习题 1.2 16

设 $a_1 = a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$, $n = 2, 3, \ldots$.

令 $x_n = \frac{a_{n+1}}{a_n}$, 证明数列 $\{x_n\}$ 收敛于 $\frac{1}{2}(\sqrt{5}+1)$.

Proof. **法 1 (需要用到单调有界原理):**

$a_n:\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots$ 单调上升.

$x_n = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{1}{x_{n-1}}$.

$x_{n+2} - x_n = \left[1 + \frac{1}{x_{n+1}}\right] - \left[1 + \frac{1}{x_{n-1}}\right] = -\frac{x_{n+1} - x_{n-1}}{x_{n+1}x_{n-1}}$.

同理: $x_{n+1} - x_{n-1} = -\frac{x_n - x_{n-2}}{x_n x_{n-2}}$ 代入上式得

$$x_{n+2} - x_n = -\frac{x_{n+1} - x_{n-1}}{x_{n+1}x_{n-1}} = \frac{x_n - x_{n-2}}{x_{n+1}x_{n-1}x_n x_{n-2}},$$

从而数列 $\{x_n\}$ 的奇数次项和偶数次项均单调.

由 $x_n:\ 1,\ 2,\ \frac{3}{2},\ \frac{5}{3}, \ldots$ 知数列 $\{x_n\}$ 的奇数次项单调上升, 偶数次项单调下降.

$x_n = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$ 且 a_n 单调上升得 $1 < x_n < 2$.

根据单调有界原理，数列 $\{x_n\}$ 的奇数次项和偶数次项均收敛.

对于奇数次项 (或偶数次项) $x_n = 1 + \frac{1}{x_{n-1}} = 1 + \frac{1}{1 + \frac{1}{x_{n-2}}}$, 设 $\lim_{n\to\infty} x_n = t$, 两边取极限得

$$t = 1 + \frac{1}{1 + \frac{1}{t}},$$

解得 $t = \frac{1}{2}(\sqrt{5}+1)$ 或 $t = \frac{1}{2}(-\sqrt{5}+1)$ (舍). $\square$

Proof. **法 2 (需要用到上下确界的知识):**

a_n : 1, 1, 2, 3, 5, 8, 13, 21, ... 单调上升.

$x_n = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$ 且 a_n 单调上升得 $1 < x_n < 2$.

则数列 $\{x_n\}$ 有上下确界, 不妨记为 T 和 t.

$x_n = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{1}{x_{n-1}}$, 等式两边同时取上确界得

$$T = 1 + 1/t,$$

类似，两边同时取下确界得

$$t = 1 + 1/T.$$

联立上述两式并舍去负值得 $T = t = \frac{1}{2}(\sqrt{5} + 1)$. $\square$

Proof. **法 3 (需要用到线性代数的知识):**

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

由

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1-\sqrt{5} \\ \sqrt{5}-1 & 2 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}+1}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 2 & 1-\sqrt{5} \\ \sqrt{5}-1 & 2 \end{pmatrix}^{-1}$$

得

$$\begin{aligned}
&\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \\
&= \begin{pmatrix} 2 & 1-\sqrt{5} \\ \sqrt{5}-1 & 2 \end{pmatrix} \begin{pmatrix} [\frac{\sqrt{5}+1}{2}]^{n-1} & 0 \\ 0 & [\frac{1-\sqrt{5}}{2}]^{n-1} \end{pmatrix} \begin{pmatrix} 2 & 1-\sqrt{5} \\ \sqrt{5}-1 & 2 \end{pmatrix}^{-1} \\
&= \frac{1}{10-2\sqrt{5}} \begin{pmatrix} 2 & 1-\sqrt{5} \\ \sqrt{5}-1 & 2 \end{pmatrix} \begin{pmatrix} [\frac{\sqrt{5}+1}{2}]^{n-1} & 0 \\ 0 & [\frac{1-\sqrt{5}}{2}]^{n-1} \end{pmatrix} \begin{pmatrix} 2 & \sqrt{5}-1 \\ 1-\sqrt{5} & 2 \end{pmatrix}
\end{aligned}$$

代入第一个式子计算得 $a_n = \frac{1}{10-2\sqrt{5}}\Big[4[\frac{\sqrt{5}+1}{2}]^{n-1} + (6-2\sqrt{5})[\frac{1-\sqrt{5}}{2}]^{n-1}\Big]$.
从而

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \frac{\frac{1}{10-2\sqrt{5}}\Big[4[\frac{\sqrt{5}+1}{2}]^{n} + (6-2\sqrt{5})[\frac{1-\sqrt{5}}{2}]^{n}\Big]}{\frac{1}{10-2\sqrt{5}}\Big[4[\frac{\sqrt{5}+1}{2}]^{n-1} + (6-2\sqrt{5})[\frac{1-\sqrt{5}}{2}]^{n-1}\Big]} \\
&= \frac{\sqrt{5}+1}{2}.
\end{aligned}$$

□

Proof. **法 4 (需要用到级数的知识):**
由 $\frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} < 2$ 得
级数 $f(x) = \sum_{k=1}^{\infty} a_k x^k$ 在区间 $(-1/2, 1/2)$ 内收敛.

$$\begin{aligned}
f(x) = a_1x^1 + &a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\
xf(x) = &a_1x^2 + a_2x^3 + a_3x^4 + \cdots \\
\Rightarrow\ f(x) + xf(x) = a_1x^1 + &a_3x^2 + a_4x^3 + a^5x^4 + \cdots \\
\Rightarrow\ f(x) + xf(x) = a_1x^1 + &\big(f(x) - a_1x^1 - a_2x^2\big)/x
\end{aligned}$$

即

$$xf(x) + x^2f(x) = a_1x^2 + f(x) - a_1x^1 - a_2x^2,$$

代入 a_1, a_2 并化简得

$$f(x) = \frac{-x}{x^2 + x - 1} = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \frac{\sqrt{5}+1}{2}x} - \frac{1}{1 + \frac{\sqrt{5}-1}{2}x}\right),$$

用幂级数展开得 (化简后, 法 3 中的 a_n 应与下面的 a_n 一样)

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

所以

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\sqrt{5}+1}{2}.$$

□