit space

$$V(X, Y; L) = \{(f, g, h)\hat{}|(f, g, h) \in W(X, Y; L)\},$$

where

$$(f, g, h)\hat{} = \{(af, gc^{-1}, ch(\Sigma a)^{-1})|(a, c) \in \text{Aut}(X) \times \text{Aut}(Y)\}.$$ 

Let $\text{Hom}_\mathcal{R}(X, L)_Y$ be the subset of $\text{Hom}_\mathcal{R}(X, L)$ consisting of morphisms $l : X \to L$ whose mapping cone Cone($l$) is isomorphic to $Y$. Consider the action of the group $\text{Aut}(X)$ on $\text{Hom}_\mathcal{R}(X, L)_Y$ by $d \cdot l = dl$; the orbit is denoted by $l^*$ and the orbit space is denoted by $\text{Hom}_\mathcal{R}(X, L)_Y^*$.

Dually, one can also consider the subset $\text{Hom}_\mathcal{R}(L, Y)_X$ of $\text{Hom}_\mathcal{R}(L, Y)$ with
the group action $\text{Aut}(Y)$ and the orbit space $\text{Hom}_R(L, Y)^*_X$. The following proposition is an observation due to [17].

**Proposition 4.1.** $|V(X, Y; L)| = |\text{Hom}_R(X, L)\rangle_Y = |\text{Hom}_R(L, Y)^*_X|.$

We assume further that $\mathcal{R}$ is 2-periodic, that is, $\mathcal{R}$ is Krull–Schmidt and $\Sigma^2 \cong 1$.

Let $\text{Gr}(\mathcal{R})$ be the Grothendieck group of $\mathcal{R}$ and $I_{\mathcal{R}}(\cdot, \cdot)$ be the symmetric Euler form of $\mathcal{R}$. For an object $M$ of $\mathcal{R}$, we denote by $[M]$ the isoclass of $M$ and by $h_M = \dim_k M$ the canonical image of $[M]$ in $\text{Gr}(\mathcal{R})$. Let $\mathfrak{h}$ be the subgroup of $\text{Gr}(\mathcal{R}) \otimes \mathbb{Q}$ generated by $h_M/d(M), M \in \text{ind } \mathcal{R}$, where $d(M) = \dim_k(\text{End}(X)/\text{rad End}(X))$. One can naturally extend the symmetric Euler form to $\mathfrak{h} \times \mathfrak{h}$. Let $n$ be the free abelian group with basis $\{u_X| X \in \text{ind } \mathcal{R}\}$. Let

$$\mathfrak{g}(\mathcal{R}) = \mathfrak{h} \oplus n,$$

a direct sum of $\mathbb{Z}$-modules. Consider the quotient group

$$\mathfrak{g}(\mathcal{R})/(q - 1)\mathfrak{g}(\mathcal{R}).$$

Let $F^L_{X,Y} = |V(X, Y; L)|$. Then, by Peng and Xiao [14], we know that $\mathfrak{g}(\mathcal{R})/(q - 1)$ is a Lie algebra over $\mathbb{Z}/(q - 1)\mathbb{Z}$, called the Ringel–Hall Lie algebra of $\mathcal{R}$. The Lie operation is defined as follows:

1. for any indecomposable objects $X, Y \in \mathcal{R},$

$$[u_X, u_Y] = \sum_{L \in \text{ind } \mathcal{R}} (F^L_{Y,X} - F^L_{X,Y})u_L - \delta_X,\Sigma_Y h_X/d(X),$$

where $\delta_X,\Sigma_Y = 1$ for $X \cong \Sigma Y$ and 0 else;

2. $[\mathfrak{h}, \mathfrak{h}] = 0$;

3. for any objects $X, Y \in \mathcal{R}$ with $Y$ indecomposable,

$$[h_X, u_Y] = I_{\mathcal{R}}(h_X, h_Y)u_Y, \quad [u_Y, h_X] = -[h_X, u_Y].$$

If the triangulated category $R$ is proper, that is, any indecomposable object of $R$ has non-trivial class in the Grothendieck group, then the sum over $L \in \text{ind } R$ in (1) is necessarily zero when $Y \cong \Sigma X$.

4.2. The Ringel–Hall Lie algebra of $S_w/\Sigma^2$: the case $w$ is odd

Applying Theorem 3.6 to the case $a = 1, w$ being odd and $n = 2$, we have the following lemma.

**Lemma 4.2.** The 2-periodic orbit category $S_w/\Sigma^2$ is triangulated equivalent to the root category $D^b(T_1)/\Sigma^2$ of the standard homogeneous tube $T_1$.

We recall that the standard homogeneous tube $T_1$ is the category of finite-dimensional nilpotent representations of the Jordan quiver (the quiver with one vertex and one loop). For each positive integer $n$, there is an indecomposable representation $\langle n \rangle$ of length $n$, and up to isomorphism all indecomposable representations are of this form. Let $\mathcal{R} = D^b(T_1)/\Sigma^2$ denote the root category of $T_1$. The image of $\langle n \rangle$ in $\mathcal{R}$ will still be denoted by $\langle n \rangle$, and its suspension will be denoted by $\langle -n \rangle$. The Grothendieck group of $\mathcal{R}$ is free of rank 1 generated by the canonical image $\xi$ of $\langle 1 \rangle$. To compute the Ringel–Hall Lie algebra of $\mathcal{R}$, we need the following well-known fact on the bounded derived category of a hereditary abelian category.
Lemma 4.3. Let $\mathcal{A}$ be a hereditary abelian category and $D^b(\mathcal{A})$ be the bounded derived category. Let $f: X \to Y$ be a morphism of $\mathcal{A}$; then

$$X \xrightarrow{(\pi,a)} Y \xrightarrow{\cok f \oplus \ker f \{e,l\}^f} \Sigma X$$

is a triangle of $D^b(\mathcal{A})$.

The following description of the Ringel–Hall Lie algebra of $\mathcal{R}$ is known to the experts, but we failed to find a precise statement in the literature. It can be easily deduced by using Lemma 4.3 (cf. [12, 21]).

Proposition 4.4. The Ringel–Hall Lie algebra $g(\mathcal{R})_{q-1}$ of $\mathcal{R}$ has a $\mathbb{Z}/(q-1)$-basis $\{u_{\langle n \rangle}|n \in \mathbb{Z}\backslash\{0\}\} \cup \{z\}$ with structure constants given by

1. $[u_{\langle n \rangle}, u_{\langle -n \rangle}]=nz$, for $n \in \mathbb{Z}\backslash\{0\}$;
2. $[u_{\langle n \rangle}, u_{\langle m \rangle}]=0$, for $n \neq m \in \mathbb{Z}\backslash\{0\}$;
3. $[z, u_{\langle n \rangle}]=0$, for $n \in \mathbb{Z}\backslash\{0\}$.

We have an ‘integral’ version of $g(\mathcal{R})_{q-1}$. Let $\Omega$ be the set of isomorphism classes of finite field extensions of $k$. For any $E \in \Omega$, let $\langle n \rangle^E = \langle n \rangle \otimes_k E$ be the extension of $\langle n \rangle$ over field $E$ and $\langle -n \rangle^E$ be the suspension of $\langle n \rangle^E$. One can define the Ringel–Hall Lie algebra $g(\mathcal{R})_{\langle E \rangle}^{-1}$ similarly. Consider the product of Lie algebras

$$\mathcal{L} := \prod_{E \in \Omega} g(\mathcal{R})_{\langle E \rangle}^{-1}.$$

Let $U_{\langle \pm n \rangle} = (\ldots, u_{\langle \pm n \rangle} e, \ldots)\in \mathbb{N}$, $Z = (\ldots, z^E, \ldots)\in E$. Consider the Lie subalgebra $g$ of $\mathcal{L}$ generated by $U_{\langle \pm n \rangle}$ and $Z$; we also call $g$ the Ringel–Hall Lie algebra of $\mathcal{R}$. It is easy to see that the rational extension $g \otimes \mathbb{Q}$ of $g$ is isomorphic to the infinite-dimensional Heisenberg Lie algebra. Indeed, $\{-1/nU_{\langle n \rangle}|n \in \mathbb{N}\} \cup \{Z\} \cup \{U_{\langle -n \rangle}|n \in \mathbb{N}\}$ is a Chevalley basis.

Combining Lemma 4.2 and Proposition 4.4, we have the following proposition.

Proposition 4.5. The Ringel–Hall algebra of $S_w/\Sigma^2$ for $w$ odd is isomorphic to the infinite-dimensional Heisenberg Lie algebra.

4.3. The Ringel–Hall Lie algebra of $S_w/\Sigma^2$; the case $w$ is even

Applying Theorem 3.6 to the case $a = 1$, $w$ being even and $n = 2$, we have the following lemma.

Lemma 4.6. The 2-periodic orbit category $S_w/\Sigma^2$ is triangulated equivalent to the cluster tube of rank 2.

Let us first recall the definition of the cluster tube of rank 2. Let $\Delta$ be the cyclic quiver with two vertices. Let $T_2$ be the category of finitely generated nilpotent right $k\Delta$-modules. Let $D = D^b(T_2)$ be the bounded derived category of $T_2$, $\tau$ be the AR-translation functor and $\Sigma$ be the suspension functor of $D^b(T_2)$. The cluster tube of rank 2, denoted by $\mathcal{C}$, is defined as the orbit category $D^b(T_2)/\tau^{-1} \circ \Sigma$ (cf. [1, 2]). In particular, for objects $X$ and $Y$ of $\mathcal{C}$, the morphism space $\mathcal{C}(X, Y)$ is

$$\mathcal{C}(X, Y) = D(X, Y) \oplus D(X, \tau^{-1}\Sigma Y).$$
The composite functor $\mathcal{T}_2 \to \mathcal{D} \to \mathcal{C}$ is bijective on isoclasses of objects and preserves indecomposability. Thus, we have $\text{ind}\mathcal{C} = \text{ind}\mathcal{T}_2 = \{\langle n \rangle, \langle -n \rangle | n \in \mathbb{N}\}$, where $\langle n \rangle$ is the unique indecomposable $k\Delta$-module of length $n$ with socle the simple module corresponding to the vertex 1, and $\langle -n \rangle$ the unique indecomposable $k\Delta$-module of length $n$ with socle the simple module corresponding to the vertex 2. We have $\tau\langle n \rangle = \langle -n \rangle$ in $\mathcal{T}_2$ (here $\tau^2 = 1$), and $\Sigma\langle n \rangle \cong \tau\langle n \rangle = \langle -n \rangle$ in $\mathcal{C}$ (here $\tau \cong \Sigma$ and $\tau^2 \cong \Sigma^2 = 1$). The Grothendieck group of $\mathcal{C}$ is free of rank 1 generated by the canonical image of $(1)$; see [1]. It is easy to see that the images of $\langle \pm 2n \rangle$ in the Grothendieck group are 0, so the triangulated category $\mathcal{C}$ is not proper.

To compute the Ringel–Hall Lie algebra of $\mathcal{C}$, we need some auxiliary results. Let $X, Y$ be two objects of $\mathcal{T}_2$, viewed as objects of $\mathcal{C}$, and $f \in \mathcal{D}(X, Y)$ and $g \in \mathcal{D}(X, \tau^{-1}\Sigma Y)$. Let $Z$ be an object of $\mathcal{T}_2$ such that $\Sigma Z$ is a cone of $f + g$. As discussed in [20, Section 4.6], we can compute $Z$ using the long exact sequence

$$
\tau^{-1}X \xrightarrow{\tau^{-1}f} \tau^{-1}Y \longrightarrow Z \longrightarrow X \xrightarrow{f} Y,
$$

where the short exact sequence

$$
0 \longrightarrow \text{cok}(\tau^{-1}f) \longrightarrow Z \longrightarrow \ker(f) \longrightarrow 0
$$

is induced from $g$ by the inclusion $\ker(f) \hookrightarrow X$ and the quotient $\tau^{-1}Y \twoheadrightarrow \text{cok}(\tau^{-1}f)$. Namely, we have the following commutative diagram:

$$
g : \begin{array}{ccc}
0 & \longrightarrow & \tau^{-1}Y \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{cok}(\tau^{-1}f) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tau^{-1}f \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Z \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \ker(f) \\
0 & \longrightarrow & 0
\end{array}
$$

where the square in the left-upper corner is a push-out and the square in the right-lower corner is a pull-back. The next proposition follows easily.

**Proposition 4.7.** Let $f \in \mathcal{D}(X, Y), g \in \mathcal{D}(X, \tau^{-1}\Sigma Y)$. For any non-zero $t \in k$, we have $\text{Con}(f + tg) \cong \text{Con}(f + g)$. In particular, $\text{Con}(f + g)$ is indecomposable if and only if $\text{Con}(f + tg)$ is indecomposable.

For $f \in \mathcal{D}(X, Y)$ and $g \in \mathcal{D}(X, \tau^{-1}\Sigma Y)$, let $(f + g)^*$ be the orbit of $f + g$ by the action of $\text{Aut}_\mathcal{C}(X)$ and $(f + g)$ be the orbit of $f + g$ by the action of $\text{Aut}_\mathcal{C}(Y)$. We have the following easy observations.

1. For any $g \in \mathcal{D}(X, \tau^{-1}\Sigma Y)$ and any $h \in \mathcal{D}(Y, \tau^{-1}\Sigma Z)$, we have $h \circ g = 0$ in $\mathcal{C}$.
2. For $g \in \mathcal{D}(X, \tau^{-1}\Sigma Y)$, the orbit $g^*$ is contained in $\mathcal{D}(X, \tau^{-1}\Sigma Y)$.

**Lemma 4.8.** Let $X, Y \in \text{ind}\mathcal{C}, f \in \mathcal{D}(X, Y), g \in \mathcal{D}(X, \tau^{-1}\Sigma Y)$. Assume that $(f + g)^* = h^*$ for some $h \in \mathcal{D}(X, Y)$; then the mapping cone $\text{Con}(f + g)$ of $f + g$ in $\mathcal{C}$ is indecomposable if and only if $f$ is injective or surjective.

**Proof.** Suppose that $\text{Con}(f + g)$ is indecomposable; then, by the condition $(f + g)^* = h^*$, we know that $\text{Con}(h)$ is indecomposable. Thus, $h$ is injective or surjective. Since $(f + g)^* = h^*$, there exists $\phi_X \in \text{Aut}_\mathcal{C}(X)$ such that $(f + g)\phi_X = h$. We can write $\phi_X$ as $\phi + x$, where
\( \phi \in \text{Aut}_D(X) \) and \( x \in D(X, \tau^{-1}\Sigma X) \). In particular, we have \( f \circ \phi = h \) and \( f \) is injective or surjective.

Suppose that \( f \) is injective or surjective. The assumption \( (f + g)^* = h^* \) implies that \( h \) is injective or surjective respectively. Therefore, the mapping cone of \( f + g \) is indecomposable. \( \square \)

**Remark 4.9.** For any \( f \in D(X,Y), g \in D(X,\tau^{-1}\Sigma Y) \), if \( f \) is surjective, then \( (f + g)^* = f^* \); if \( f \) is injective, then \( (f + g)_* = f_* \). Indeed, assume that \( f \) is surjective; then

\[
\text{ker } f \longrightarrow X \overset{f}{\longrightarrow} Y \longrightarrow \Sigma \text{ker } f
\]

is an exact triangle in the derived category \( D \). Applying the functor \( D(\tau\Sigma^{-1}X,-) \) to the triangle above, we get a long exact sequence

\[
D(\tau\Sigma^{-1}X, \text{ker } f) \longrightarrow D(\tau\Sigma^{-1}X,X) \xrightarrow{D(\tau\Sigma^{-1}X,f)} D(\tau\Sigma^{-1}X,Y) \longrightarrow D(\tau\Sigma^{-1}X,\Sigma \text{ker } f) \longrightarrow \cdots
\]

where the last term vanishes for the heredity of \( T_2 \). Hence, for any \( g \in D(\tau\Sigma^{-1}X,Y), \) there is a \( v \in D(\tau\Sigma^{-1}X,X) \) such that \( g = f \circ v \). Now taking \( 1 \in D(X,X) \) and \( u := -\tau^{-1}\Sigma v \in D(X,\tau^{-1}\Sigma X) \), one gets \( (f + g)(1 + u) = f \), which implies the desired results. Similarly, one can show the result for the case in which \( f \) is injective.

**Lemma 4.10.** Let \( X,Y \in \text{ind}C \) and \( f \in D(X,Y), g \in D(X,\tau^{-1}\Sigma Y) \). Suppose \( (f + g)^* \neq h^* \) for any \( h \in D(X,Y) \). Then \( (f + sg)^* \neq (f + tg)^* \) for any non-zero \( s \neq t \in k \).

**Proof.** We first remark that if \( (f + g)^* \neq h^* \) for any \( h \in D(X,Y) \), then \( (f + sg)^* \neq h^* \) for any \( h \in D(X,Y) \) and \( 0 \neq t \in k \).

Suppose that \( (f + sg)^* = (f + tg)^* \) for some \( s \neq t \in k \). There exists \( \phi_X \in \text{Aut}_C(X) \) such that \( (f + sg)\phi_X = (f + tg) \). Since \( X \) is indecomposable, we can write \( \phi_X = a + b + c \), where \( a \in k, b \in \text{rad } \text{End}_D(X), c \in D(X,\tau^{-1}\Sigma X) \). The equality \( (f + sg)(a + b + c) = (f + tg) \) implies \( f(a + b) = f \) and \( -fc = g(s(a + b) - t) \). In particular, one has \( a = 1 \) and \( s - t + sb \in \text{Aut}_D(X) \) since \( s \neq t \) and \( sb \in \text{rad } \text{End}_D(X) \). Thus, we have \( g = -fc(s - t + sb)^{-1} \) and \( (f + g)(1 - c(s - t + sb)^{-1}) = f \), which is a contradiction.

**Lemma 4.11.** Let \( X,Y \in \text{ind}C \) and \( f,h \in D(X,Y), g,k \in D(X,\tau^{-1}\Sigma Y) \). If \( (f + g)^* = (h + k)^* \), then \( (f + tg)^* = (h + tk)^* \) for any \( 0 \neq t \in k \).

**Proof.** Assume that \( \phi_X = a + b + c \), where \( a \in k, b \in \text{rad } \text{End}_D(X), h \in D(X,\tau^{-1}\Sigma X) \) and \( (f + g)\phi_X = h + k \). An easy calculation shows that \( f(a + b) = h \) and \( fc + g(a + b) = k \). One can take \( \psi = a + b + tc \) and verify that \( (f + tg)\psi = (h + tk) \).

Let \( X, Y \) and \( L \) be objects in \( \text{ind}C \). By definition and Proposition 4.1, we have \( F^L_Y = |\text{Hom}_C(X,L)^\vee_Y| \). The set \( \text{Hom}_C(X,L)^\vee_Y \) admits a natural partition:

\[
\text{Hom}_C(X,L)^\vee_Y = S_1 \cup S_2 \cup S_3,
\]

where

\[
S_1 = \{ f^* | f \in D(X,L) \text{ such that } \text{Con}(f) \cong Y \},
\]

\[
S_2 = \{ (f + g)^* | f \neq f \in D(X,L), g \in D(X,\tau^{-1}\Sigma L) \text{ such that } \text{Con}(f + g) \cong Y \} \setminus S_1,
\]

\[
S_3 = \{ g^* | 0 \neq g \in D(X,\tau^{-1}\Sigma L) \text{ such that } \text{Con}(g) \cong Y \}.
\]
Therefore,

\[ F^L_{YX} = |S_1| + |S_2| + |S_3|. \]

It follows from Proposition 4.7 and Lemmas 4.10 and 4.11 that \(|S_2|\) is divisible by \(q - 1\) and hence is 0 in \(\mathbb{Z}/(q - 1)\). However, note that \(S_2\) is not necessarily empty. Namely, for some \(X\) and \(L\) there are non-zero morphisms \(f \in D(X, L)\) and \(g \in D(X, \tau^{-1} \Sigma L)\) such that \((f + g)^* \neq h^*\) for any \(h \in D(X, L)\) and the mapping cone of \(f + g\) in \(C\) is indecomposable. For example, one considers \(X = \langle 4 \rangle\) and \(L = \langle 3 \rangle\). Let \(f : X \to \langle 2 \rangle \to L\) and \(g \in D(X, \tau^{-1} \Sigma L)\) given by the short exact sequence

\[ 0 \to \langle -3 \rangle \to \langle -7 \rangle \to \langle 4 \rangle \to 0. \]

One checks that the cone of \(f + g\) is \(\langle -3 \rangle\).

It is not hard to see that

\[ S_1 = \text{Hom}_D(X, L)^*_Y \cup \text{Hom}_D(X, L)^*_\Sigma^{-1}Y, \]

\[ S_3 = \text{Hom}_D(X, \Sigma^{-1}L)^*_\Sigma^{-1}Y. \]

Thus, the numbers \(|S_1|\) and \(|S_3|\) are essentially characterized by the following proposition due to Peng–Xiao [12].

**Proposition 4.12.** Let \(X, Y, Z\) be indecomposable modules of \(T_2\). Then in the triangulated category \(D = D^b(T_2)\), we have

1. \(|V_D(X, Y; Z)| = 1\) if and only if there is a triangle of the form \(X \to Z \to Y \to \Sigma X\); otherwise \(|V_D(X, Y; Z)| = 0\);
2. \(|V_D(Z, \Sigma X; Y)| = |V_D(X, Y; Z)| |\text{Hom}_D(Z, X)|^2 / |\text{Aut}_D(Z)||\text{Hom}_D(Y, X)|;
3. \(|V_D(\Sigma^{-1}Y, Z; X)| = |V_D(X, Y; Z)| |\text{Aut}_D(X)| |\text{Hom}_D(Y, Z)|^2 / |\text{Aut}_D(Z)||\text{Hom}_D(Y, X)|.\]

Now we are in a position to determine the structure of the Ringel–Hall Lie algebra \(g(C)_{(q-1)}\) of \(C \cong S^w / \Sigma^2\) for \(w\) even. Let \(\mathcal{L}\) be the Lie algebra over \(\mathbb{Z}/(q - 1)\) with basis \(\{z, u_n, n \in \mathbb{Z}\setminus\{0\}\}\) and structure constants given by (in the following \(x, y \in \mathbb{N}, m, n \in \mathbb{Z}\))

\[ [u_m, u_n] = 0 \text{ for } m \text{ and } n \text{ even; } \]
\[ [u_m, u_n] = 0 \text{ for } m \text{ and } n \text{ both odd of the same sign; } \]
\[ [u_2x, u_{2y-1}] = \begin{cases} u_2(x+y) - 1 + u_2(y-x) - 1, & x < y, \\ u_2(x+y) - 1 - u_2(y-x) + 1, & x \geq y; \end{cases} \]
\[ [u_2x, u_{2y+1}] = \begin{cases} -u_2(2x+y) + 1 - u_2(y-x), & x < y, \\ -u_2(2x+y) + 1 + u_2(y-x) - 1, & x \geq y; \end{cases} \]
\[ [u_{-2x}, u_{2y-1}] = \begin{cases} u_2(x+y) - 1 - u_2(y-x), & x < y, \\ u_2(x+y) - 1 + u_2(y-x) + 1, & x \geq y; \end{cases} \]
\[ [u_{-2x}, u_{2y+1}] = \begin{cases} u_2(2x+y) - 1 + u_2(y-x), & x < y, \\ u_2(2x+y) - 1 - u_2(y-x) - 1, & x \geq y; \end{cases} \]
\[ [u_{2x-1}, u_{2y+1}] = \begin{cases} u_2(x+y) - 2 - u_2(y-x), & x < y, \\ u_2(x+y) - 2 - u_2(y-x) - 2, & x = y, \\ u_2(x+y) - 2 - u_2(y-x) - 2, & x > y; \end{cases} \]
\[ [\bar{z}, u_n] = \begin{cases} 0, & \text{for } n \text{ even} \\ 4u_n, & \text{for } n \text{ positive odd} \\ -4u_n, & \text{for } n \text{ negative odd.} \end{cases} \]
Remark 4.13. By the definition relations (1)(3) – (6)(8) of \(L\), one can show that each 
\(u_{2x} + u_{-2x}(x \in \mathbb{N})\) is central in \(L\). A direct computation shows that they form a basis of the centre.

Theorem 4.14. The assignment \(z \mapsto h_{(1)}, u_n \mapsto u_{(n)}, n \in \mathbb{Z} \setminus \{0\}\) linearly extends to a Lie algebra isomorphism from \(L\) to \(\mathfrak{g}(\mathcal{C})_{(q-1)}\).

Proof. Let us check that \(u_{(n)}, n \in \mathbb{Z} \setminus \{0\}\) satisfy (3). Similarly, one checks that they together with \(h_{(1)}\) satisfy other relations. Let \(x, y \in \mathbb{N}\) and \(l \in \mathbb{Z} \setminus \{0\}\). By the arguments before Proposition 4.12, we have

\[
F_{(2y-1),(2x)}^{(l)} = |V_{m}((2x), (2y-1); (l))| + |V_{m}((2x), \Sigma^{-1}(2y-1); (l))| \\
+ |V_{m}((2x), \Sigma^{-1}(2y-1); l)| \\
= |V_{m}((2x), (2y-1); (l))| + |V_{m}((2x), \Sigma^{-1}(2y-1); l)| \\
+ |V_{m}((2x), \Sigma^{-1}(2y-1); l)| \\
= |V_{m}((2x), (2y-1); (l))| + |V_{m}((2x), \Sigma^{-1}(2y-1); l)| \\
+ |V_{m}((2x), \Sigma^{-1}(2y-1); l)| \\
= |V_{m}((2x), (2y-1); (l))| + |V_{m}((2x), \Sigma^{-1}(2y-1); l)|
\]

where the last congruence follows from Proposition 4.12(2) and (3) and the fact that the number of automorphisms of an indecomposable object in \(D\) is the product of \(q - 1\) and a power of \(q\).

Now it follows from Proposition 4.12(1) that

\[
|V_{m}((2x), (2y-1); (l))| = \begin{cases} 1 & \text{if } l = 2(x + y) - 1, \\ 0 & \text{else,} \end{cases}
\]

\[
|V_{m}((2x), (2y-1); (l))| = 0,
\]

\[
|V_{m}((2x), (2y-1); (l))| = \begin{cases} 1 & \text{if } x < y \text{ and } l = 2(y - x) - 1, \\ 0 & \text{else.} \end{cases}
\]

Therefore, we have

\[
F_{(2y-1),(2x)}^{(l)} = \begin{cases} 1 & \text{if } l = 2(x + y) - 1, \\ 1 & \text{if } x < y \text{ and } l = 2(y - x) - 1 \pmod{q - 1}, \\ 0 & \text{else.} \end{cases}
\]

Similarly, we have

\[
F_{(2y-1),(2x)}^{(l)} = \begin{cases} 1 & \text{if } x \geq y \text{ and } l = 2(x - y) + 1 \pmod{q - 1}, \\ 0 & \text{else.} \end{cases}
\]

The desired result follows immediately from the definition of the bracket:

\[
[u_{(2x)}, u_{(2y-1)}] = \sum_{l \in \mathbb{Z} \setminus \{0\}} (F_{(2y-1),(2x)}^{(l)} - F_{(2x),(2y-1)}^{(l)}) u_{(l)}.
\]

Similar to the case of \(\omega\) being odd, we also have the ‘integral’ version \(\mathfrak{g}\) of \(\mathfrak{g}(\mathcal{C})_{(q-1)}\) for \(\mathcal{C}\). Namely, \(\mathfrak{g}\) is a Lie algebra over \(\mathbb{Z}\) with a basis \(\{Z, U_n, n \in \mathbb{Z} \setminus \{0\}\}\). Note that Theorem 4.14 is independent of the base filed \(k\). By the definition of the integral Ringel-Hall Lie algebra \(\mathfrak{g}\), one can deduce that \(\mathfrak{g}\) is isomorphic to the Lie algebra over \(\mathbb{Z}\) with basis \(\{z, u_n, n \in \mathbb{Z} \setminus \{0\}\}\) and structure constants given by (1) – (8). We take the quotient of this Lie algebra by its centre and extend the scalars to \(\mathbb{Q}\). The resulting Lie algebra has a basis.
\begin{equation}
\{ a_x | x \in \mathbb{N} \cup \{0\} \} \cup \{ b_y, c_y | y \in \mathbb{N} - \frac{1}{2} \} \ (a_x = \bar{a}_{2x} \ \text{for} \ x \in \mathbb{N}, \ a_0 = \frac{1}{2} \bar{z}, \ b_y = \bar{a}_{2y}, \ c_y = \bar{a}_{-2y} \ \text{for} \ y \in \mathbb{N} - \frac{1}{2}) \end{equation}

\begin{enumerate}
\item \( [a_x, a_{x'}] = 0, [b_y, b_{y'}] = 0, [c_y, c_{y'}] = 0; \)
\item \( [a_x, b_y] = b_{y+x} + \text{sgn}(y-x)b_{|y-x|}; \)
\item \( [a_x, c_y] = -c_{y+x} - \text{sgn}(y-x)c_{|y-x|}; \)
\item \( [b_y, c_{y'}] = 2a_{y+y'} - 2a_{|y-y'|}; \)
\end{enumerate}

where, for an integer \( r \), \( \text{sgn}(r) = 1 \) if \( r \) is positive and \( \text{sgn}(r) = -1 \) if \( r \) is negative.

\section{Appendix A. An explicit equivalence}

Let \( k \) be a field and \( d \) be a non-zero integer. Let \( \Gamma \) be the graded algebra \( k[t] \) with \( \text{deg}(t) = d \), viewed as a dg algebra with trivial differential. Let \( n \) be a positive integer. In Section 3, we proved that the orbit category of \( D_{fd}(\Gamma)/\Sigma^n \) admits a canonical triangle structure and is triangle equivalent to a certain orbit category of the bounded derived category of a standard tube. In this appendix, we construct an explicit equivalence, provided that \( n \) is even.

\subsection{A.1. Induced functors}

We follow \cite{8}. Let \( \mathcal{C} \) and \( \mathcal{C}' \) be a \( k \)-linear category, \( F : \mathcal{C} \to \mathcal{C} \) and \( F' : \mathcal{C}' \to \mathcal{C}' \) be auto-equivalences, and \( \mathcal{C}/F \) and \( \mathcal{C}'/F' \) be the corresponding orbit categories. Let \( (\Phi, \alpha) \) be an \((F,F')\)-equivariant functor, that is, \( \Phi : \mathcal{C} \to \mathcal{C}' \) is a \( k \)-linear functor and \( \alpha : \Phi F \to F' \Phi \) is a natural isomorphism. Then \( (\Phi, \alpha) \) induces a \( k \)-linear functor \( \Phi : \mathcal{C}/F \to \mathcal{C}'/F' \): for \( f \in \text{Hom}_{\mathcal{C}}(X, F\alpha Y) \) the image \( \Phi(f) \) is the composition

\[ \Phi X \xrightarrow{\Phi f} \Phi F\alpha Y \xrightarrow{\alpha_{FP^{-1}Y}} F\Phi FP^{-1}Y \xrightarrow{FP^{-1}\alpha_Y} F\Phi Y. \]

In particular, the \((F,F)\)-equivariant functor \((\text{id}_{\mathcal{C}}, \epsilon F)\) induces a functor \( \mathcal{C}/F \to \mathcal{C}/F \), denoted by \( \Delta(\epsilon) \). If \( \mathcal{C} \) is triangulated and \( F \) is a triangle auto-equivalence, then the suspension functor of \( \mathcal{C}/F \) is induced by the \((F,F)\)-equivariant functor \((\Sigma, \phi)\), where \( \phi : \Sigma F \to F \Sigma \) is the natural isomorphism in the triangle structure of \( F \).

Let \( F_1, F_2 : \mathcal{C} \to \mathcal{C} \) be two \( k \)-linear functors endowed with commutation morphisms

\[ \phi_{ij} : F_i F_j \to F_j F_i, \ i, j = 1, 2. \]

We assume that \( \phi_{ii} = \epsilon_i 1_{F_i F_i} \) and that \( \phi_{ij} \) is the inverse of \( \phi_{ji} \) for \( i, j = 1, 2 \). Let \( F = F_1 F_2 \). Then the above commutation morphisms yield two \((F,F)\)-equivariant functors \((F_1, F_1 \phi_{12} : F_1 F_1 F_2 \to F_1 F_2 F_1)\) and \((F_2, \phi_{21} F_2 : F_2 F_1 F_2 \to F_1 F_2 F_2)\). It follows from \cite{8, Section 2.2} that the induced functors \( \tilde{F}_1 \) and \( \tilde{F}_2 \) satisfy: \( \tilde{F}_1 \tilde{F}_2 \cong \Delta(\epsilon_1 \epsilon_2 1_F) \).

\subsection{A.2. The universal property of triangulated orbit categories}

Let us be given a triangle auto-equivalence \( F : \mathcal{C} \to \mathcal{C} \), a triangle functor \( \Phi : \mathcal{C} \to \mathcal{C}' \) and a natural isomorphism \( \Phi \cong \Phi \circ F \). Assume that they all admit dg lifts, and that the orbit category \( \mathcal{C}/F \) admits a canonical triangle structure. Then, by Keller \cite[Section 9.4]{7}, the triangle functor \( \Phi \) induces a triangle functor

\[ \Phi : \mathcal{C}/F \to \mathcal{C}' \]

with a natural isomorphism \( \Phi \cong \Phi \circ \pi_C \).

\textbf{Lemma A.1.} Keep the above notation and assumptions.

(a) If \( \Phi \) is essentially surjective, so is \( \tilde{\Phi} \).
(b) If \( \Phi \) induces bijections for any objects \( X \) and \( Y \) of \( \mathcal{C} \)

\[
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(X, F^p Y) \longrightarrow \text{Hom}_{\mathcal{C}'}(\Phi X, \Phi Y),
\]

then the functor \( \tilde{\Phi} \) is fully faithful.

**Proof.** (a) This is because, on objects, \( \tilde{\Phi} \) takes the same value as \( \Phi \).

(b) For objects \( X \) and \( Y \) of \( \mathcal{C} \) and for an integer \( p \), the functor \( \Phi \) gives us a map

\[
\text{Hom}_{\mathcal{C}}(X, F^p Y) \xrightarrow{\Phi(X,F^p Y)} \text{Hom}_{\mathcal{C}'}(\Phi X, \Phi F^p Y) \sim \longrightarrow \text{Hom}_{\mathcal{C}'}(\Phi X, \Phi Y).
\]

Summing them up for all integers \( p \) yields a map

\[
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(X, F^p Y) \longrightarrow \text{Hom}_{\mathcal{C}'}(\Phi X, \Phi Y),
\]

which is precisely the map \( \tilde{\Phi}(X,Y) \). Therefore, \( \tilde{\Phi} \) is fully faithful under the assumption. \( \square \)

**Remark A.2.** In this appendix, we apply Lemma A.1 without checking the existence of the required dg lift: this is standard in all cases.

### A.3. Hereditary graded algebras

Let \( \Gamma \) be a hereditary graded \( k \)-algebra. Let \( \text{Grmod}(\Gamma) \) be the category of graded modules over the graded algebra \( \Gamma \) and \( \text{grmod}(\Gamma) \) be its subcategory of finite-dimensional graded modules. Let \( (1) \) be the degree-shifting functor of \( \text{Grmod}(\Gamma) \). We grade our algebras and modules cohomologically, and so, for a graded \( \Gamma \)-module \( M = \bigoplus_{p \in \mathbb{Z}} M_p \), the degree shifting \( (M(1)) = M^{p+1} \). Consider \( \Gamma \) as a dg \( k \)-algebra with trivial differential, and let \( \mathcal{D}(\Gamma) \) and \( \mathcal{D}_{fd}(\Gamma) \), respectively, denote the derived category and the finite-dimensional derived category. The two categories \( \mathcal{D}(\Gamma) \) and \( \text{Grmod}(\Gamma) \) are closely related.

**Lemma A.3.** The functor \( H^* : \mathcal{D}(\Gamma) \to \text{Grmod}(\Gamma) \) by taking an object \( X \in \mathcal{D}(\Gamma) \) to its total cohomology \( \oplus_{n \in \mathbb{Z}} H^n(X) \) induces a bijection from the isoclasses of indecomposable objects of \( \mathcal{D}(\Gamma) \) to those of \( \text{Grmod}(\Gamma) \) and satisfies \( H^* \circ \Sigma = (1) \circ H^* \).

**Proof.** Apply [10, Theorem 3.1] to the triangulated category \( \mathcal{D}(\Gamma) \) and its compact generator \( \Gamma \). \( \square \)

A complex of graded \( \Gamma \)-modules can be viewed as a bicomplex. Let \( X \) be a complex of graded \( \Gamma \)-modules and \( \text{Tot}(X) \) be the associated total complex which can be viewed as a dg \( \Gamma \)-module. Then it is clear that \( \text{Tot} \) induces a triangle functor form \( \mathcal{D}(\text{grmod}(\Gamma)) \) to \( \mathcal{D}(\Gamma) \), which restricts to a triangle functor form \( \mathcal{D}^b(\text{grmod}(\Gamma)) \) to \( \mathcal{D}_{fd}(\Gamma) \). Let us still denote it by \( \text{Tot} \).

**Lemma A.4.** (a) The functor \( \text{Tot} \) is essentially surjective.

(b) We have a natural isomorphism of triangle functors \( \text{Tot} \circ \Sigma \circ (-1) \cong \text{Tot} \).

(c) For two objects \( X \) and \( Y \) of \( \mathcal{D}(\text{Grmod}(\Gamma)) \), we have a bijection induced by \( \text{Tot} \)

\[
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(\text{Grmod}(\Gamma))}(X, \Sigma^p Y(-p)) \longrightarrow \text{Hom}_{\mathcal{D}(\Gamma)}(\text{Tot}X, \text{Tot}Y).
\]
Proof. The statement (a) is a consequence of Lemma A.3 and the statement (b) follows immediately from the definition of Tot. The existence of the bifunctorial map
\[ \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{D(\text{Grmod}(\Gamma))}(X, \Sigma^p Y(-p)) \rightarrow \text{Hom}_{D(\Gamma)}(\text{Tot} X, \text{Tot} Y) \]
was shown in the proof of Lemma A.1. It remains to prove the bijectivity. Since Tot commutes with infinite direct sums, by infinite d\'evissage it suffices to prove this for \( X = \Gamma \) and \( Y = \Sigma^q \langle q' \rangle \) for all integers \( q \) and \( q' \). We have
\[
\text{Left-hand side} = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{D(\text{Grmod}(\Gamma))}(\Gamma, \Sigma^p \Sigma^q \langle q' \rangle(-p)) = \text{Hom}_{D(\Gamma)}(\Gamma, \Sigma^q \langle q' \rangle) = H^0 \Gamma(q + q') = \text{Hom}_{D(\Gamma)}(\Gamma, \Sigma^q + q' \Gamma) = \text{Right-hand side}. \]

The functor \( \Sigma \circ \langle -1 \rangle \) is a triangle auto-equivalence of \( D^b(\text{grmod}(\Gamma)) \). It satisfies the conditions in Theorem 2.1, and hence the orbit category \( D^b(\text{grmod}(\Gamma))/\Sigma \circ \langle -1 \rangle \) admits a canonical triangle structure.

**Proposition A.5.** The orbit category \( D^b(\text{grmod}(\Gamma))/\Sigma \circ \langle -1 \rangle \) is triangle equivalent to \( D_{fd}(\Gamma) \).

**Proof.** This follows from Lemmas A.1 and A.4. \( \square \)

**A.4. Graded modules over \( \Gamma \) and quiver representations**

Let \( \overrightarrow{A}_\infty \) be the following quiver of type \( A_\infty \):
\[
\cdots \xleftarrow{i-1} i \xleftarrow{i+1} \cdots
\]

Let \( Q = Q^d \) be the disjoint union of \(|d|\)-copies of \( \overrightarrow{A}_\infty \), whose vertices are labelled \((j, i)\), \( 0 \leq j \leq |d| - 1, i \in \mathbb{Z} \). Let \( \sigma \) be the unique automorphism of \( Q \), which takes the following values on vertices:
\[
\sigma(j, i) = \begin{cases} 
(j - 1 - \left\lfloor \frac{j - 1}{|d|} \right\rfloor |d|, i + \text{sgn}(d) \left\lfloor \frac{j - 1}{|d|} \right\rfloor) & \text{if } 1 \leq j \leq |d| - 1, \\
(|d| - 1, i + \text{sgn}(d)) & \text{if } j = 0,
\end{cases}
\]

where \( \text{sgn}(d) \) is the sign of \( d \), and \( \lfloor x \rfloor \) is the greatest integer smaller than or equal to \( x \). Pushing out along the automorphism \( \sigma \) is an auto-equivalence of \( \text{Rep}(Q) \), still denoted by \( \sigma \).

For a finite-dimensional representation \( M \) written as a tuple \( M = (M_0, M_1, \ldots, M_{|d|-1}) \), we have \( \sigma(M) = (M_1, M_2, \ldots, M_{|d|-1}, \tau^{-\text{sgn}(d)} M_0) \), where \( \tau \) is the Auslander–Reiten translation of \( \text{rep}(\overrightarrow{A}_\infty) \). The following is an easy observation.

**Lemma A.6.** There is an equivalence of categories between \( \text{Grmod}(\Gamma) \) and \( \text{Rep}(Q) \) such that it restricts to an equivalence between \( \text{grmod}(\Gamma) \) and \( \text{rep}(Q) \) and the following diagrams
are commutative:

\[
\begin{array}{ccc}
\text{Grmod}(\Gamma) & \sim & \text{Rep}(Q) \\
\downarrow^{(1)} & & \downarrow^{\sigma} \\
\text{grmod}(\Gamma) & \sim & \text{rep}(Q) \\
\end{array}
\]

A.5. A covering functor

Let \( Q \) and \( \sigma \) be as in the preceding subsection. Recall that \( n \) is a positive integer. Let \( m \) be the greatest common divisor of \( n \) and \( |d| \), and let \( n' = n/m, d' = d/m \). Let \( c \) be the inverse of \( d' \) modulo \( n' \).

Let \( \bar{Q} \) be the quotient quiver of \( Q \) under the automorphism \( \sigma^n \). Precisely, \( \bar{Q} \) is the disjoint union of \( m \)-copies of \( \Delta_{n'} \), where \( \Delta_{n'} \) is the cyclic quiver with \( n' \) vertices, that is, the quiver

\[
\begin{array}{c}
0 \\
\vdots \\
1 \leftrightarrow 2 \\
\end{array}
\]

The vertices of \( \bar{Q} \) are labelled \((j, i)\), \(0 \leq j \leq m - 1, 0 \leq i \leq n' - 1\). Moreover, the covering map \( C : Q \to \bar{Q} \) is given by the unique map between quivers, which takes the following value on vertices:

\[
C(j, i) = \left( j - \left\lfloor \frac{j}{m} \right\rfloor m, i - \left\lfloor \frac{i}{n'} \right\rfloor n' \right).
\]

The map \( C \) induces a pair of adjoint triangle functors

\[
\begin{array}{c}
\mathcal{D}(\text{Rep } Q) \xrightarrow{C_*} \mathcal{D}(\text{Rep } \bar{Q}), \\
\mathcal{D}(\text{Rep } \bar{Q}) \xleftarrow{C^*} \mathcal{D}(\text{Rep } Q)
\end{array}
\]

where \( C_* \) is the pull-back functor and \( C^* \) is the push-out functor:

\[
C_*(X)_{(\bar{i}, \bar{j})} = \bigoplus_{(i,j) \in \bar{Q}_0} X_{(i,j)}, \quad \text{for } X \in \mathcal{D}(\text{Rep } Q) \text{ and } (\bar{i}, \bar{j}) \in \bar{Q}_0.
\]

The functor \( C_* \) is essentially surjective, and restricts to a triangle functor \( \mathcal{D}^b(\text{rep } Q) \to \mathcal{D}^b(\text{rep } \bar{Q}) \) (here by \( \text{rep} \) we mean the category of finite-dimensional nilpotent representations). By abuse of notation, we denote \( C = C_* \). It is easy to prove the following lemma.

**Lemma A.7.** Let \( X \) be an object of \( \mathcal{D}(\text{Rep } Q) \). Then

\[
C^*C(X) = \bigoplus_{p \in \mathbb{Z}} \sigma^{np}(X).
\]

Let \( \bar{\sigma} : \bar{Q} \to \bar{Q} \) be the unique automorphism of \( \bar{Q} \) taking the following value on vertices:

\[
\bar{\sigma}(j, i) = \left\{ \begin{array}{ll}
(j - 1, i) & \text{if } 1 \leq j \leq m - 1, \\
(m - 1, i + c - \left\lfloor \frac{i + c}{n'} \right\rfloor n') & \text{if } j = 0.
\end{array} \right.
\]
Then the following diagram is commutative:
\[ \begin{array}{ccc}
Q & \xrightarrow{C} & \tilde{Q} \\
\downarrow{\sigma^{-1}} & & \downarrow{\sigma^{-1}} \\
Q & \xrightarrow{C} & Q
\end{array} \]
and induces a commutative diagram of triangle functors:
\[ \begin{array}{ccc}
D^{b}(\text{grmod} \Gamma) & \xrightarrow{\sim} & D^{b}(\text{rep} \ Q) \\
\downarrow{\Sigma \circ (-1)} & & \downarrow{\Sigma \circ \sigma^{-1}} \\
D^{b}(\text{grmod} \ Gamma) & \xrightarrow{\sim} & D^{b}(\text{rep} \ Q) \\
\end{array} \xrightarrow{C} \begin{array}{ccc}
D^{b}(\text{rep} \ Q) & \xrightarrow{C} & D^{b}(\text{rep} \ Q) \\
\downarrow{\Sigma \circ \sigma^{-1}} & & \downarrow{\Sigma \circ \sigma^{-1}} \\
D^{b}(\text{rep} \ Q) & \xrightarrow{C} & D^{b}(\text{rep} \ Q).
\end{array} \]

Here, by abuse of notation, we denote by \( \tilde{\sigma} \) the push-out functor along the automorphism \( \sigma \).
By Theorem 2.1, the orbit category \( D^{b}(\text{rep} \ Q)/\Sigma \circ \sigma^{-1} \) admits a canonical triangle structure. Thus, by Lemma A.1 and Proposition A.5, we obtain a chain of triangle functors
\[ D_{fd}(\Gamma) \xrightarrow{\sim} D^{b}(\text{grmod} \ Gamma)/\Sigma \circ (-1) \xrightarrow{\sim} D^{b}(\text{rep} \ Q)/\Sigma \circ \sigma^{-1} \xrightarrow{C} D^{b}(\text{rep} \ Q)/\Sigma \circ \sigma^{-1}. \]

Let \( \Phi \) be the composition of the above three triangle functors.

**Lemma A.8.** Assume that \( n \) is even.

(a) We have a natural isomorphism of triangle functors \( \Phi \circ \Sigma^{n} \simeq \Phi \).

(b) The functor \( \Phi \) is essentially surjective.

(c) The triangle functor \( \Phi \) induces a bijection for any objects \( M \) and \( N \) of \( D_{fd}(\Gamma) \)
\[ \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{D_{fd}(\Gamma)}(M, \Sigma^{p}N) \longrightarrow \text{Hom}_{D^{b}(\text{rep} \ Q)/\Sigma \circ \sigma^{-1}}(\Phi(M), \Phi(N)). \]

**Proof.** (a) The triangle structures of \( \Sigma \) and \( \hat{\sigma} \) yield the following commutation morphisms:
\[ \begin{array}{ccc}
-1_{\Sigma^{2}} : \Sigma \circ \Sigma & \rightarrow & \Sigma \circ \Sigma, \\
1_{\hat{\sigma}^{-2}} : \hat{\sigma}^{-1} \circ \hat{\sigma}^{-1} & \rightarrow & \hat{\sigma}^{-1} \circ \hat{\sigma}^{-1}, \\
\phi_{12} : \Sigma \circ \hat{\sigma}^{-1} & \rightarrow & \hat{\sigma}^{-1} \circ \Sigma, \\
\phi_{21} : \hat{\sigma}^{-1} \circ \Sigma & \rightarrow & \Sigma \circ \hat{\sigma}^{-1}.
\end{array} \]

By Subsection A.1, we have two induced (triangle) auto-equivalences of \( D^{b}(\text{rep} \ Q)/\Sigma \circ \sigma^{-1} \), which, by abuse of notation, will still be denoted by \( \Sigma \) and \( \hat{\sigma}^{-1} \). Moreover, \( \Sigma \circ \hat{\sigma}^{-1} \cong \Delta(-1_{\Sigma \circ \sigma^{-1}}) \).

Now since \( \Phi \) is a triangle functor, it follows that
\[ \Phi \circ \Sigma^{n} \cong \Sigma^{n} \circ \Phi \cong \Delta((-1)^{n}1_{\Sigma \circ \sigma^{-1}}) \circ \hat{\sigma}^{n} \circ \Phi \cong \Phi. \]

(b) In view of Lemma A.1, this is because \( C \) is essentially surjective.

(c) It suffices to prove that the functor \( C \) induces a bijection for any object \( X \) and \( Y \) of \( D^{b}(\text{rep} \ Q) \):
\[ \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{D^{b}(\text{rep} \ Q)}(X, \sigma^{p}Y) \longrightarrow \text{Hom}_{D^{b}(\text{rep} \ Q)}(C(X), C(Y)). \]

By Lemma A.7 the space on the left is isomorphic to \( \text{Hom}_{D^{b}(\text{rep} \ Q)}(X, C^{*}C(Y)) \). Thus, the bijectivity of the map under investigation follows from the adjointness of \( C \) and \( C^{*} \).

**Proposition A.9.** Let \( n \in \mathbb{N} \) be even. We have a triangle equivalence
\[ \Phi : D_{fd}(\Gamma)/\Sigma^{n} \longrightarrow D^{b}(\text{rep} \ Q)/\Sigma \circ \sigma^{-1}. \]
A.6. The characterization

Recall that $m$ is the greatest common divisor of $n$ and $|d|$, $n' = n/m$, $d' = d/m$ and $c$ is the inverse of $d'$ modulo $n'$.

Let $\tau$ be the Auslander–Reiten translation of $D^b(\text{rep } \Delta_{n'})$. It follows from Theorem 2.1 that the orbit category $D^b(\text{rep } \Delta_{n'})/\tau^c \circ \Sigma^m$ admits a canonical triangle structure. Let $\pi : D^b(\text{rep } \Delta_{n'}) \to D^b(\text{rep } \Delta_{n'})/\tau^c \circ \Sigma^m$ denote the canonical projection functor. Recall that $\tilde{Q}$ is the disjoint union of $m$-copies of $\Delta_{n'}$. So an object $X$ of $D^b(\text{rep } \tilde{Q})$ can be written as an ordered sequence $X = (X_0, \ldots, X_{m-1})$, where $X_0, \ldots, X_{m-1} \in D^b(\text{rep } \Delta_{n'})$. We define a triangle functor $\Pi : D^b(\text{rep } \tilde{Q}) \to D^b(\text{rep } \Delta_{n'})$ by setting $\Pi(X) = \bigoplus_{j=0}^{m-1} \Sigma^{m-1-j} X_j$. Let $\Psi = \pi \circ \Pi$ be the composition.

**Lemma A.10.** The functor $\Psi : D^b(\text{rep } \tilde{Q}) \to D^b(\text{rep } \Delta_{n'})/\tau^c \circ \Sigma^m$ induces a triangle equivalence

$$\tilde{\Psi} : D^b(\text{rep } \tilde{Q})/\Sigma \circ \tilde{\sigma}^{-1} \to D^b(\text{rep } \Delta_{n'})/\tau^c \circ \Sigma^m.$$

**Proof.** We claim that the following diagram is commutative:

$$
\begin{array}{ccc}
D^b(\text{rep } \tilde{Q}) & \xrightarrow{\Psi} & D^b(\text{rep } \Delta_{n'})/\tau^c \circ \Sigma^m \\
\downarrow{\Sigma \circ \sigma^{-1}} & & \downarrow{\pi} \\
D^b(\text{rep } \tilde{Q}) & \xrightarrow{\Psi} & D^b(\text{rep } \Delta_{n'})/\tau^c \circ \Sigma^m.
\end{array}
$$

Indeed, for $X = (X_0, \ldots, X_{m-1}) \in D^b(\text{rep } \tilde{Q})$, we have

$$
\Psi(X) = \pi \left( \bigoplus_{j=0}^{m-1} \Sigma^{m-1-j} X_j \right),
$$

$$
\Psi \circ \Sigma \circ \sigma^{-1}(X) = \Psi(\tau^c \circ \Sigma X_{m-1}, \Sigma X_1, \ldots, \Sigma X_{m-2}) = \pi \left( \tau^c \circ \Sigma^m X_{m-1} \oplus \bigoplus_{j=0}^{m-2} \Sigma^{m-1-j} X_j \right).
$$

The natural isomorphism $\Psi \simeq \Psi \circ \Sigma \circ \sigma^{-1}$ is then induced from the natural isomorphism $\pi \simeq \pi \circ \tau^c \circ \Sigma^m$. It is clear that $\tilde{\Psi}$ is essentially surjective because so is $\Psi$. Moreover, $\tilde{\Psi}$ induces an identity of morphism spaces

$$
\text{Hom}_{D^b(\text{rep } \Delta_{n'})/\tau^c \circ \Sigma^m}(\Psi X, \Psi Y) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{D^b(\text{rep } \tilde{Q})}(X, (\Sigma \circ \sigma^{-1})^p Y).
$$

Applying Lemma A.1 yields the desired result: $\tilde{\Psi}$ is a triangle equivalence. It remains to prove the above identity. Writing $X = (X_0, \ldots, X_{m-1})$ and $Y = (Y_0, \ldots, Y_{m-1})$, it follows by induction that, for $p \in \mathbb{Z}$,

$$
(\Sigma \circ \sigma^{-1})^p Y = \Sigma^p(\tau^{cp} Y_{m-j_p}, \ldots, \tau^{cp} Y_{m-1}, \tau^{c(p'-1)} Y_0, \ldots, \tau^{c(p'-1)} Y_{m-j_p-1}),
$$

where $0 < j_p < m$.\qed
where \( p' = \lceil p/m \rceil \) is the smallest integer greater than or equal to \( p/m \), and \( j_p = m + p - mp' \). Therefore (below \((?, ?) = \text{Hom}_{D^b(\text{rep } \overline{\Sigma}_n)}(? , ?)\))

\[
\text{Right-hand side} = \bigoplus_{p \in \mathbb{Z}} \left( \bigoplus_{j=0}^{m-1} (X_j, \Sigma^{p'} \circ \tau^{cp'} Y_{m-j_p+j}) \right)
\]

\[
\text{Right-hand side} = \bigoplus_{p' \in \mathbb{Z}, j_p = 1} \left( \bigoplus_{j=0}^{m-1} (X_j, \Sigma^{p'} \circ \tau^{cp'} Y_{m-j_p+j}) \right)
\]

\[
= \bigoplus_{p' \in \mathbb{Z}, j_p = 1} \left( \bigoplus_{j=0}^{m-1} (X_j, \Sigma^{p'} \circ \tau^{cp'} Y_{m-j_p+j}) \right)
\]

On the other hand, we have

\[
\text{Left-hand side} = \bigoplus_{p \in \mathbb{Z}} (\Psi X, (\tau^c \circ \Sigma^m)^p \Psi Y)
\]

\[
\text{Left-hand side} = \bigoplus_{p \in \mathbb{Z}} \left( \bigoplus_{j=0}^{m-1} \Sigma^{m-1-j} X_j, (\tau^c \circ \Sigma^m)^p \bigoplus_{j=0}^{m-1-j} \Sigma^{m-1-j} Y_j \right)
\]

\[
= \bigoplus_{p \in \mathbb{Z}} \bigoplus_{j=1}^{m-1} X_j, \tau^{cp} \circ \Sigma^{j-j'+mp} Y_{j'} \right)
\]

Combining Proposition A.9 and Lemma A.10, we obtain the main result of this section.

**Theorem A.11.** Let \( n \in \mathbb{N} \) be even. Then we have a triangle equivalence

\[
\Psi \circ \Phi : D_{fd}(\Gamma)/\Sigma^n \rightarrow D^b(\text{rep } \overline{\Sigma}_n)/\tau^c \circ \Sigma^m.
\]

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