IMAGINARY QUADRATIC FIELDS WITH ONO NUMBER 3

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Abstract: In this paper, we prove that an imaginary quadratic field F has class group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if the Ono number of F is 3 and F has exactly 3 ramified primes under the Extended Riemann Hypothesis (ERH). In addition, we give the list of all imaginary quadratic fields with Ono number 3.

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1. **Introduction**

Let F be an imaginary quadratic field with discriminant D, and

$$f_D(x) = \begin{cases} x^2 + x + \frac{1-D}{4}, & \text{if } D \equiv 1 \mod 4 \\ x^2 - \frac{D}{4}, & \text{if } D \equiv 0 \mod 4 \end{cases}$$

The Ono number p_D of F is defined as $p_D = \max\{\Omega(f_D(i)),\ i \in \mathbb{Z} \cap [0, \frac{-D}{4} - 1]\}$, where $\Omega(f_D(i))$ is the number of prime factors (counting multiplicity) of $f_D(i)$. One can see [7] for details. Let h_D be the class number of $\mathbb{Q}(\sqrt{D})$. Then by the Frobenius-Rabinowitch Theorem ([3, 8]), $h_D = 1$ if and only if $p_D = 1$. Sasaki proved in [12] that $h_D \geq p_D$ and $h_D = 2$ if and only if $p_D = 2$. F. Sairaiji and K. Shimizu proved some very useful inequalities between class numbers and Ono numbers in [9, 10]. J. Cohen and J. Sonn conjectured in [2] that $h_D = 3$ if and only if $p_D = 3$ and D is a prime. Their conjecture is proved to be true in [4].

We will show in this paper that the ono number is not only related to the class number, but also related to the structure of the class group. Let $\mathcal{C}_D$ be the class group of F, t_D the number of ramified primes in F. We will prove that $\mathcal{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if $p_D = 3$ and $t_D = 3$. In addition, we give the list of all imaginary quadratic fields with Ono number 3.

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2. Main results

We use the same notations as in Section 1. Let q_D be the smallest prime number which splits completely in $\mathbb{Q}(\sqrt{D})$. By the genus theory of Gauss, if $\mathcal{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $t_D = 3$. By Theorem 2.2 of [11] (or cf. [6]), we have $p_D = 3$. Next we will prove that if $t_D = 3$ and $p_D = 3$, then $\mathcal{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Theorem 2.1. (ERH). If $p_D = 3$, then $|D| \leq 9.662 \times 10^{10}$.

Proof. Assume that $p_D = 3$. By Theorem 2.1 of [10] (cf. [5]; p.111) or Lemma 17 and Theorem 18 of [2], $|D/4| < q_D^3$. By Theorem 3.3 of [10], we have $|D| \leq 10^{15}$. By Table 3 of [1], $q_D \leq (1.881 \cdot \log|D| + 6.18)^2$ if $25 \leq \log|D| \leq 100$. So we have $|D/4| < (1.881 \cdot \log|D| + 6.18)^6$ which implies that $|D| \leq 9.662 \times 10^{10}$. $\square$

By GP/PARI, one can prove that if $|D| \leq 9.662 \times 10^{10}$, then $q_D \leq 167$. In fact, there are only two imaginary quadratic fields satisfying $|D| \leq 9.662 \times 10^{10}$ and $q_D > 157$. They are

$$q_{-30942935860} = 167,$$

$$q_{-18936628027} = 163.$$

Theorem 2.2. (ERH). If $p_D = 3$, then $|D| < 18629852$.

Proof. Since $q_D \leq 167$ and $|D/4| < q_D^3$ (Theorem 2.1 of [10]), we have

$$|D| < 4 \cdot 167^3 = 18629852.$$

$\square$

We can determine all imaginary quadratic fields with Ono number 3 with the help of computer. We give the complete list on next page. By that table, we have the following theorem.

Theorem 2.3. (ERH). The ideal class group $\mathcal{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if $t_D = 3$ and $p_D = 3$.

If the ideal class group $\mathcal{C}_D \simeq \mathbb{Z}/4\mathbb{Z}$, then $t_D = 2$ by the genus theory. By Sasaki's inequality ([12]) $h_D \geq p_D$, we have $p_D = 3$ or 4. However the inverse direction is not correct, i.e., $t_D = 2$ and $p_D = 3$ or 4 do not imply that $\mathcal{C}_D \simeq \mathbb{Z}/4\mathbb{Z}$. The last case of the table is a counterexample. In that case, $t_{-3763} = 2$ and $p_{-3763} = 3$, while $\mathcal{C}_{-3763} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

In the following table, (2)(2) means $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, (4) means $\mathbb{Z}/4\mathbb{Z}$, (3) means $\mathbb{Z}/3\mathbb{Z}$ and (2)(3) means $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

Table of imaginary quadratic fields with Ono number 3

(ordered by the square free part of D)

No.	$-D$	t_D	p_D	$\mathcal{C}_D$	No.	$-D$	t_D	p_D	$\mathcal{C}_D$
1	$84 = 2^2 \cdot 3 \cdot 7$	3	3	(2)(2)	27	$1012 = 2^2 \cdot 11 \cdot 23$	3	3	(2)(2)
2	23	1	3	(3)	28	283	1	3	(3)
3	$120 = 2^3 \cdot 3 \cdot 5$	3	3	(2)(2)	29	307	1	3	(3)
4	31	1	3	(3)	30	331	1	3	(3)
5	$132 = 2^2 \cdot 3 \cdot 11$	3	3	(2)(2)	31	$355 = 5 \cdot 71$	2	3	(4)
6	$168 = 2^3 \cdot 3 \cdot 7$	3	3	(2)(2)	32	379	1	3	(3)
7	$184 = 2^3 \cdot 23$	2	3	(4)	33	$435 = 3 \cdot 5 \cdot 29$	3	3	(2)(2)
8	$228 = 2^2 \cdot 3 \cdot 19$	3	3	(2)(2)	34	$483 = 3 \cdot 7 \cdot 23$	3	3	(2)(2)
9	59	1	3	(3)	35	499	1	3	(3)
10	$280 = 2^3 \cdot 5 \cdot 7$	3	3	(2)(2)	36	547	1	3	(3)
11	$292 = 2^2 \cdot 73$	2	3	(4)	37	$555 = 3 \cdot 5 \cdot 37$	3	3	(2)(2)
12	$312 = 2^3 \cdot 3 \cdot 13$	3	3	(2)(2)	38	$595 = 5 \cdot 7 \cdot 17$	3	3	(2)(2)
13	$328 = 2^3 \cdot 41$	2	3	(4)	39	$627 = 3 \cdot 11 \cdot 19$	3	3	(2)(2)
14	83	1	3	(3)	40	643	1	3	(3)
15	$340 = 2^2 \cdot 5 \cdot 17$	3	3	(2)(2)	41	$715 = 5 \cdot 11 \cdot 13$	3	3	(2)(2)
16	$372 = 2^2 \cdot 3 \cdot 31$	3	3	(2)(2)	42	$723 = 3 \cdot 241$	2	3	(4)
17	$408 = 2^3 \cdot 3 \cdot 17$	3	3	(2)(2)	43	$763 = 7 \cdot 109$	2	3	(4)
18	107	1	3	(3)	44	$795 = 3 \cdot 5 \cdot 53$	3	3	(2)(2)
19	$520 = 2^3 \cdot 5 \cdot 13$	3	3	(2)(2)	45	883	1	3	(3)
20	$532 = 2^2 \cdot 7 \cdot 19$	3	3	(2)(2)	46	907	1	3	(3)
21	139	1	3	(3)	47	$1003 = 17 \cdot 59$	2	3	(4)
22	$708 = 2^2 \cdot 3 \cdot 59$	3	3	(2)(2)	48	$1243 = 11 \cdot 113$	2	3	(4)
23	$760 = 2^3 \cdot 5 \cdot 19$	3	3	(2)(2)	49	$1387 = 19 \cdot 73$	2	3	(4)
24	$772 = 2^2 \cdot 193$	2	3	(4)	50	$1435 = 5 \cdot 7 \cdot 41$	3	3	(2)(2)
25	$195 = 3 \cdot 5 \cdot 13$	3	3	(2)(2)	51	$1555 = 5 \cdot 311$	2	3	(4)
26	211	1	3	(3)	52	$3763 = 53 \cdot 71$	2	3	(2)(3)

References

[1] E. Bach, Explicit bounds for primality testing and related problems, Math. Comp. 55 (1990), 355C380.

[2] J. Cohen and J. Sonn, On the Ono invariants of imaginary quadratic fields, J. Number Theory 95 (2002), 259C267.

[3] F. G. Frobenius, Über quadratische Formen, die viele Primzahlen darstellen, Sitzungsber. Königliche Akad. Wiss. Berlin (1912) 966-980.

[4] H. Gu, D. Gu and Y. Liu, Ono invariants of imaginary quadratic fields with class number three, J. Number Theory 127 (2007), no. 2, 262-271.

[5] H. Möller, Verallgemeinerung eines Satzes von Rabinowitsch über imaginär-quadratische Zahlkörper, J. Reine Angew. Math. 285 (1976), 100-113.

[6] R. A. Mollin, Quadratics, CRC Press, (1995).

[7] T. Ono, Arithmetic of algebraic groups and its applications, St. Paul's International Exchange Series Occasional Papers VI, St. Paul's university, Tokyo (1986).

[8] G. Rabinowitch, Eindeutigkeit der Zerlegung in Primzahlfaktoren in quadratischen Zahlkörpern, J. Reine Angew. Math. 142 (1913), 153-164.

[9] F. Sairaiji and K. Shimizu, A note on Ono's numbers associated to imaginary quadratic fields, Proc. Japan Acad. 77A (2001), 29-31.

[10] F. Sairaiji and K. Shimizu, An inequality between class numbers and Ono's numbers associated to imaginary quadratic fields, Proc. Japan Acad. 78A (2002), 105-108.

[11] K. Shimizu, Imaginary quadratic fields whose exponents are less than or equal to 2, Math. J. Okayama Univ., 50 (2008), 85-99.

[12] R. Sasaki, On a lower bound for the class number of an imaginary quadratic field, Proc. Japan Acad. 62A (1986), 37-39.