



The 3-adic regulators and wild kernels

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Abstract

For any number field, J.-F. Jaulent introduced a new invariant called the group of logarithmic classes in 1994. This invariant is proved to be closely related to the wild kernels of number fields. In this paper, we show how to compute the kernel of the natural homomorphism from the group of logarithmic classes to the group of p -ideal classes by computing the p -adic regulator which is a classical invariant in number theory. As an application, we prove Gangl's conjecture on 9-rank of the tame kernel of imaginary quadratic field $\mathbb{Q}(\sqrt{-9k-3})$.

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1. Introduction

The 2-primary part of the tame kernel $K_2\mathcal{O}_F$ of a quadratic number field F has been intensively studied. And there are also some results concerning the p -primary part of the tame kernel when p is odd, however there are few results on the p^2 -rank of the tame kernel of a number field not containing a primitive p th root of unity.

There are some general results on the p -primary part of the tame kernel of number fields in [18] and [13]. Based on his numerical computations, Gangl proposed in [6] some conjectures,

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which, in the case $p = 3$, relate the divisibility of order of the tame kernel of imaginary quadratic number fields by 3 or 9 to the divisibility of class numbers of the same imaginary quadratic number fields by 3. There is a list of the conjectural tame kernels and the wild kernels of imaginary quadratic number fields with discriminant larger than -5000 in [3].

In 1992, Browkin studied the p -rank of the tame kernels of quadratic number fields by the reflection theorem in [1]. He proved two of Gangl’s conjectures. However, one of these conjectures remains open, i.e., if 9 divides the cardinality of $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-9k-3})})$, then 3 divides the class number of $\mathbb{Q}(\sqrt{-9k-3})$, where k is a positive integer and $3k + 1$ is square-free.

Later in 1994, J.-F. Jaulent introduced a new invariant called the group of logarithmic classes in [8]. The arithmetic of this logarithmic group can give some information on the wild kernels of number fields. One can see [4,8–12] and [16] for details. Especially, Pauli and Soriano-Gafiuk can describe the p -rank of the wild kernel of quadratic number field $\mathbb{Q}(\sqrt{d})$ by the logarithmic p -class group of the quadratic number field $\mathbb{Q}(\sqrt{-3d})$ without assuming $\mathbb{Q}(\sqrt{d})$ contains a primitive p th root of unity in [16].

In this paper, we use the 3-adic regulators to compute the difference between the logarithmic 3-class group and the 3-ideal class group. Our computation relies on the results of Kishi in [14]. The p -adic regulators can not give all information of the difference between the logarithmic 3-class group and the 3-ideal class group in general. However in some cases, we can determine the kernel of the natural homomorphism from the logarithmic 3-class group to the 3-ideal class group by the 3-adic regulator. Then by a theorem of Pauli and Soriano-Gafiuk on the wild kernel and logarithmic class group, we can prove Gangl’s conjecture.

2. The logarithmic ℓ -class group

In the following context, we will briefly review the definition of logarithmic ℓ -class group and some theorems proved in [4] and [16]. We will use the notation as in [4] and [16]. For the details of logarithmic ℓ -class group, one can see [4,8–12] and [16].

Let ℓ be a prime number and Log_ℓ be the ℓ -adic logarithm function defined in [19]. For any prime number p , we define

$$\text{deg}_\ell p = \begin{cases} \text{Log}_\ell p & \text{for } p \neq \ell; \\ \ell & \text{for } p = \ell \neq 2; \\ 4 & \text{for } p = \ell = 2. \end{cases}$$

For $a \in \mathbb{Q}_p^\times \simeq p^\mathbb{Z} \times \mathbb{F}_p^\times \times (1 + 2p\mathbb{Z}_p)$ denote by $\langle a \rangle$ the projection of a to $(1 + 2p\mathbb{Z}_p)$. Let F be a number field and \wp a prime ideal of \mathcal{O}_F over p . Let F_\wp be the completion of F with respect to \wp . For any $a \in F^*$, it is defined in [16] that

$$g_\wp(\alpha) := \frac{\text{Log}_p(\mathbf{N}_{F_\wp/\mathbb{Q}_p}(\alpha))}{[F_\wp : \mathbb{Q}_p] \cdot \text{deg}_p p}.$$

The logarithmic ramification index \tilde{e}_\wp is defined in [16] as follows. The p -part of the logarithmic ramification index \tilde{e}_\wp is $[g_\wp(F_\wp^*) : \mathbb{Z}_p]$. For all primes q with $q \neq p$, the q part of \tilde{e}_\wp is the q part of the ramification index e_\wp of \wp . The logarithmic inertia degree \tilde{f}_\wp is defined by the relation $\tilde{e}_\wp \tilde{f}_\wp = e_\wp f_\wp = \text{deg}(F_\wp/\mathbb{Q}_p)$, where f_\wp is the classical inertia degree. For any place \wp , we define $\text{deg}_\ell \wp$ to be $\tilde{f}_\wp \text{deg}_\ell p$.

For any $x \in \mathcal{R}_F = \mathbb{Z}_\ell \otimes_{\mathbb{Z}} F^*$,

$$\tilde{v}_\wp(x) := -\frac{\text{Log}_p(\mathbf{N}_{F_\wp/\mathbb{Q}_p}(x))}{\text{deg}_I \wp}.$$

The group of ℓ -ideals is defined to be

$$\mathcal{I}d_{F,\ell} := \left\{ I = \prod_{\wp \nmid \ell} \wp^{\alpha_\wp} \mid \alpha_\wp = 0 \text{ for almost all } \wp \right\}$$

in [16]. And

$$\tilde{\mathcal{I}}d_{F,\ell} := \{ I \in \mathcal{I}d_{F,\ell} \mid \text{deg}_\ell I = 0 \}$$

is the subgroup of ℓ -ideals of degree 0. The group

$$\tilde{\mathcal{P}}_{F,\ell} := \left\{ I = \prod_{\wp \nmid \ell} \wp^{v_\wp(a)} \mid a \in \mathcal{R}_F \text{ and } \tilde{v}_\wp(a) = 0 \text{ for any } \wp \mid \ell \right\}$$

is the subgroup of principal ℓ -ideals having logarithmic valuation 0 at all ℓ -adic places. The group of logarithmic ℓ -classes is isomorphic to the quotient of the latter two:

$$\tilde{\mathcal{C}}\ell_{F,\ell} \simeq \mathcal{I}d_{F,\ell} / \tilde{\mathcal{I}}d_{F,\ell}.$$

Let \wp_1, \dots, \wp_s be the ℓ -adic places of F . Let $\tilde{\mathcal{C}}\ell(\ell)$ be the ℓ group of logarithmic divisor classes of degree zero:

$$\tilde{\mathcal{C}}\ell(\ell) := \left\{ [I] \in \tilde{\mathcal{C}}\ell \mid I = \sum_{i=1}^s a_i \wp_i \text{ with } \text{deg}_F(I) = 0 \right\}.$$

Let $\mathcal{C}\ell'$ be the ℓ -group of the ℓ -ideal classes, i.e., the ℓ -part of $\mathcal{C}\ell / \langle [\wp_1], \dots, [\wp_s] \rangle$, where $\mathcal{C}\ell$ is the ordinary class group.

Lemma 2.1. (See [4,5].) *Let*

$$\theta : \tilde{\mathcal{C}}\ell \longrightarrow \mathcal{C}\ell', \quad \sum_{\wp} m_\wp \wp \mapsto \prod_{\wp \nmid \ell} \wp^{(1/\lambda_\wp)m_\wp},$$

where λ_\wp is the quotient of the logarithmic valuation \tilde{v}_\wp over the ordinary valuation v_\wp . The sequence

$$0 \longrightarrow \tilde{\mathcal{C}}\ell(\ell) \longrightarrow \tilde{\mathcal{C}}\ell \xrightarrow{\theta} \mathcal{C}\ell' \longrightarrow \text{Coker } \theta \longrightarrow 0$$

is exact.

Let I be an ideal of a number field F . Recall that the I -units are defined to the elements in F^* such that $v_{\wp}(a) = 0$ for any $\wp \nmid I$, where v_{\wp} is the ordinary \wp -valuation such that $v_{\wp}(\pi) = 1$ for any uniformizer of F_{\wp} .

Lemma 2.2. (See [4, Lemma 13], Generators and Relations of $\tilde{\mathcal{C}}\ell(\ell)$.) Let \wp_1, \dots, \wp_s be the ℓ -adic places of F . Assume that $s > 1$. Reorder the \wp_i such that

$$v_{\ell}(\deg(\wp_1)) = \min_{1 \leq i \leq s} v_{\ell}(\deg(\wp_i)).$$

Let $\gamma_1, \dots, \gamma_r$ be a basis of the ℓ -units of F . Then the group $\tilde{\mathcal{C}}\ell(\ell)$ is given by the generators

$$[I_i] := \left[\wp_i - \frac{\deg(\wp_i)}{\deg(\wp_1)} \wp_1 \right] \quad (i = 2, \dots, s)$$

with relations $\sum_{i=2}^s \tilde{v}_{\wp_i}(\gamma_j)[I_i] = 0$.

3. Computing the 3-adic regulator for a real quadratic number field $\mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod{3}$

In the case $3 \nmid d$, Kishi classifies in [14] a fundamental unit ε of $F = \mathbb{Q}(\sqrt{d})$ into eight types:

- (i) $\text{Tr}_F \varepsilon \equiv 0 \pmod{9}$;
- (ii) $N_F \varepsilon = -1, \text{Tr}_F \varepsilon \equiv \pm 1, \pm 2 \pmod{9}$;
- (iii) $N_F \varepsilon = 1, \text{Tr}_F \varepsilon \equiv \pm 1 \pmod{9}$;
- (iv) $N_F \varepsilon = 1, \text{Tr}_F \varepsilon \equiv \pm 2 \pmod{27}$;
- (v) $N_F \varepsilon = 1, \text{Tr}_F \varepsilon \equiv \pm 2 \pmod{9} \not\equiv \pm 2 \pmod{27}$;
- (vi) $\text{Tr}_F \varepsilon \equiv \pm 3 \pmod{9}$;
- (vii) $N_F \varepsilon = -1, \text{Tr}_F \varepsilon \equiv \pm 4 \pmod{9}$;
- (viii) $N_F \varepsilon = 1, \text{Tr}_F \varepsilon \equiv \pm 4 \pmod{9}$.

By [14], (iii) and (viii) do not occur; (ii) and (vii) occur only when $d \equiv 2 \pmod{3}$.

The following theorem was proved by Herz in 1966. In 2000, Kishi gave another proof in [14].

Theorem 3.1. (See [7, Theorem 6]; [14, Theorem 4.4].) Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field with class number $h(d)$. Let ε be a fundamental unit of F and let $h(-3d)$ denote the class number of the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-3d})$. Then $3|h(-3d)$ if and only if at least one of the following conditions holds:

- (a₁) $N_F \varepsilon = 1, \text{Tr}_F \varepsilon \equiv \pm 2 \pmod{27}$;
- (a₂) $\text{Tr}_F \varepsilon \equiv 0 \pmod{9}$;
- (a₃) $N_F \varepsilon = -1, \text{Tr}_F \varepsilon \equiv \pm 4 \pmod{9}$;
- (b) $3|h(d)$.

Assume that a number field F has r_1 real places and r_2 pairs of complex places. Let $r = r_1 + r_2 - 1$. Let $\sigma_1, \dots, \sigma_{r_1}, \sigma_{r_1+1}, \overline{\sigma_{r_1+1}}, \dots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+r_2}}$ be all the embeddings of F into \mathbb{C}_p .

Let $\delta_i = 1$ if σ_i is real, $\delta_i = 2$ if σ_i is complex. Let $\{\varepsilon_1, \dots, \varepsilon_r\}$ be a basis for the units of F modulo roots of unity. Then the p -adic regulator is defined to be

$$R_p(F) = \det(\delta_i \text{Log}_p(\sigma_i \varepsilon_j))_{1 \leq i, j \leq r}.$$

One can see §5.5 of [19] for details.

In the following context, we will assume that $F = \mathbb{Q}(\sqrt{d})$ with positive $d \equiv 1 \pmod{3}$. In this case, we have $r = 1$. Let ε be a fundamental unit and E the group generated by ε . Then the p -adic regulator is

$$R_p(F) = \text{Log}_p(\varepsilon).$$

Let

$$E_{(3)^2} := \{u \in E \mid u \equiv 1 \pmod{(3)^2}\}.$$

Let v_3 be the ordinary valuation in \mathbb{C}_3 such that $v_3(3) = 1$.

Remark. Although we use the same symbol E as in [14], the meaning of it is slightly different. In [14], the E contains also the roots of unity in F . While in this paper, it contains only the free abelian group generated by the fundamental unit ε . There is a misprint in the second table of page 8 of [14]. In case (vii), the norm of ε should be -1 .

Lemma 3.2. *Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field with $d \equiv 1 \pmod{3}$. With notation in the above paragraph, we have*

$$v_3(R_3(F)) + v_3(|E/E_{(3)^2}|) \geq 2,$$

where $|E/E_{(3)^2}|$ is the cardinality of the group $E/E_{(3)^2}$. Furthermore, if $v_3(|E/E_{(3)^2}|) = 1$, then $v_3(R_3(F)) = 1$.

Proof. Since $d \equiv 1 \pmod{3}$, 3 splits completely in F . Choose a fundamental unit $\varepsilon = 1 + a$, where a belongs to some prime ideal \wp over 3. Since 3 splits completely, $F_\wp = \mathbb{Q}_3$. By Lemma 5.5 of [19], $v_3(R_3(F)) = v_3(\text{Log}_3(1 + a)) = v_3(a)$.

Assume $|E/E_{(3)^2}| = n$. Then $(1 + a)^n \in E_{(3)^2}$. For any $x = x_1 + x_2\sqrt{d} \in F$ with $x_1, x_2 \in \mathbb{Q}$, let $\bar{x} = x_1 - x_2\sqrt{d}$ be the conjugate of x .

(1) If $N_F \varepsilon = \varepsilon \bar{\varepsilon} = 1$, then $a = \varepsilon - 1 \in \wp$ and the conjugate $\bar{a} = \bar{\varepsilon} - 1 = \varepsilon^{-1}(1 - \varepsilon) \in \wp$. Hence $a \in \wp$ which implies $a \in \wp \bar{\wp} = (3)$. So if $\varepsilon^n \in E_{(3)^2}$ and $N_F \varepsilon = 1$, then $1 + na + na^2 + \dots \in E_{(3)^2}$. Hence $na \in (3)^2$, i.e., $v_3(a) + v_3(n) \geq 2$. So

$$v_3(R_3(F)) + v_3(|E/E_{(3)^2}|) \geq 2.$$

Furthermore if $v_3(a) = 1$, then $3|n$ is a sufficient and necessary condition to make $(1 + a)^n \in E_{(3)^2}$. So $|E/E_{(3)^2}| = 3$ or 6 , i.e., $v_3(n) = 1$. If $v_3(a) \geq 2$, then $n = 1$ or 2 , i.e., $v_3(n) = 0$. Hence if $v_3(|E/E_{(3)^2}|) = 1$, then $v_3(R_3(F)) = 1$.

(2) If $N_F \varepsilon = \varepsilon \bar{\varepsilon} = -1$, then we have $1 + a + \bar{a} + a\bar{a} = -1$. So

$$\text{Tr}_F \varepsilon = (1 + a) + \overline{(1 + a)} = 2 + a + \bar{a} = -a\bar{a} \in (3).$$

So ε must be of type (i) or (vi) in the table in the beginning of this section.

If ε is of type (i), i.e., $\text{Tr}_F \varepsilon \equiv 0 \pmod{9}$, then $(3)|(a)$ or $\wp^2|(a)$. If $(3)|(a)$, then the proof is the same as in (1). If $\wp^2|(a)$, then a is not a uniformizer of $F_\wp = \mathbb{Q}_3$. Hence $v_3(R_3(F)) = v_3(\text{Log}_3(1+a)) = v_3(a) \geq 2$.

If a is of type (vi), then by the second table in page 8 in [14], $v_3(|E/E_{(3)^2}|) = 1$. Since $\text{Tr}_F \varepsilon = -a\bar{a} \equiv \pm 3 \pmod{9}$, $\wp|(a)$ but $\wp^2 \nmid (a)$. So a is a uniformizer of $F_\wp = \mathbb{Q}_3$. Hence $v_3(R_3(F)) = 1$. \square

Corollary 3.3. *Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field with $d \equiv 1 \pmod{3}$. Then $v_3(R_3(F)) = 1$ for the cases (v) and (vi) of the table in the beginning of this section and $v_3(R_3(F)) = 2$ for the cases (i) and (iv). Then other cases do not occur.*

Proof. It follows from Lemma 3.2 and the third table in the page 8 in [14]. \square

4. The wild kernels $WK_2(F)$ of quadratic number fields

Let F be a number field. The tame kernel $K_2\mathcal{O}_F$ is the kernel of the tame symbols and the wild kernel $WK_2(F)$ is defined to be the kernel of the norm residue symbols. One can see [15] for details.

Lemma 4.1. *(See [16, Example 5].) If $\ell = 3$ and $F = \mathbb{Q}(\sqrt{d})$ with $d \in \mathbb{Z}$ square-free, then*

$$3\text{-rank } WK_2(F) = 3\text{-rank } \widetilde{C}\ell_{\mathbb{Q}(\sqrt{-3d})}.$$

Lemma 4.2. *(See [2, Table 1].) The index $i_F := (K_2(\mathcal{O}_F) : WK_2(F))$ always divides 6. And 3 divides i_F if and only if $d \equiv -3 \pmod{9}$, $d \neq -3$.*

Let $F = \mathbb{Q}(\sqrt{3k+1})$ be a real quadratic number field. Then 3 splits completely in F . Let \wp_1 and \wp_2 be the primes over 3. Then $F_{\wp_i} = \mathbb{Q}_3$, where $i = 1$ or 2. Let \wp be \wp_1 or \wp_2 .

With notation in Section 2, we have

$$g_\wp(F_\wp^*) := \frac{\text{Log}_3(N_{F_\wp/\mathbb{Q}_3}(F_\wp^*))}{[F_\wp : \mathbb{Q}_3] \cdot \text{deg}_3 3} = \frac{\text{Log}_3\langle \mathbb{Q}_3^* \rangle}{\text{deg}_3 3} = \mathbb{Z}_3.$$

By definition, $\tilde{e}_\wp = [g_\wp(F_\wp^*) : \mathbb{Z}_3] = 1$. Since $\tilde{e}_\wp \tilde{f}_\wp = e_\wp f_\wp = \text{deg}(F_\wp/\mathbb{Q}_p) = 1$, we have $\tilde{f}_\wp = 1$. Hence $\text{deg}_3 \wp = \tilde{f}_\wp \text{deg}_3 3 = 3$. So for any $x \in \mathcal{R}_F = \mathbb{Z}_3 \otimes_{\mathbb{Z}} F^*$,

$$\tilde{v}_\wp(x) := -\frac{\text{Log}_3(N_{F_\wp/\mathbb{Q}_3}(x))}{\text{deg}_3 \wp} = -\frac{\text{Log}_3(x)}{3}.$$

By Lemma 2.2, the group $\widetilde{C}\ell(\ell)$ is generated by one generator $I = [\wp_2 - \wp_1]$. This generator satisfies three relations. One of these three relations is

$$\tilde{v}_\wp(\varepsilon)[I] = 0.$$

Since

$$\tilde{v}_{\wp}(\varepsilon) = -\frac{\text{Log}_3(\varepsilon)}{3} = -\frac{R_3(F)}{3},$$

by the proof of Lemma 3.2, we have the following lemma.

Lemma 4.3. *Let $F = \mathbb{Q}(\sqrt{3k+1})$ be a real quadratic field and ε a fundamental unit of F . Then if $v_3(R_3(F)) = 1$, then $v_3(\tilde{v}_{\wp}(\varepsilon)) = 0$. Hence if $v_3(R_3(F)) = 1$, then $\widetilde{C\ell}(\ell)$ is trivial for $\ell = 3$.*

Theorem 4.4. *Let $F = \mathbb{Q}(\sqrt{-9k-3})$ be an imaginary quadratic number field. If 9 divides the cardinality of $K_2(\mathcal{O}_F)$, then 3 divides the class number of F .*

Proof. If 9 divides the cardinality of $K_2(\mathcal{O}_F)$, there are two possible cases, one is the 9-rank of $K_2(\mathcal{O}_F)$ greater than 0, another is the 3-rank of $K_2(\mathcal{O}_F)$ greater than 1.

(1) If the 3-rank of $K_2(\mathcal{O}_F)$ is greater than 1, then 3 divides the class number of F by Theorem 5.6 of [1].

(2) If the 9-rank of $K_2(\mathcal{O}_F)$ and the 3-rank of $K_2(\mathcal{O}_F)$ are both 1, then the 3-rank of $WK_2(F)$ is 1 by Lemma 4.2. By Lemma 4.1, this implies the 3-rank of $\widetilde{C\ell}_{\mathbb{Q}(\sqrt{3k+1})}$ is 1.

The fundamental unit of $\mathbb{Q}(\sqrt{3k+1})$ can be one of cases (i), (iv), (v), (vi) of the table in the beginning of Section 3. If it is (i) or (iv), then 3 divides the class number of F by Theorem 3.1. If it is (v) or (vi), then by Corollary 3.3, $v_3(R_3(\mathbb{Q}(\sqrt{3k+1}))) = 1$. Hence by Lemma 4.3, $\widetilde{C\ell}(\ell)_{\mathbb{Q}(\sqrt{3k+1})}$ is trivial. So there is an injective homomorphism from $\widetilde{C\ell}_{\mathbb{Q}(\sqrt{3k+1})}$ to $C\ell'_{\mathbb{Q}(\sqrt{3k+1})}$ by Lemma 2.1 for $\mathbb{Q}(\sqrt{3k+1})$. So the 3-rank of $C\ell'_{\mathbb{Q}(\sqrt{3k+1})}$ is positive. Hence 3 divides the class number of $\mathbb{Q}(\sqrt{3k+1})$.

By the reflection theorem of Scholz [17]:

$$3\text{-rank } C\ell_{\mathbb{Q}(\sqrt{-9k-3})} \geq 3\text{-rank } C\ell_{\mathbb{Q}(\sqrt{3k+1})} > 0.$$

So 3 divides the class number of F . \square

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