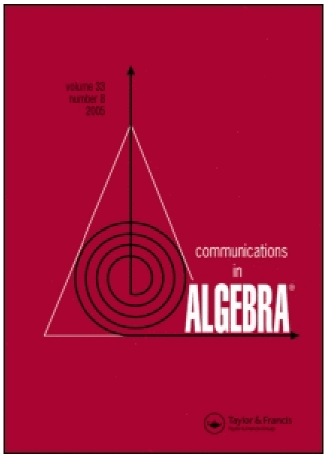

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Xuejun Guo [a]
[a] Department of Mathematics, Nanjing University, Najing, China

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A REMARK ON K_2 OF THE RINGS OF INTEGERS OF TOTALLY REAL NUMBER FIELDS

Xuejun Guo
Department of Mathematics, Nanjing University, Najing, China

Let F be a totally real number field with degree $n = [F : \mathbb{Q}] \geq 3$. Mazur and Urbanowicz proved that if

$$K_2\mathcal{O}_F \simeq (\mathbb{Z}/2\mathbb{Z})^{[F:\mathbb{Q}]} \qquad (*)$$

and F is not $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ or $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$, then F must be one of the 14 cases listed in Mazur and Urbanowicz (1992). In this article, it is proved that 3 of these 14 cases don't satisfy $(*)$, while all the other cases satisfy $(*)$. Hence we find all totally real number fields which satisfy $(*)$.

Key Words: Birch–Tate conjecture; Totally real number fields; Zeta function.

Mathematics Subject Classification (2000): 11R70; 19F15.

1. INTRODUCTION

Let F be a totally real number field with degree $[F : \mathbb{Q}] = n$, $\mathcal{O}_F$ the ring of integers of F. The Birch–Tate conjecture states that

$$|K_2(\mathcal{O}_F)| = w_2(F)|\zeta_F(-1)|,$$

where

$$w_2(F) = 2 \prod_{1 \text{ prime}} l^{n_l},$$

and $n(l)$ is the largest integer n such that F contains $\mathbb{Q}(\zeta_{l^n} + \zeta_{l^n}^{-1})$, the maximal real subfield of $\mathbb{Q}(\zeta_{l^n})$.

This conjecture is proved in Mazur and Wiles (1984) for totally real abelian number fields up to 2-torsion. In Kolster (1986) proved this conjecture holds if the 2-subgroup of $K_2(\mathcal{O}_F)$ is elementary Abelian. Later, Wiles (1990) proved that the Birch–Tate conjecture also gives the correct powers of 2 for totally real abelian number fields. One can see details in Kolster's appendix to Rognes and Weibel (2000).

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Address correspondence to Xuejun Guo, Department of Mathematics, Nanjing University, Najing 210093, China; E-mail: guoxj@nju.edu.cn

Since $[F:\mathbb{Q}] = n$, there is a subgroup H_2^0F of $K_2(\mathcal{O}_F)$ such that the quotient group $K_2(\mathcal{O}_F)/H_2^0F$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. One can see the definition of H_2^0F in Gras (1986). Note that this subgroup is denoted by $K_2^+(\mathcal{O}_F)$ in Keune (1989). This means $K_2(\mathcal{O}_F)$ is at least $(\mathbb{Z}/2\mathbb{Z})^n$. It is very interesting to find all totally real number fields F satisfying $(*)$.

Browkin and Hurrelbrink (1984) proved that there are only four real quadratic fields satisfying $(*)$

$$F = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5}), \text{and } \mathbb{Q}(\sqrt{13}).$$

Hurrelbrink (1982) and Kirchheimer (1981) proved that

$$F = \mathbb{Q}(\zeta_9 + \zeta_9^{-1}) \qquad \text{and} \qquad \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$$

also satisfy $(*)$.

If $(*)$ is satisfied, the Birch–Tate conjecture holds by Theorem 3.4 of Kolster (1986). Hence one can use the Birch–Tate conjecture to compute the $|K_2(\mathcal{O}_F)|$. Mazur and Urbanowicz (1992) proved that if a totally real number field F with degree $[F:\mathbb{Q}] = 3$ is not the maximal subfield of a cyclotomic field and satisfies $(*)$, then $F = \mathbb{Q}(x)$, where x satisfies one of the following 14 equations:

(1) $x^3 - x^2 - 3x + 1 = 0$,
(2) $x^3 - x^2 - 4x - 1 = 0$,
(3) $x^4 - x^3 - 3x^2 + x + 1 = 0$,
(4) $x^4 - x^3 - 4x^2 + 4x + 1 = 0$,
(5) $x^4 - 4x^2 + x + 1 = 0$,
(6) $x^4 - 5x^2 + 5 = 0$,
(7) $x^4 - 4x^2 + 2 = 0$,
(8) $x^5 + 2x^4 - 5x^3 - 2x^2 + 4x - 1 = 0$,
(9) $x^5 - 5x^3 + x^2 + 5x - 1 = 0$,
(10) $x^5 - 5x^3 + x^2 + 3x + 1 = 0$,
(11) $x^5 - 2x^4 - 4x^3 + 3x^2 + 2x - 1 = 0$,
(12) $x^5 + 3x^4 - 4x^3 - 5x^2 + 5x - 1 = 0$,
(13) $x^5 - 6x^3 + 3x^2 + 2x - 1 = 0$,
(14) $x^5 - 2x^4 - 6x^3 + 3x^2 + 6x - 1 = 0$.

In this article, we prove that if $F = \mathbb{Q}(x)$, where x satisfies (4), (5), or (8), then $(*)$ does not hold. In case (4), $K_2(\mathcal{O}_F) \simeq \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$. In case (7), $K_2(\mathcal{O}_F) \simeq \mathbb{Z}/5\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^4$. In case (8), $K_2(\mathcal{O}_F) \simeq \mathbb{Z}/5\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^5$. All the other cases satisfy condition $(*)$.

In fact, although Mazur and Urbanowicz wanted to exclude the maximal real subfields of cyclotomic fields, case (8) is just $\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$. And from case (9) to case (14), the fields $\mathbb{Q}(x)$ defined by the six equations are just the different embeddings in $\mathbb{R}$ of a same real number field.

Note that we don't assume Birch–Tate conjecture hold for all totally real number fields. We use only the proved fact that odd-part of Birch–Tate conjecture holds for all totally real number fields. As for the 2-primary part of $K_2(\mathcal{O}_F)$, we use Theorem 3.1 of Kolster (1986) to compute the 2^k-rank of $K_2(\mathcal{O}_F)$. The computations have been performed by PARI/GP 2.2.12.

2. DESCRIPTION OF THE COMPUTATION

In this section, the number field $\mathbb{Q}(x)$ defined by the equation $f_i = 0$ in case (i) in the Introduction will be denoted by F_i. PARI/GP can compute the Galois group of a polynomial and the value of the Dedekind zeta function of F_i at -1. By the formula of $w_2(F)$ in the introduction, it is also easy to compute the value. In the next, we will just list the value of $w_2(F_i)|\zeta_{F_i}(-1)|$.

Since F_2, F_4, F_6, F_7, and F_8 are totally real abelian number fields, the Birch–Tate conjecture holds. By PARI/GP, we know that $w_2(F_i)|\zeta_{F_i}(-1)| = 24 \times \frac{1}{3}, 120 \times \frac{4}{15}, 24 \times \frac{2}{3}, 32 \times \frac{5}{6}, 264 \times \frac{20}{33}$, for $i = 2, 4, 6, 7, 8$, respectively. Hence $|K_2(\mathcal{O}_{F_i})| = 8, 32, 16, 5 \times 16, 5 \times 32$, for $i = 2, 4, 6, 7, 8$, respectively. Since there is a surjective homomorphism from $K_2(\mathcal{O}_{F_i})$ to $(\mathbb{Z}/2\mathbb{Z})^{[F_i:\mathbb{Q}]}$, we know that

$$
\begin{aligned}
&K_2(\mathcal{O}_{F_2}) \simeq (\mathbb{Z}/2\mathbb{Z})^3,\\
&K_2(\mathcal{O}_{F_4}) \simeq (\mathbb{Z}/2\mathbb{Z})^5 \quad \text{or} \quad \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3,\\
&K_2(\mathcal{O}_{F_6}) \simeq (\mathbb{Z}/2\mathbb{Z})^4,\\
&K_2(\mathcal{O}_{F_7}) \simeq \mathbb{Z}/5\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^4,\\
&K_2(\mathcal{O}_{F_8}) \simeq \mathbb{Z}/5\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^5.
\end{aligned}
$$

To determine the structure of $K_2(\mathcal{O}_{F_4})$, we need to compute the 2-rank of $K_2(\mathcal{O}_{F_4})$. There is a general formula for the 2-rank of $K_2(\mathcal{O}_F)$ in Tate (1976):

$$
2\text{-rank } K_2(\mathcal{O}_F) = r_1 + g_2 - 1 + 2\text{-rank } Cl(\mathcal{O}_F[1/2]), \tag{2.1}
$$

where r_1 is the number of real places of F, and g_2 is the number of dyadic places of F. By PARI/GP, we get $g_2(F_4) = 1$, 2-rank $Cl(\mathcal{O}_F[1/2]) = 0$. So 2-rank $K_2(\mathcal{O}_{F_4}) = 4$. Hence $K_2(\mathcal{O}_{F_4}) \simeq \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$. To make certain, one can also use the Theorem 3.1 of Kolster (1986) to compute the 4-rank of $K_2(\mathcal{O}_{F_4})$, which is 1.

As for F_1, F_3, F_5, they are not Galois number fields. The $w_2(F_i)|\zeta_{F_i}(-1)| = 24 \times \frac{1}{3}, 120 \times \frac{2}{15}, 24 \times \frac{2}{3}$, for $i = 1, 3, 5$. So $K_2(\mathcal{O}_{F_i})$ is a 2-group for $i = 1, 3, 5$. To compute the 2-rank of $K_2(\mathcal{O}_{F_i})$ by (2.1), we need to know the number of dyadic primes of F_i, and the class group of $\mathcal{O}_{F_i}$. We use PARI/GP to compute the decomposition of the polynomial f_i in $\mathbb{Q}_2[x]$. In all the three cases, f_i are irreducible in $\mathbb{Q}_2[x]$. So $g_2 = 1$. And all of the three class groups are trivial. So the 2-rank $K_2(\mathcal{O}_{F_1}) = 3$, 2-rank $K_2(\mathcal{O}_{F_3}) = 4$, and 2-rank $K_2(\mathcal{O}_{F_5}) = 4$. Next we will compute the 4-rank.

Let m be the number of dyadic primes of F, which decompose in $F(\sqrt{-1})$, $A(F(\sqrt{-1})/F)$ the 2-Sylow-subgroup of the relative S-class-group of $F(\sqrt{-1})/F$, where S consists of all infinite and all dyadic primes of F. Kolster (1986) proved that

$$
2^n\text{-rank } K_2(\mathcal{O}_F) = m + 2^{n-1}\text{-rank } A(F(\sqrt{-1})/F)/\text{im}({}_2A(F)), \tag{2.2}
$$

where ${}_2A(E)$ consists of the elements of order ≤ 2 in the S-class-group of F. The three polynomials $f_1(x + \sqrt{-1})f_1(x - \sqrt{-1}), f_3(x + \sqrt{-1})f_3(x - \sqrt{-1})$, and

$f_5(x+\sqrt{-1})f_5(x-\sqrt{-1})$ are all irreducible in $\mathbb{Q}[x]$. So they define the fields $F_i(\sqrt{-1})$ for $i = 1, 3$, and 5, respectively. By PARI/GP, we can compute m and class number in (2.2). The result is 4-rank $K_2(\mathcal{O}_{F_i}) = 0$ for $i = 1, 3, 5$, which implies that (∗) holds for these 3 cases.

Using the function "factornf" in PARI/GP, we find that f_9 has always a root in $\mathbb{Q}[x]/f_i$ for $10 \le i \le 14$. Hence the last 6 polynomials f_i $(9 \le i \le 14)$ essentially define the same totally real number field. The value of ζ_{F_9} at -1 is $4/3$, $w_2(F_9) = 24$, $g_2 = 1$, the class number of F_9 is 1 and the class number of $F_9[\sqrt{-1}]$ is 3. Hence the $K_2(\mathcal{O}_{F_i}) = (\mathbb{Z}/2\mathbb{Z})^5$ for $9 \le i \le 14$.

We write the above results as a theorem.

Theorem 2.1. *If F is a totally real number field, then F satisfies the condition*

$$K_2\mathcal{O}_F \simeq (\mathbb{Z}/2\mathbb{Z})^{[F:Q]}$$

if and only if F is one of the 13 *fields* (*isomorphic fields are seen as the same field*): $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$, $\mathbb{Q}(\zeta_7+\zeta_7^{-1})$, $\mathbb{Q}(\zeta_9+\zeta_9^{-1})$, $\mathbb{Q}(x)$, *where x satisfies one of the following* 6 *equations*:

(1) $x^3 - x^2 - 3x + 1 = 0$,
(2) $x^3 - x^2 - 4x - 1 = 0$,
(3) $x^4 - x^3 - 3x^2 + x + 1 = 0$,
(4) $x^4 - 4x^2 + x + 1 = 0$,
(5) $x^4 - 5x^2 + 5 = 0$,
(6) $x^5 - 5x^3 + x^2 + 5x - 1 = 0$.

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