SOLVABLE LATTICE SUMS AND QUADRATIC DIRICHLET L-VALUES

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Abstract. We study how to express the Epstein zeta functions of binary quadratic forms in terms of a finite number of quadratic Dirichlet L-values. We prove a conjecture raised by Zucker and Robertson in 1984.

1. Introduction

Lattice sums are not only interesting mathematical subjects, but also very useful in physics and chemistry. One can see [GZ80] for a historical survey of lattices sums. Recall that for a positive definite binary quadratic form $F(x,y) = [A,B,C] := Ax^2 + Bxy + Cy^2$, the Epstein zeta functions is defined as

$$L(A,B,C,s) = \sum_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} (Ax^2 + Bxy + Cy^2)^{-s}, \Re(s) > 1.$$

Zucker and Robertson calculated many $L(A,B,C,1)$ by Dirichlet L-functions values at 1 in [ZR75], and they further calculated the values of many Epstein zeta functions corresponding to principal forms at general points in [ZR76]. Glasser and Zucker defined in [GZ80] a lattice sum to be solvable if it can be expressed in terms of a finite number of one-dimensional sums, i.e., Dirichlet L-series,

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

where χ is a quadratic Dirichlet character modulo N. They discussed many positive definite quadratic forms whose class group $\mathrm{Cl}(D)$ is 2-torsion when they calculated some lattice sums, and gave 111 examples connecting Epstein zeta functions and quadratic Dirichlet L-functions in [GZ80]. The followings are from the Table VI of [GZ80]. Note that $\chi_D = \left(\frac{D}{\cdot}\right)$ is the Kronecker character.

(1) $L(1,0,1,s) = 4\zeta(s)L(\chi_{-4})$,

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(2) $L(1,0,3,s) = 2(1+2^{1-2s})\zeta(s)L(\chi_{-3})$,
(3) $L(1,0,5,s) = \zeta(s)L(s,\chi_{-20}) + L(s,\chi_{-4})L(s,\chi_{+5})$,
(4) $L(1,0,15,s) = (1-2^{1-s}+2^{1-2s})\zeta(s)L(s,\chi_{-15})$
$+(1-2^{1-s}+2^{1-2s})L(s,\chi_{-3})L(s,\chi_{+5})$,
(5) $L(1,0,78,s) = \frac{1}{2}(\zeta(s)L(s,\chi_{-312}) + L(s,\chi_{-3})L(s,\chi_{+104})$
$+L(s,\chi_{13})L(s,\chi_{-24}) + L(s,\chi_{+8})L(s,\chi_{-39}))$,
(6) $L(1,1,1,s) = \zeta(s)L(s,\chi_{-3})$,
(7) $L(1,1,19,s) = (1+5^{1-2s})\zeta(s)L(s,\chi_{-3}) + L(s,\chi_{+5})L(s,\chi_{-15})$.

The discriminants of quadratic forms in (1), (3), (5) and (6) are fundamental, while none of the discriminants of quadratic forms in (2), (4) or (7) are fundamental. Glasser and Zucker raised the following conjecture on the character in the express form.

Conjecture 1.1. [GZ80] *Let Q_D be the set of all prmitive binary quadratic form $[A,B,C] := Ax^2+Bxy+Cy^2$ of discriminant D. The Epstein zeta functions of $[A,B,C] \in Q_D$ can be expressed by quadratic Dirichlet L-functions if and only if $[A,B,C]$ lives in a class of quadratic forms with one class per genus i.e.* $\mathrm{Cl}(D)$ *is* 2*-torsion.*

The "if" part of conjecture 1.1 was first proposed in [G73]. Zucker and Robertson in [ZR84] discovered a few strange counterexamples to the "only if" part. In those counterexamples, there are two different classes $[F] = [A,B,C]$ and $[f] = [A,-B,C]$ in the same genus, i.e., $\mathrm{ord}([f]) = \mathrm{ord}([F]) = 4$ and $\mathrm{Cl}(D) \cong (\mathbb{Z}/2\mathbb{Z})^r \oplus \mathbb{Z}/4\mathbb{Z}$.

Although the original conjecture is not true, Guillera and Rogers suggests in [GR14] that the original conjecture is ‘more or less’ correct.

In this paper, we prove the following theorem.

Theorem 1.2. *For a positive definite binary quadratic form $F = [A,B,C]$ whose discriminant $D := B^2-4AC = f_D^2 d_K$, the Epstein zeta functions can be expressed by*

$$L(A,B,C,s) = \sum_{\psi} C(s,\psi)L(s,\psi,K)$$

for some quadratic Hecke characters $\psi \in X_K^0(f_\psi)$ if and only if F satisfies one of the following conditions:

(1) *one class per genus, i.e., the class group* $\mathrm{Cl}(D)$ *of F is* 2*-torsion,*
(2) $\mathrm{Cl}(D) = (\mathbb{Z}/2\mathbb{Z})^n \oplus \mathbb{Z}/4\mathbb{Z}$, *and $[F]$ is an element of order* 4 *in* $\mathrm{Cl}(D)$,
where $C(s,\psi) := a_\psi \sum_{nf_\psi^2|f_D^2} c_n(\psi)n^{-s}$, $a_\psi,\ c_n(\psi) \in \mathbb{R}$, $f_\psi|f_D$ is the conductor of ψ. Moreover, if D is a fundamental discriminant, then $C(s,\psi) = a_\psi$.

Theorem 1.2 shows Glasser and Zucker's conjecture is true if the condition of the original conjecture is changed to (1) and (2) in Theorem 1.2. Since the quadratic Hecke L-functions can always be expressed by quadratic Dirichlet L-functions by Theorem 3.4, our theorem completely settle down the problem when a lattice sum is solvable.

We will prove Theorem 1.2 by considering the CM-modular form space $M_1^{CM}(\Gamma_0(N), \chi_D)$ for some N, D.

In this paper, "quadratic character" means the characters satifies $\chi^2 = 1$. In other words, the trivial character χ_0 can be regarded as an "quadratic character".

Notation: In this paper, D is a negative integer, K is an imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, $\mathcal{O}_K$ is the ring of integers of K, d_K is the discriminant of K, $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, and $\chi_D := \left(\frac{D}{\cdot}\right)$ is the Kronecker character, $\widehat{\mathrm{Cl}(D)}$ is the character group of $\mathrm{Cl}(D)$.

2. Preliminaries

2.1. **Hecke L-functions.** An *order* $\mathcal{O}$ in K is a subring $\mathcal{O} \subset K$ which is a finitely generated $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$, $m := [\mathcal{O}_K : \mathcal{O}]$ is called the *condutor* of $\mathcal{O}$, and $D = m^2 d_K$ is by definition the discriminant of $\mathcal{O}$. Let $\mathfrak{m}$ be an ideal of an imaginary quadratic field K and ℓ a non-negative integer. Let $I(\mathcal{O}, \mathfrak{m})$ be the group of fractional $\mathcal{O}$-ideals prime to $\mathfrak{m}$ and $P(\mathcal{O}, \mathfrak{m})$ the subgroup of $I(\mathcal{O}, \mathfrak{m})$ generated by principal ideals. We define

$$P_{K,\mathbb{Z}}(\mathfrak{m}) := \{\alpha\mathcal{O}_K | \alpha \in \mathcal{O}_K, \text{ and } \alpha \equiv a \bmod \mathfrak{m}\mathcal{O}_K, \ a \in \mathbb{Z}, \ (a, \mathfrak{m}) = 1\},$$

$$P_{K,1}(\mathfrak{m}) := \{\alpha\mathcal{O}_K | \alpha \in K^\times, \alpha \equiv 1 \bmod \mathfrak{m}\}.$$

Recall that

$$\mathrm{Cl}(\mathcal{O}) \cong I(\mathcal{O}, (m))/P(\mathcal{O}, (m)) \cong I(\mathcal{O}_K, (m))/P_{K,\mathbb{Z}}((m)).$$

We have definitions of Hecke character and q-series associated to Hecke character according to [F]:

Definition 2.1. *A Hecke character ψ of K modulo $\mathfrak{m}$ with ∞-type ℓ is a homomorphism*

$$\psi : I(\mathcal{O}_K, \mathfrak{m}) \to \mathbb{C}^\times$$

such that for all $\alpha \in P_{K,1}(\mathfrak{m})$, we have

$$\psi(\alpha\mathcal{O}_K) = \alpha^\ell.$$

The ideal $\mathfrak{m}$ is called the condutor of ψ if it is minimal in the sense that if ψ is defined modulo $\mathfrak{m}'$, then $\mathfrak{m}|\mathfrak{m}'$.

Definition 2.2. *For any Hecke character $\psi \in X_K^{\ell}(\mathfrak{m})$, we define*

$$f_\psi(\tau) := \sum_{(\mathfrak{a},\mathfrak{m})=1} \psi(\mathfrak{a}) q^{\mathcal{N}(\mathfrak{a})}.$$

and Hecke L-function

$$L(s,\psi,K) := \sum_{(\mathfrak{a},\mathfrak{m})=1} \psi(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{-s}.$$

Theorem 2.3. (Hecke, Shimura) *Let $\psi \in X_K^0(\mathfrak{m})$, f_ψ a Hecke eigenform of weight 1, level $N = \mathcal{N}(\mathfrak{m})|d_K|$. Then $f_\psi \in M_1(\Gamma_0(N), \chi_{d_K})$, and f_ψ is a newform if and only if $\mathfrak{m}$ is the conductor of ψ.*

2.2. **Binary theta series.** Let $Q(D)$ (Q_D) be the set of all (primitive) binary quadratic form $[A,B,C] := Ax^2 + Bxy + Cy^2$ of discriminant $D = B^2 - 4AC$, where primitive meansgcd$(A,B,C) = 1$. And we have a natural action $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ by $gF(x,y) = F(ax+by, cx+dy)$ for $F \in Q_D$.

We can identify an order with a positive definite quadratic form by the following map according to [C]:

$$\mathcal{I} : [a,b,c] \mapsto \mathrm{span}_{\mathbb{Z}} \left(a, \frac{b-\sqrt{D}}{2} \right)$$

induces an isomorphism $\mathrm{Cl}(D) \cong \mathrm{Cl}(\mathcal{O})$ for some order $\mathcal{O}$ whose discriminant is D.

Let $f(\tau)$ be a modular form of weight k with respect to some congruence subgroup Γ. We have a fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ at cusp $i\infty$. Recall that the L-function associated to f be $L(s,f) := \sum_{n=1}^{\infty} a_n n^{-s}$ via the Mellin transform. Two modular forms are same if and only if their related L-functions is same.

Denote $r(n,F) = \#\{(x,y) \in \mathbb{Z}^2 | F(x,y) = n\}$ for a positive definite binary quadratic form

$$F(x,y) = [a,b,c] := ax^2 + bxy + cy^2, \quad a,b,c \in \mathbb{Z},\ a > 0.$$

The theta series $\theta_F(\tau) = \sum_{(x,y)\in\mathbb{Z}^2} q^{F(x,y)} = \sum_{n=0}^{\infty} r(n,F) q^n$ is in the space $M_1(\Gamma_0(|D|), \chi_D)$ by the work of Hecke [H26], Weber [W93] and Schoeneberg [S39], where $D = b^2 - 4ac$.

Now we consider a fixed discriminant $D = f_D^2 d_K < 0$, where $K = \mathbb{Q}(\sqrt{D})$, $d_K = \mathrm{disc}(K)$, and f_D positive integer. Let $\Theta_D = \langle \theta_F | F \in Q_D \rangle$ and $\Theta(D) = \langle \theta_F | F \in Q(D) \rangle$ be the $\mathbb{C}$-spaces generated by the θ-series. It's easy to see $\Theta_D \subset M_1(\Gamma_0(|D|), \chi_D)$.

By Dirichlet and Weber result in [W82], one can show that the set $\{\theta_F | F \in Q_D/\mathrm{GL}_2(\mathbb{Z})\}$ is a basis of Θ_D. In particular,

$$\dim\Theta_D = \overline{h_D} := \# Q_D/\mathrm{GL}_2(\mathbb{Z}).$$

By Gauss's composition of binary forms, $\mathrm{Cl}(D) = Q_D/\mathrm{SL}_2(\mathbb{Z})$. Let $\widehat{\mathrm{Cl}(D)}$ be the character group of $\mathrm{Cl}(D)$. Let $h_D = \# Q_D/\mathrm{SL}_2(\mathbb{Z})$, $g_D = \#\mathrm{Cl}(D)/\mathrm{Cl}(D)^2$. Then we have

$$\overline{h_D} = \frac{h_D + g_D}{2}.$$

For a character $\chi \in \widehat{\mathrm{Cl}(D)}$, put

$$\theta_\chi(\tau) = \frac{1}{w_D} \sum_{F \in \mathrm{Cl}(D)} \chi(F)\theta_F(\tau) = \sum_{n=0}^{\infty} a_n(\chi) q^n \ \in \ \Theta_D,$$

where $w_D = 2$ for $D < 4$ and $w_{-3} = 6, w_{-4} = 4$.

Let Hecke algebra $\mathbb{T}(D)$ be the algebra generated by the Hecke operators T_p with $(p, D) = 1$.

Definition 2.4. *Let $f \in M_k(\Gamma_0(N), \psi)$ be a $\mathbb{T}(N)$-eigenfunction with eigencharacter $\lambda_f : \mathbb{T}(D) \to \mathbb{C}$. We say that f has CM (complex multiplication) by a Dirichlet character θ of conductor M if*

$$\lambda_f(T_p) = 0 \text{ for all } p \nmid NM \text{ with } \theta(p) \neq 1.$$

Let $M_k^{CM}(\Gamma_0(N), \psi; \theta)$ be the space generated by all $\mathbb{T}(N)$-eigenfunctions $f \in M_k(\Gamma_0(N), \psi)$ which have CM by θ.

Theorem 2.5 ([E12], Theorem 1, [E14], Theorem 1)**.** *Let θ_χ defined as above, then*

(1) $\{\theta_\chi\}_{\chi \in \widehat{\mathrm{Cl}(D)}}$ generates Θ_D and forms a basis of Θ_D by $\theta_\chi = \theta_{\overline{\chi}}$, θ_χ is a Hecke eigenfunction. And θ_χ is a cusp form if and only if χ is not a quadratic character.

(2) All of $\theta_\chi(m\tau)$ generated the $\Theta(D)$ for $\theta_\chi \in \Theta_{D_\chi}$ is a newform and $m^2 D\chi | D$. Moreover, $\Theta(D) = M_1^{CM}(\Gamma_0(N), \chi_D) := M_1^{CM}(\Gamma_0(N), \chi_D; \chi_D)$ and $\dim\Theta(D) = \sum_{f | d_D} 2^{\omega(f)} \overline{h}_{D/f^2}$, where $\omega(f)$ denotes the number of distinct prime divisors of f.

This theorem implies the following interesting result.

Theorem 2.6 ([E12], Theorem 2)**.** *We have $\Theta_D = \Theta_D^E \oplus \Theta_D^S$, where $\Theta_D^E = \Theta_D \cap E_1(\Gamma_0(|D|), \chi_D)$ denote the Eisenstein spaces part and $\Theta_D^S = \Theta_D \cap S_1(\Gamma_0(|D|), \chi_D)$ denotes the cusp space part of Θ_D. Moreover, $\{\theta_\chi | \chi \in \widehat{\mathrm{Cl}(D)}[2]\}$ is a basis of Θ_D^E, and that $\{\theta_\chi | \chi^2 \neq 1\}$ is a basis of Θ_D^S.*

If D is a fundamental discriminant, i.e. if $D = d_K$, then each θ_χ is a newform. However, in the general case this is no longer true for every $\chi \in \widehat{\mathrm{Cl}(D)}$, actually we have:

Theorem 2.7 ([E12], Theorem 3)**.** *Let $\chi \in \mathrm{Cl}(D)$, where $D = f_D^2 d_K$.*

(1) *There is a unique divisor $f_\chi | f_D$ and a unique primitive character $\chi_{pr} \in \mathrm{Cl}(D_\chi)$, where $D_\chi = f_\chi^2 d_K$, such that $\chi = \chi_{pr} \circ \overline{\pi}_{D,D_\chi}$.*

(2) *The form $\theta_{\chi_{pr}} \in \Theta_{D_\chi}$ is a newform of level $|D_\chi|$. Moreover, there exist constants $c_n(\chi) \in \mathbb{R}$ such that*

$$\theta_\chi(\tau) = \sum_{n | \overline{f_\chi^2}} c_n(\chi) \theta_{\chi_{pr}}(n\tau),$$

where $\overline{f_\chi^2} = f_D / f_\chi$.

If a newform f has real coefficients and its weight k is odd, then f has CM by [S09]. Moreover f_ψ has CM for any quadratic Hecke characters $\psi \in X_K^0(\mathfrak{m})$ and $f_\psi \in \Theta_D$, $D = \mathcal{N}(\mathfrak{m}) d_K$.

3. Main Results

In this section, we will prove the main theorem of this paper.

Theorem 3.1. *For a positive definite quadratic form $F = [A, B, C]$ whose discriminant is $D = B^2 - 4AC = f_D^2 d_K$, the Epstein zeta function can be expressed by linear combination of $L(s, \theta_\chi)$ for some quadratic characters χ if and only if F satisfies one of the following conditions:*

(1) *one class per genus, i.e. the class group $\mathrm{Cl}(D)$ of F is 2-torsion.*

(2) *$\mathrm{Cl}(D) = (\mathbb{Z}/2\mathbb{Z})^n \oplus \mathbb{Z}/4\mathbb{Z}$, and F is an element of order 4 in $\mathrm{Cl}(D)$.*

By (1) of Theorem 2.5, θ_D is generated by $\theta_{\chi_i}(\tau) := \frac{1}{w_D} \sum_{F \in \mathrm{Cl}(D)} \chi_i(F) \theta_F(\tau)$, where $\chi_i \in \widehat{\mathrm{Cl}(D)}$, $i = 1, 2, ..., h_D$. In other words, we can express each θ_F for $F \in \mathrm{Cl}(D)$ by θ_χ. Actually, $\theta_F(\tau) = \frac{w_D}{h_D} \sum_{\chi \in \widehat{\mathrm{Cl}(D)}} \overline{\chi(F)} \theta_\chi(\tau)$ since

$$\sum_{\chi \in \widehat{\mathrm{Cl}(D)}} \chi(F) = \begin{cases} h_D, & F \text{ is identity,} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\theta_{F_j} &= \frac{w_D}{h_D}\sum_{\chi\in\widehat{\mathrm{Cl}(D)}}\overline{\chi(F_j)}\theta_\chi \\ &= \frac{w_D}{h_D}\left(\sum_{\chi^2=1}\chi(F_j)\theta_\chi+\sum_{\chi^2\neq1}\overline{\chi(F_j)}\theta_\chi\right) \\ &= \frac{w_D}{h_D}\left(\sum_{\chi^2=1}\chi(F_j)\theta_\chi+\sum_{\substack{\chi^2\neq1\\(\chi,\overline{\chi})}}(\chi(F_j)+\overline{\chi(F_j)})\theta_\chi\right),\end{aligned}\tag{3.1}$$

where $(\chi,\overline{\chi})$ means take pairs of characters in this summation. Then we find a representation of θ_{F_j} by basis in Θ_D.

Proof. (of Theorem 3.1) If $\mathrm{Cl}(D)$ is 2-torsion, then all characters are quadratic; and if $\mathrm{Cl}(D)=(\mathbb{Z}/2\mathbb{Z})^n\oplus(\mathbb{Z}/4\mathbb{Z})$ and $\mathrm{ord}(F)=4$, then there is a character χ_0 such that $\chi_0(F)=\pm i$, and for each non-quadratic character χ, $\chi\overline{\chi_0}$ is quadratic by the fact $\widehat{\mathrm{Cl}(D)}\simeq\mathrm{Cl}(D)$. Hence we have $\chi(F)+\overline{\chi(F)}=0$ for each non-quadratic character χ. So $\theta_F=\frac{w_D}{h_D}\sum\limits_{\chi^2=1}\chi(F)\theta_\chi$ by $(\star)$. Note that two modular forms are equal if and only if the associated L-functions of them are equal. The Epstein zeta function of F is the L-function of θ_F. Hence the Epstein zeta function of F can be expressed by a combination of $L(s,\theta_\chi)$ for some quadratic characters if the condition (1) or (2) is satisfied.

Conversely, if the Epstein zeta function can be expressed by linear combination of $L(s,\theta_\chi)$ for some quadratic characters, then θ_F can be expressed by sum of θ-series for some quadratic characters, i.e. $\theta_F=\sum\limits_{\chi^2=1}c_\chi\theta_\chi$. Hence either $\widehat{\mathrm{Cl}(D)}$ is a 2-torsion group of characters; or $\chi(F)+\overline{\chi(F)}=0$ (i.e., $\chi(F)=\pm i$) for all non-quadratic character χ by (3.1). If $\widehat{\mathrm{Cl}(D)}$ is a 2-torsion group, then conditions (1) holds for $\mathrm{Cl}(D)$ is isomorphic to $\widehat{\mathrm{Cl}(D)}$.

If $\chi(F_i)=\pm i$ for each non-quadratic characters χ, then we will prove that condition (2) holds. We will prove it in two steps. At first, we will prove that $\mathrm{Cl}(D)$ is a 2-group and there is no element of order 8 in $\mathrm{Cl}(D)$. Then we will prove that there is only one $\mathbb{Z}/4\mathbb{Z}$ summand in the decomposition of $\mathrm{Cl}(D)$.

If $h_D=mp^e$ for some odd prime p, $(m,p)=1$, then exists a character $\chi_p\in\widehat{\mathrm{Cl}(D)}$ such that $\mathrm{ord}(\chi_p)=p$. But $\chi_p(F_i)^p=(\pm i)^p\neq1$, a contradiction. So $\mathrm{Cl}(D)$ is a 2-group and $h_D=2^n$ for some integer n. If $G\in\mathrm{Cl}(D)$ and $\mathrm{ord}(G)=8$, then exists a character $\chi_G\in\widehat{\mathrm{Cl}(D)}$ and $\mathrm{ord}(\chi_G)=8$ which implies χ_G^2 is a non-quadratic character. Hence $\mathrm{Cl}(D)$ hasn't any elements of order 8.

Now we will show $\mathrm{Cl}(D) = (\mathbb{Z}/2\mathbb{Z})^n \oplus \mathbb{Z}/4\mathbb{Z}$. If $\mathrm{Cl}(D) = (\mathbb{Z}/2\mathbb{Z})^n \oplus (\mathbb{Z}/4\mathbb{Z})^m$, where $m > 1$, then there are two characters $\chi_1, \chi_2 \in \widehat{\mathrm{Cl}(D)}$, such that $\chi_1, \chi_2, \chi_1\chi_2$ are non-quadratic characters. But if $\chi_1(F) = \pm i$, $\chi_2(F) = \pm i$, then $\chi_1\chi_2(F) = \pm 1 \neq \pm i$, which is impossible. Since there is a character χ such that $\chi(F) = \pm i$ and $\mathrm{Cl}(D)$ doesn't have any element of order 8, we have $\mathrm{ord}(F) = 4$. □

To prove Theorem 1.2, we need the following lemma.

Lemma 3.2. *For a positive definite quadratic form $F(x, y) = [A, B, C]$ whose discriminant $D := B^2 - 4AC = f_D^2 d_K$, the Epstein zeta function*

$$L(A, B, C; s) = \sum_{\psi} C(s, \psi)L(s, \psi, K)$$

for some quadratic Hecke character $\psi \in X_K^0(f_D)$ if and only if $L(A, B, C; s)$ can be expressed by linear combination of $L(s, \theta_\chi)$ for some quadratic characters χ. Moreover, $C(s, \psi) := a_\psi \sum_{n\mathcal{N}(f_\psi)|f_D^2} c_n(\psi)n^{-s}$, and a_ψ, $c_n(\psi) \in \mathbb{R}$, f_ψ is the conductor of ψ.

Proof. If $L(A, B, C; s) = \sum_{\psi} C(s, \psi)L(s, \psi, K)$ for some quadratic Hecke characters $\psi \in X_K^0(f_D)$, then $\theta_F = \sum_{\psi} a_\psi c_n(\psi) f_\psi(n\tau)$, where $f_\psi = \theta_{\chi_{pr}}$, $\chi_{pr}([F]) = \psi(\mathcal{I}([F])\mathcal{O})$ by [S09]. By (2) of Theorem 2.5, we know that $\theta_F \in \Theta_D$ is an Eisenstein series. Hence we can express θ_F by $\{\theta_\chi | \chi \in \widehat{\mathrm{Cl}(D)}[2]\}$ in Θ_D^E by Theorem 2.6. Then the Epstein zeta functions can be express by linear combination of $L(s, \theta_{\chi_i})$, which every χ_i are quadratic characters.

If $L(A, B, C; s)$ can be expressed by linear combination of $L(s, \theta_\chi)$ for some quadratic characters χ, then we have $L(s, \theta_\chi) = C(s, \chi)L(s, \theta_{\chi_{pr}}) = C(s, \chi)L(s, \psi, K)$ for some primitive quadratic character χ_{pr} and primitive Hecke character ψ by Theorem 2.7.

□

Then we have the main result in this paper.

Theorem 3.3. *For positive definite quadratic form $F(x, y) = [A, B, C]$ whose discriminant $D := B^2 - 4AC = f_D^2 d_K$, the Epstein zeta function can be expressed by*

$$L(A, B, C; s) = \sum_{\psi} C(s, \psi)L(s, \psi, K)$$

for some quadratic Hecke characters $\psi \in X_K^0(f_\psi)$ if and only if F satisfies one of the following conditions:

(1) *one class per genus, i.e. the class group* $\mathrm{Cl}(D)$ *of F is 2-torsion,*

(2) $\mathrm{Cl}(D) = (\mathbb{Z}/2\mathbb{Z})^n \oplus \mathbb{Z}/4\mathbb{Z}$ *and* $[F]$ *is an element of order* 4 *in* $\mathrm{Cl}(D)$, *where* $C(s,\psi) := a_\psi \sum_{nf_\psi^2 | f_D^2} c_n(\psi) n^{-s}$, $a_\psi,\ c_n(\psi) \in \mathbb{R}$, $f_\psi | f_D$ *is the conductor of* ψ. *Moreover, if* $D = d_K$ *is a fundamental discriminant, then* $C(s,\psi) = a_\psi$.

We write the Epstein zeta function $L(A,B,C;s)$ as summation of quadratic Hecke L-functions in Theorem 3.3. However the Epstein zeta function $L(A,B,C;s)$ in Conjecture 1.1 is expressed by quadratic Dirichlet L-functions. The following theorem tells us that quadratic Hecke L-functions can always be expressed by quadratic Dirichlet L-functions. Hence Conjecture 1.1 is proved.

Recall that for fundamental discriminants d_1, d_2, the genus character $\chi^{(D)}_{d_1,d_2}(Q)$ on the set of all "not negative-definite" primitive binary quadratic forms of discriminant $\Delta = d_1 d_2 f_0^2$ is defined as

$$\chi^{(D)}_{d_1,d_2}([a,b,c]) := \prod_{\text{prime discriminant } q^* | d_1} \chi^{(q^*)}([a,b,c]),$$

$$\chi^{(q^*)}([a,b,c]) := \begin{cases} \chi_{q^*}(a), & \text{if } (a,q^*) = 1, \\ \chi_{q^*}(c), & \text{if } ((c,q^*) = 1. \end{cases} \tag{3.2}$$

Since χ_{q^*} is a Kronecker character which is quadratic and a genus character is a product of Kronecker characters, one can see that a genus characters is also quadratic. By Section 7.2 of [KM20], all of the quadratic characters of $\mathrm{Cl}(D)$ are also genus characters.

Theorem 3.4 ([KM20],Theorem 1)**.** *Suppose that discriminant* $D = m^2 d_K = d_1 d_2 f_0^2$ *for two fundamental discriminants* d_1, d_2 *and* $f_0 \in \mathbb{N}$. *Let genus character* $\chi^{(D)}_{d_1,d_2}$ *on* $\mathrm{Cl}(D)$ *be the associated quadratic character with* $\phi \in X^0_K((m)) \simeq \mathrm{Cl}(\mathcal{O})$. *Then we have the factorization*

$$L(s,\phi,K) = \epsilon(s,\chi^{(D)}_{d_1,d_2}) L(s,\chi_{d_1}) L(s,\chi_{d_2}).$$

where $\epsilon(s,\chi^{(D)}_{d_1,d_2}) =$

$$\prod_{\substack{p|f_0 \\ p\ prime}} \frac{(1-\chi_{d_1}(p)p^{-s})(1-\chi_{d_2}(p)p^{-s}) - p^{m_p-1-2m_p s}(p^{1-s}-\chi_{d_1}(p))(p^{1-s}-\chi_{d_2}(p))}{1-p^{1-2s}},$$

m_p *is a positive integer such that* p^{m_p} *is the highest power of* p *dividing* f_0 *if* $f_0 > 1$, *while the empty product is understood as being* 1 *if* $f_0 = 1$.

In next section, we will use the symbol χ_{d_1,d_2} for $\chi^{(D)}_{d_1,d_2}$ if $f_0 = 1$.

4. Some Examples

In this section, we will present some examples. Moreover, we can calculate each solvable lattice sum by quadratic Dirichlet L-function values. The Epstein zeta functions of principal forms in these examples has been included in the [GZ80]. Note that Example 4.2 is the counterexample to the original conjecture proposed by Zucker and Robertson in [ZR84]. More details to calculate $C(s,\chi)$ can be found in Chapter 5 of [E12].

Some information of $\mathrm{Cl}(D)$ come from [SR13].

Example 4.1. *Let $D=-84$. Then $\mathrm{Cl}(-84)\cong(\mathbb{Z}/2\mathbb{Z})^2$ has four quadratic form classes $F_1(x,y)=[1,0,21]$, $F_2(x,y)=[2,2,11]$, $F_3(x,y)=[3,0,7]$, $F_4(x,y)=[5,4,5]$. And $\chi_{+1,-84}=1$, $\chi_{-7,+12}$, $\chi_{-4,+21}$ and $\chi_{-3,+28}$ are all quadratic characters in $\widehat{\mathrm{Cl}(-39)}$. By (3.1), we have*

$$\theta_{F_1}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{-7,+12}(F_1)\theta_{\chi_{-7,+12}}+\chi_{-4,+21}(F_1)\theta_{\chi_{-4,+21}}+\chi_{-3,+28}(F_4)\theta_{\chi_{-3,+28}}),$$
$$\theta_{F_2}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{-7,+12}(F_2)\theta_{\chi_{-7,+12}}+\chi_{-4,+21}(F_2)\theta_{\chi_{-4,+21}}+\chi_{-3,+28}(F_4)\theta_{\chi_{-3,+28}}),$$
$$\theta_{F_3}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{-7,+12}(F_3)\theta_{\chi_{-7,+12}}+\chi_{-4,+21}(F_3)\theta_{\chi_{-4,+21}}+\chi_{-3,+28}(F_4)\theta_{\chi_{-3,+28}}),$$
$$\theta_{F_4}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{-7,+12}(F_4)\theta_{\chi_{-7,+12}}+\chi_{-4,+21}(F_4)\theta_{\chi_{-4,+21}}+\chi_{-3,+28}(F_4)\theta_{\chi_{-3,+28}}).$$

Note that $37=F_1(4,1)$, $11=F_2(0,1)$, $7=F_3(0,1)$ and $17=F_4(2,-1)$. By (3.2), we have

$$\theta_{F_1}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{-7}(37)\theta_{\chi_{-7,+12}}+\chi_{-4}(37)\theta_{\chi_{-4,+21}}+\chi_{-3}(37)\theta_{\chi_{-3,+28}}),$$
$$\theta_{F_2}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{-7}(11)\theta_{\chi_{-7,+12}}+\chi_{-4}(11)\theta_{\chi_{-4,+21}}+\chi_{-3}(11)\theta_{\chi_{-3,+28}}),$$
$$\theta_{F_3}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{+12}(7)\theta_{\chi_{-7,+12}}+\chi_{-4}(7)\theta_{\chi_{-4,+21}}+\chi_{-3}(7)\theta_{\chi_{-3,+28}}),$$
$$\theta_{F_4}=\frac{1}{2}(\theta_{\chi_{+1,-84}}+\chi_{-7}(17)\theta_{\chi_{-7,+12}}+\chi_{-4}(17)\theta_{\chi_{-4,+21}}+\chi_{-3}(17)\theta_{\chi_{-3,+28}}).$$

Then by Theorem 1.2 and Mellin transform, we have

$$\begin{aligned}L(1,0,21,s)=&\frac{1}{2}(\zeta(s)L(s,\chi_{-84})+L(s,\chi_{-7})L(s,\chi_{+12})\\&+L(s,\chi_{-4})L(s,\chi_{+21})+L(s,\chi_{-3})L(s,\chi_{+28})),\\L(2,2,11,s)=&\frac{1}{2}(\zeta(s)L(s,\chi_{-84})+L(s,\chi_{-7})L(s,\chi_{+12})\\&-L(s,\chi_{-4})L(s,\chi_{+21})-L(s,\chi_{-3})L(s,\chi_{+28})),\\L(3,0,7,s)=&\frac{1}{2}(\zeta(s)L(s,\chi_{-84})-L(s,\chi_{-7})L(s,\chi_{+12})\\&-L(s,\chi_{-4})L(s,\chi_{+21})+L(s,\chi_{-3})L(s,\chi_{+28})),\\L(5,4,5,s)=&\frac{1}{2}(\zeta(s)L(s,\chi_{-84})-L(s,\chi_{-7})L(s,\chi_{+12})\\&+L(s,\chi_{-4})L(s,\chi_{+21})-L(s,\chi_{-3})L(s,\chi_{+28})).\end{aligned}$$

Example 4.2. *This example is listed in* [ZR84]. *Note that* $D=-39$ *is fundamental,* $\mathrm{Cl}(-39)\cong\mathbb{Z}/4\mathbb{Z}$ *has four quadratic form classes* $F_1(x,y)=[1,1,10]$, $F_2(x,y)=[3,3,4]$, $F_3(x,y)=[2,1,5]$, $F_4(x,y)=[2,-1,5]$, *where* $\mathrm{ord}(F_3)=\mathrm{ord}(F_4)=4$. *And* $\chi_{+1,-39}=1$, $\chi_{-3,+13}$ *are all quadratic characters in* $\widehat{\mathrm{Cl}(-39)}$. *By* (3.1), *we have*

$$\theta_{F_3}=\frac{1}{4}(\theta_{\chi_{+1,-39}}+\chi_{-3,+13}(F_3)\theta_{\chi_{-3,+13}}),$$
$$\theta_{F_4}=\frac{1}{4}(\theta_{\chi_{+1,-39}}+\chi_{-3,+13}(F_4)\theta_{\chi_{-3,+13}}).$$

Since $5=F_3(0,1)=F_4(0,1)$ *and* $\chi_{13}(5)=-1$, *we have*

$$\theta_{F_3}=\theta_{F_4}=\frac{1}{4}(\theta_{\chi_{1,-39}}-\theta_{\chi_{-3,13}})=\frac{1}{2}(\theta_{\chi_{+1}}\theta_{\chi_{-39}}-\theta_{\chi_{-3}}\theta_{\chi_{+13}})$$

by (3.2). *Then by Mellin transform, we have*

$$L(2,1,5,s)=L(2,-1,5,s)=\frac{1}{2}(\zeta(s)L(s,\chi_{-39})-L(s,\chi_{-3})L(s,\chi_{+13})).$$

Example 4.3. *Let* $D=-28=2^2(-7)$ *(not fundamental). Then* $\mathrm{Cl}(-28)=\{F=[1,0,7]\}$ *is trivial. The theta series* $\theta_F=2\theta_{\chi^{(28)}_{+1,-7}}$. *Now we consider the CM modular forms space* Θ_{-7}. *Note that* -7 *is fundamental,* $\mathrm{Cl}(-7)=\{f=[1,1,2]\}$ *and* $\theta_{\chi_{+1,-7}}$ *is a newform. Moreover, we have* $C(s,\chi^{(-28)}_{+1,-7})=1-2^{1-s}+2^{1-2s}$. *Then by Mellin transform, we have*

$$L(1,0,7,s)=2(1-2^{1-s}+2^{1-2s})\zeta(s)L(s,\chi_{-7}).$$

Example 4.4. *Let* $D=-60=2^2(-15)$ *(not fundamental). Then* $\mathrm{Cl}(-60)=\{F=[1,0,15],G=[3,0,5]\}$ *and* $\widehat{\mathrm{Cl}(-60)}=\{\chi^{(60)}_{+1,-15},\chi^{(60)}_{-3,+5}\}$. *Then*

$$\theta_F=\theta_{\chi^{(60)}_{+1,-15}}+\theta_{\chi^{(60)}_{-3,+5}},$$
$$\theta_G=\theta_{\chi^{(60)}_{+1,-15}}-\theta_{\chi^{(60)}_{-3,+5}}.$$

Let $\theta_{\chi_{+1,-15}},\theta_{\chi_{-3,+5}}$ *be two newforms corresponding to* $\chi_{+1,-15},\chi_{-3,+5}\in\widehat{\mathrm{Cl}(-15)}$ *in* Θ_{-15}. *We have* $C(s,\chi^{(-60)}_{+1,-15})=1-2^{1-s}+2^{1-2s}$, $C(s,\chi^{(-60)}_{-3,+5})=1+2^{1-s}+2^{1-2s}$. *Then by Mellin transform, we have*

$$\begin{aligned}L(1,0,15,s)&=(1-2^{1-s}+2^{1-2s})\zeta(s)L(s,\chi_{-15})\\&\quad+(1+2^{1-s}+2^{1-2s})L(s,\chi_{+3})L(s,\chi_{-5}),\\L(3,0,5,s)&=(1-2^{1-s}+2^{1-2s})\zeta(s)L(s,\chi_{-15})\\&\quad-(1+2^{1-s}+2^{1-2s})L(s,\chi_{+3})L(s,\chi_{-5}).\end{aligned}$$

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