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The Mahler measure of $x + 1/x + y + 1/y + 4 \pm 4\sqrt{2}$ and Beilinson's conjecture

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In this note we study the Mahler measures of reciprocal polynomials $P_k = x + 1/x + y + 1/y + k$ for $k = 4 \pm 4\sqrt{2}$. We prove identities relating the Mahler measure of $P_{4\pm4\sqrt{2}}$ to special values of the L-functions of weight 2 cusp forms of level 64. We also express the Beilinson regulator which is a 2×2 determinant of Mahler measures as special value of L-function of the elliptic curve defined by $P_{4+4\sqrt{2}} = 0$.

Keywords: Mahler measure; L-function; elliptic curve; Beilinson's conjecture.

Mathematics Subject Classification 2020: 11F67, 11R06, 19F27

1. Introduction

Let $f \in \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ be a Laurent polynomial with complex coefficients. Then the (logarithmic) Mahler measure of f is defined as

$$m(f) = \int_0^1 \cdots \int_0^1 \log|f(\mathrm{e}^{2\pi i\theta_1}, \ldots, \mathrm{e}^{2\pi i\theta_n})|d\theta_1 \cdots d\theta_n.$$

The Mahler measure is related to many areas, such as number theory, representation theory, K-theory, etc. One can see [8] and [13] for the review on Mahler measures.

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In this article, we will study the Mahler measure of Laurent polynomials of the form

$$P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k,$$

where $k \in \mathbb{C}$ is a constant. The completion of the curve $P_k(x,y) = 0$ is an elliptic curve for $k \neq 0, \pm 4$. This elliptic curve has a Weierstrass form

$$E_k : Y^2 = X^3 + \left(\frac{k^2}{4} - 2\right) X^2 + X.$$

The rational transformation is

$$x = \frac{kX - 2Y}{2X(X-1)}, \quad y = \frac{kX + 2Y}{2X(X-1)}.$$

One can see [17, §3.4] for details.

Let

$$m(k) = m(P_k(x,y)).$$

Deninger firstly noticed the connection between the Mahler measure and Beilinson's regulator map. He conjectured in [11] that

$$m(1) = cL'(E_1, 0)$$

for some $c \in \mathbb{Q}^*$ where E_1 is the elliptic curve with conductor 15 defined by $P_1(x,y) = 0$. Numerical evidence of Boyd suggests that $c = 1$.

Inspired by Deninger's work, Boyd also numerically computed in [5] the Mahler measure $m(k)$ for $k = 1, 2, \ldots, 40$. Rodriguez-Villegas computed many more cases of this family. They raised many conjectures on the relation between the Mahler measures $m(k)$ and the L-values of the elliptic curves E_k. One can see [5, Table 1] and [24, Table 4] for details. All of their conjectures are numerically verified.

In 1997, Rodriguez-Villegas proved that

$$m(4\sqrt{2}) = L'(E_{4\sqrt{2}}, 0), \quad \mathrm{Cond}(E_{4\sqrt{2}}) = 64,$$
$$m(4/\sqrt{2}) = L'(E_{4/\sqrt{2}}, 0), \quad \mathrm{Cond}(E_{4/\sqrt{2}}) = 32,$$

where $\mathrm{Cond}(E)$ means the conductor of E. These two elliptic curves have complex multiplication. Hence one can use Bloch's method in [3] to write the corresponding values of Beilinson regulator as L-values. Rodriguez-Villegas also proved conjectures for $m(4i)$, $m(2\sqrt{2})$, $m(3\sqrt{2})$. For a comprehensive list of proven identities of $m(k)$ and their references, see [27, Table 1].

One can also use the functional equations of Mahler measures to get new identities. Kurokawa and Ochiai proved in [14] that if k is a nonzero real number, then

$$2m\left(2\left(k + \frac{1}{k}\right)\right) = m(4k^2) + m\left(\frac{4}{k^2}\right). \tag{1.1}$$

Lalin and Rogers proved in [17] that for any k such that $0 < |k| < 1$,

$$m\left(2\left(k+\frac{1}{k}\right)\right)+m\left(2i\left(k-\frac{1}{k}\right)\right)=m\left(\frac{4}{k^2}\right). \tag{1.2}$$

By (1.1) and (1.2), Samart proved in [26, Theorem 2.2] that

$$m\left(\sqrt{8+6\sqrt{2}}\right)=L'(f_{64},0)+L'(f_{32},0),$$

$$m\left(\sqrt{8-6\sqrt{2}}\right)=L'(f_{64},0)-L'(f_{32},0),$$

where

$$f_{64}(\tau)=\frac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)}\in S_2(\Gamma_0(64)),$$

$$f_{32}(\tau)=\eta^2(4\tau)\eta^2(8\tau)\in S_2(\Gamma_0(32))$$

and $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau):=q^{1/24}\prod_{n=1}^{\infty}(1-q^n),\quad q=e^{2\pi i\tau}.$$

In this paper, we first study the Mahler measure of $m(4 \pm 4\sqrt{2})$ and prove the following theorem in Sec. 3.

Theorem 1.1. *The following identities are true*:

$$m(4+4\sqrt{2})=\frac{16\sqrt{2}}{\pi^2}L(f,2), \tag{1.3}$$

$$m(4-4\sqrt{2})=\frac{16\sqrt{2}}{\pi^2}L(g,2), \tag{1.4}$$

where $f(\tau)=\frac{\eta(8\tau)\eta(16\tau)^5}{\eta(32\tau)^2}$ *and* $g(\tau)=2\frac{\eta(8\tau)^3\eta(32\tau)^2}{\eta(16\tau)}$ *are cusp forms of weight* 2 *on the group* $\Gamma_1(64)$. *Moreover, let* g_{64} *and* $\bar{g}_{64}$ *be the newforms of weight* 2 *on* $\Gamma_1(64)$ *with character* $\left(\frac{8}{\bullet}\right)$ [16, *Modular form* 64.2.*b*.*a*.33.2 and 64.2.*b*.*a*.33.1] *respectively. Then we have*

$$f=\frac{1}{2}(g_{64}+\bar{g}_{64}),\quad g=\frac{1}{2i}(g_{64}-\bar{g}_{64}).$$

Let E be the elliptic curve defined by $P_{4+4\sqrt{2}}=0$. Then E has complex multiplication, but E is not isogenous to a base change of elliptic curve over $\mathbb{Q}$ as in Samart's theorem. Furthermore, we will give two independent elements in $K_2(E;\mathbb{Z})$ such that the Beilinson's regulator of these two elements is $4L''(E,0)$ in Theorem 4.3.

Recently, the first named author, Tao and Wei [28] propose a systematic way to find more examples of CM elliptic curves over real quadratic fields in this family, and extend the method in this article to study their Mahler measure and Beilinson's conjecture.

2. Modular Mahler Measure

Recall that the classical modular lambda function is

$$\lambda(\tau) = 16\frac{\eta(\tau/2)^8\eta(2\tau)^{16}}{\eta(\tau)^{24}}.$$

The q-expansion of λ is

$$\lambda(2\tau) = 16q - 128q^2 + 704q^3 - 3072q^4 + 11488q^5 - 38400q^6 + 117632q^7 + \cdots,$$

where $q = e^{2\pi i\tau}$. The modular function λ satisfies the following identities (see [10, Chap. VII, §7]),

$$\begin{aligned} \lambda(\tau) &= \lambda(\tau+2) = \lambda(\tau/(1-2\tau)),\\ \lambda(\tau+1) &= \frac{\lambda(\tau)}{\lambda(\tau)-1},\\ \lambda(-1/\tau) &= 1-\lambda(\tau),\\ \lambda(\tau/(1-\tau)) &= \frac{1}{\lambda(\tau)}. \end{aligned} \tag{2.1}$$

Let χ_{-4} be the character

$$\chi_{-4}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4,\\ -1 & \text{if } n \equiv 3 \pmod 4,\\ 0 & \text{if } n \equiv 0 \pmod 2). \end{cases}$$

Rodriguez-Villegas proved in [24, §IV, Example 2] that

$$m(k) = m(x+1/x+y+1/y+k) = \operatorname{Re}\left(\frac{16\operatorname{Im}(\tau)}{\pi^2}\sum_{m,n\in\mathbb{Z}}{}^{'}\frac{\chi_{-4}(n)}{(4m\tau+n)^2(4m\overline{\tau}+n)}\right), \tag{2.2}$$

where

$$k = \frac{4}{\sqrt{\lambda(2\tau)}} = \frac{\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8}.$$

Note that τ is purely imaginary if $k \in \mathbb{R}$ and $|k| > 4$ (see [17, §3.4]).

Recall that the Weber modular functions [2] are defined as follows:

$$\begin{aligned} \mathfrak{f}(\tau) &= e^{-\frac{\pi i}{24}}\frac{\eta((\tau+1)/2)}{\eta(\tau)} = \frac{\mathfrak{f}_1(2\tau)}{\mathfrak{f}_1(\tau)},\\ \mathfrak{f}_1(\tau) &= \frac{\eta(\tau/2)}{\eta(\tau)},\\ \mathfrak{f}_2(\tau) &= \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)}. \end{aligned}$$

The three functions $\mathfrak{f},\ \mathfrak{f}_1,\ \mathfrak{f}_2$ satisfy the simple relation,

$$\mathfrak{f}(\tau)\mathfrak{f}_1(\tau)\mathfrak{f}_2(\tau)=\sqrt{2}.$$

Hence

$$k=\frac{4}{\sqrt{\lambda(2\tau)}}=\frac{\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8}=\frac{16}{\mathfrak{f}_1(2\tau)^4\mathfrak{f}_2(2\tau)^8}=\mathfrak{f}(2\tau)^8\mathfrak{f}_1(2\tau)^4.$$

Note that $-\mathfrak{f}(2\tau)^{24},\ \mathfrak{f}_1(2\tau)^{24},\ \mathfrak{f}_2(2\tau)^{24}$ are exactly the three roots of the cubic equation

$$(x+16)^3-j(2\tau)\,x=0,$$

where $j(\cdot)$ is the classical j-invariant.

3. Proof of Theorem 1.1

3.1. *Proof of* (1.3)

Let $\tau=\sqrt{2}i/2$. Then by [29, Table VI],

$$\mathfrak{f}_1(2\tau)=2^{\frac{1}{4}}.$$

Since $j(\sqrt{2}i)=2^6 5^3$ [4, Example 4], we know that $-\mathfrak{f}(2\tau)^{24}$ is a root of

$$(x+16)^3-2^6 5^3 x=0.$$

The three roots of the above equation are

$$64,\quad -56-40\sqrt{2},\quad -56+40\sqrt{2}.$$

Hence $\mathfrak{f}(2\tau)^{24}$ is equal to $56+40\sqrt{2}$ or $56-40\sqrt{2}$. The numerical computation by Sage [25] finds that

$$\mathfrak{f}(2\tau)^{24}=56+40\sqrt{2},$$

which implies that

$$k=\frac{4}{\sqrt{\lambda(2\tau)}}=\mathfrak{f}(2\tau)^8\mathfrak{f}_1(2\tau)^4=4+4\sqrt{2}.$$

Hence we have

$$\lambda(2\tau)=3-2\sqrt{2}. \tag{3.1}$$

Then by (2.2), we find

$$\begin{aligned} m(k) &= \mathrm{Re}\left(\frac{16\mathrm{Im}(\tau)}{\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)}{(4m\tau+n)^2(4m\overline{\tau}+n)}\right) \\ &= \frac{8\sqrt{2}}{\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)n}{(8m^2+n^2)^2} \\ &= \frac{16\sqrt{2}}{\pi^2}L(f,2), \end{aligned}$$

where

$$f(\tau) = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}(\chi_{-4}(n)n)q^{8m^2+n^2} = q - q^9 - 6q^{17} + 5q^{25} + 12q^{33} + \cdots.$$

By [20, Theorem 1.1], we have

$$\eta(8\tau)^3 = \sum_{n=1}^{\infty}\chi_{-4}(n)nq^{n^2} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\chi_{-4}(n)nq^{n^2}. \tag{3.2}$$

Let $\theta(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2}$. By [21, Theorem 1.60], $\theta(\tau)$ has an eta quotient representation

$$\theta(\tau) = \frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2}.$$

Hence we have

$$f(\tau) = \eta(8\tau)^3\theta(8\tau) = \frac{\eta(8\tau)\eta(16\tau)^5}{\eta(32\tau)^2}.$$

3.2. *Proof of* (1.4)

Let $\mu = \sqrt{2}i$. Then by Eq. (2.1),

$$\lambda\left(\frac{-2+\sqrt{2}i}{3}\right) = \lambda\left(\frac{\mu}{1-\mu}\right) = \frac{1}{\lambda(\mu)}.$$

By Eq. (3.1), $\lambda(\mu) = 3 - 2\sqrt{2}$. Hence we have

$$\lambda\left(\frac{-2+\sqrt{2}i}{3}\right) = \frac{1}{3-2\sqrt{2}} = 3 + 2\sqrt{2},$$

which implies that

$$k = \frac{4}{\sqrt{\lambda\left(\frac{-2+\sqrt{2}i}{3}\right)}} = \frac{4}{1+\sqrt{2}} = 4\sqrt{2} - 4.$$

Let $\tau = \frac{-2+\sqrt{2}i}{6}$. Then

$$m(k) = \mathrm{Re}\left(\frac{16\mathrm{Im}(\tau)}{\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)}{(4m\tau+n)^2(4m\bar{\tau}+n)}\right)$$

$$= \mathrm{Re}\left(\frac{16\mathrm{Im}(\tau)}{\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)(4m\bar{\tau}+n)}{(4m\tau+n)^2(4m\bar{\tau}+n)^2}\right)$$

$$
\begin{aligned}
&= \mathrm{Re}\left(\frac{16\frac{\sqrt{2}}{6}}{\pi^2} \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)\left(4m\frac{-2-\sqrt{2}i}{6}+n\right)}{\left(\frac{8}{3}m^2-\frac{8}{3}mn+n^2\right)^2} \right) \\
&= \mathrm{Re}\left(\frac{8\sqrt{2}}{3\pi^2} \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)\left(6m(-2-\sqrt{2}i)+9n\right)}{(8m^2-8mn+3n^2)^2} \right) \\
&= \frac{8\sqrt{2}}{\pi^2} \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)\,(-4m+3n)}{(8m^2-8mn+3n^2)^2} \\
&= \frac{8\sqrt{2}}{\pi^2} \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-4}(n)\,(-2(2m-n)+n)}{(2(2m-n)^2+n^2)^2} \\
&= \frac{8\sqrt{2}}{\pi^2} \sum_{n\equiv l \pmod 2}{}' \frac{\chi_{-4}(n)\,(n+2l)}{(n^2+2l^2)^2} \quad (\text{where } l := n-2m) \\
&= \frac{8\sqrt{2}}{\pi^2} \sum_{n\equiv l\equiv 1(2)}{}' \frac{\chi_{-4}(n)n}{(n^2+2l^2)^2}.
\end{aligned}
$$

Let

$$
g := \frac{1}{2} \sum_{n\equiv l\equiv 1(2)} \chi_{-4}(n) n q^{n^2+2l^2} = 2q^3 - 6q^{11} + 2q^{19} + 4q^{27} + 10q^{43} + \cdots.
$$

Then

$$
m(k) = \frac{8\sqrt{2}}{\pi^2} \sum_{n\equiv l\equiv 1(2)}{}' \frac{\chi_{-4}(n)n}{(n^2+2l^2)^2} = \frac{16\sqrt{2}}{\pi^2} L(g,2).
$$

By [21, Theorem 1.60], there is the q-series of the following eta quotient

$$
\frac{\eta(16\tau)^2}{\eta(8\tau)} = \sum_{n=0}^{\infty} q^{(2n+1)^2}. \tag{3.3}
$$

Hence by (3.2) and (3.3), we have

$$
g = 2\frac{\eta(8\tau)^3\eta(32\tau)^2}{\eta(16\tau)}.
$$

By Theorems 1.64 and 1.65 of loc. cit., f and g are weight 2 cusp forms for $\Gamma_1(64)$ with character $\left(\frac{8}{\bullet}\right)$. Let g_{64} and $\bar{g}_{64}$ be the weight 2 newforms of $\Gamma_1(64)$ with the same character,

$$
\begin{aligned}
g_{64}(q) &= q + 2iq^3 - q^9 - 6iq^{11} - 6q^{17} + 2iq^{19} + 5q^{25} + 4iq^{27} + 12q^{33} + \cdots, \\
\bar{g}_{64}(q) &= q - 2iq^3 - q^9 + 6iq^{11} - 6q^{17} - 2iq^{19} + 5q^{25} - 4iq^{27} + 12q^{33} + \cdots.
\end{aligned}
$$

A simple computation by Sage finds that the dimension of the space of weight 2 cusp forms on $\Gamma_1(64)$ is two. Then $f(q) = \frac{1}{2}(g_{64}(q) + \bar{g}_{64}(q))$ and $g(q) = \frac{1}{2i}(g_{64}(q) - \bar{g}_{64}(q))$ by comparing the q and q^3 terms in the q-expansion.

Let $\Lambda_1(s) = 64^{s/2}(2\pi)^{-s}L(g_{64}, s)$ and $\Lambda_2(s) = 64^{s/2}(2\pi)^{-s}L(\bar{g}_{64}, s)$. The function equation

$$\Lambda_1(s) = \frac{1+i}{\sqrt{2}}\Lambda_2(2-s)$$

gives

$$L'(g_{64}, 0) = \frac{8\sqrt{2}}{\pi^2}(1-i)L(\bar{g}_{64}, 2), \tag{3.4}$$

$$L'(\bar{g}_{64}, 0) = \frac{8\sqrt{2}}{\pi^2}(1+i)L(g_{64}, 2). \tag{3.5}$$

Combining this with the evaluations of $m(4+4\sqrt{2})$ and $m(4-4\sqrt{2})$ above, we also have

$$m(4+4\sqrt{2}) = \mathrm{Re}L'(g_{64}, 0) + \mathrm{Im}L'(g_{64}, 0),$$
$$m(4-4\sqrt{2}) = \mathrm{Re}L'(g_{64}, 0) - \mathrm{Im}L'(g_{64}, 0).$$

4. Beilinson Regulator

Let C be a non-singular, complete, geometrically irreducible curve of genus g over a number field K. For a cycle γ in $H_1(C(\mathbb{C}), \mathbb{Z})$ with $C(\mathbb{C}) := C \times_{\mathbb{Q}} \mathbb{C}$ and an element $M = \sum_k \{f_k, g_k\}$ in the algebraic K-group $K_2^T(C)$ (see [15, §1] for the definition of this group), where f_k, g_k are rational functions on C, the *regulator integral* is defined as

$$\langle \gamma, M \rangle := \frac{1}{2\pi} \int_\gamma \sum_k \eta(f_k, g_k),$$

where $\eta(f, g) := \log|f| \mathrm{d}\arg g - \log|g| \mathrm{d}\arg f$, and we use any representative of γ avoiding the zeros and poles of the functions f_k and g_k.

Beilinson's conjecture [1] asserts that

(1) the integral subspace $K_2^T(C)_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$ [15, §1] has $\mathbb{Q}$-dimension $g[K : \mathbb{Q}]$,
(2) the leading coefficient $L^*(C, 0)$ of the Taylor expansion of the L-function of C at $s = 0$ can be expressed as a determinant of regulator integrals called *Beilinson regulator*:

$$L^*(C, 0) \sim_{\mathbb{Q}^\times} |\det(\langle \gamma_i, M_j \rangle)_{1 \leqslant i,j \leqslant g[K:\mathbb{Q}]}|,$$

where γ_i form a basis of $H_1(C(\mathbb{C}), \mathbb{Z})^-$ and M_j form a basis of $K_2^T(C)_{\mathrm{int}} \otimes \mathbb{Q}$.

Here $(\cdot)^-$ denotes the space of anti-invariants under the complex conjugation acting on $C(\mathbb{C})$. Note that $L^*(C, 0)$ has a simple relation to $L(C, 2)$ via the (conjectural) functional equation of $L(C, s)$.

Recall that an elliptic curve defined over $\overline{\mathbb{Q}}$ is a $\mathbb{Q}$-curve if it is isogenous to all its Galois conjugates. It is a quadratic $\mathbb{Q}$-curve if the elliptic curve and the isogeny are both defined over a quadratic field. Let $E = E_{4+4\sqrt{2}}$ and $E' = E_{4-4\sqrt{2}}$. Then E is birational to the $\mathbb{Q}$-curve [16, Elliptic curve 64.1-a6] with complex multiplication and E' is birational to its conjugate [16, Elliptic curve 64.1-a2]. By [16], there is a 4-isogeny between E and E' over $K = \mathbb{Q}(\sqrt{2})$. Hence E is a $\mathbb{Q}$-curve and there is a 4-isogeny $\phi : E \to E'$ between E and E' defined over K. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ be the element mapping $\sqrt{2}$ to $-\sqrt{2}$ and ${}^{\sigma}\phi : E' \to E$ be the Galois conjugate of ϕ defined by the Galois action on the defining polynomials of ϕ. Then ${}^{\sigma}\phi$ is a dual isogeny of ϕ and one can check ${}^{\sigma}\phi \circ \phi = [-4]$.

Remark 4.1. The work in progress [7] will study the parametrization of $\mathbb{Q}$-curves by modular units and its applications to the Mahler measure of $\mathbb{Q}$-curves without complex multiplication.

The paper [6] explicitly expressed L-functions of non-CM quadratic $\mathbb{Q}$-curves as a product of those of two conjugate newforms. As A. Ferraguti kindly pointed to us, this is also true for our CM quadratic $\mathbb{Q}$-curve E as we explain below.

Lemma 4.2. *The L-function of E is the product of the L-functions of g_{64} and $\bar{g}_{64}$, namely*

$$L(E, s) = L(g_{64}, s)L(\bar{g}_{64}, s).$$

Proof. E is K-isogenous to its Galois conjugate E'. Moreover, the group of K-isogenies $\mathrm{Hom}_K(E, E')$ is isomorphic to $\mathbb{Z}$. Then taking the restriction of scalars of E to K, we get an abelian surface $A/\mathbb{Q}$. Milne [19, Proposition 3] showed that the L-function of E/K is the same as the L-function of A. The universal property of the restriction of scalars tells us that the $\mathbb{Q}$-algebra of endomorphisms of A, call it $\mathrm{End}_{\mathbb{Q}}(A)$ is isomorphic (as a $\mathbb{Q}$-vector space) to $\prod_{\tau \in \mathrm{Gal}(K/Q)} \mathrm{Hom}_K({}^{\tau}E, E)$. In our case the latter is 2-dimensional. On the other hand, by [23, Theorem 2.1], it must also be a field because A is simple. In fact if it was not, then E would be the base change of a curve defined over $\mathbb{Q}$ which is not the case. It follows that $\mathrm{End}_{\mathbb{Q}}(A)$ is a quadratic number field, and therefore A is an abelian variety of GL_2-type. Assuming Serre's modularity conjecture which is now a theorem of Khare and Wintenberger, Ribet [23, Theorem 4.4] proved that there exists a weight 2 newform h for $\Gamma_1(N)$ such that A is $\mathbb{Q}$-isogenous to A_h, the abelian variety attached to h by Shimura's construction.

Now we follow the argument in [12, Remark 9] to determine the level N. First we can use Milne's formula [19, Proposition 1] to calculate the conductor $\mathcal{N}_{\mathbb{Q}}(A)$ of A

$$\mathcal{N}_{\mathbb{Q}}(A) = N_{K/\mathbb{Q}}(\mathcal{N}_K(E))d_{K/\mathbb{Q}}^2 = 64^2,$$

where $N_{K/\mathbb{Q}}$ refers to taking norm, $\mathcal{N}_K(E) = (8)$ is the conductor of E/K and $d_{K/\mathbb{Q}} = 8$ is the discriminant of K. By Carayol's theorem [9], we have $\mathcal{N}_{\mathbb{Q}}(A) = N^2$ which gives $N = 64$.

Since ${}^{\sigma}\phi \circ \phi = [-4]$ in $\mathrm{End}_K(E) = \mathbb{Z}$, we have $\mathrm{End}_{\mathbb{Q}}(A) \cong \mathbb{Q}(\sqrt{-1})$. Hence the field generated by the Fourier coefficients of h is also $\mathbb{Q}(\sqrt{-1})$. Note that we have

$$L(E,s) = L(A,s) = L(A_h,s) = L(h,s)L(\bar{h},s).$$

By checking the newforms of level 64 in LMFDB [16] and comparing L-functions, we can verify this lemma. □

Let $M_1 = \{x,y\} \in K_2^T(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_2 = \{x,y\} \in K_2^T(E') \otimes_{\mathbb{Z}} \mathbb{Q}$. It is straightforward to check that E is isomorphic over K to the curve C defined by

$$y^2 + (2x^2 + (4+4\sqrt{2})x + 1)y + x^4 = 0.$$

In [15, Proposition 5.6(2)], the authors constructed an integral element in K_2 of C/K with divisors supported on the torsion subgroup $\mathbb{Z}/4\mathbb{Z}$. By Proposition 3.4 of loc. cit., any element in K_2 with divisors supported on this torsion subgroup is a rational multiple of the above element. Since divisors of x, y are supported on on torsion subgroup of E, by the isomorphism between E and C, we have $M_1 \in K_2^T(E)_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Similarly, we also have $M_2 \in K_2^T(E')_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Note that these two elements are different, since they are in the K_2 of different curves. Let $\phi^* : K_2^T(E')_{\mathrm{int}} \to K_2^T(E)_{\mathrm{int}}$ be the map induced by ϕ. Now we will compute the Beilinson regulator of M_1 and $\phi^*(M_2)$.

Let $y_1(x)$ and $y_2(x)$ be the two roots of $P(x,y) = x+y+1/x+1/y+4+4\sqrt{2} = 0$. Suppose $|y_1(x)| \geq |y_2(x)|$. Since $\{(x,y) \in \mathbb{C}^2 | P(x,y) = 0\} \cap \{(x,y) \in \mathbb{C}^2 | |x| = |y| = 1\} = \emptyset$, we have $|y_2(x)| < 1 < |y_1(x)|$ on $|x| = 1$. The circle $|x| = 1$ could be lifted to $\gamma_E = \{(x, y_1(x)) | P(x, y_1(x)) = 0, |x| = 1\}$ which is a generator of $H_1(E;\mathbb{Z})^-$ because the complex conjugation reverses the orientation. Recall that $M_1 = \{x,y\} \in K_2^T(E)_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $m(P)$ is exactly the regulator integral $\langle \gamma_E, M_1 \rangle$ if we choose the orientation properly.

Let $Q(x,y) = x + y + 1/x + 1/y + 4 - 4\sqrt{2}$. Then E' has two conjugate branch points on the unit circle $|x| = 1$. The path on $|x| = 1$ between these two conjugate points where $\Delta_{E'}(x) = (x + \frac{1}{x} + 4 - 4\sqrt{2})^2 - 4 \geq 0$ could be lifted to a loop $\gamma_{E'} = \{(x, y(x)) | Q(x, y(x)) = 0, |x| = 1, \Delta_{E'}(x) \geq 0\}$ which is a generator of $H_1(E;\mathbb{Z})^-$ because the complex conjugation reverses the orientation. Suppose $|y_1(x)| \geq |y_2(x)|$, then $|y_2(x)| < 1 < |y_1(x)|$ on the path except at the branch points and $|y_1(x)| = 1/|y_2(x)|$. We have

$$\gamma_{E'} = \gamma_{E'}^1 \cup \gamma_{E'}^2,$$

where $\gamma_{E'}^i = \{(x, y_i(x)) | |x| = 1, \Delta_{E'}(x) \geq 0\}$. Hence $m(Q)$ is the integral of $\log(|y|)$ on $\gamma_{E'}^1$ and the regulator integral is the integral of $\log(|y|)$ on $\gamma_{E'}$ which implies that $m(Q)$ is half of the integral on $\gamma_{E'}$ which is $\langle \gamma_{E'}, M_2 \rangle$ (for more details, see [5, pp. 49–50]). Thus we have

$$\langle \gamma_E, M_1 \rangle = m(P), \quad \langle \gamma_{E'}, M_2 \rangle = 2m(Q).$$

The Beilinson regulator R of $M_1, \phi^*(M_2)$ is

$$\begin{aligned} R &= \left|\det\begin{pmatrix} \langle\gamma_E, M_1\rangle & \langle\gamma_{E'}, M_2\rangle \\ \langle\gamma_E, \phi^*(M_2)\rangle & \langle\gamma_{E'}, {}^\sigma(\phi^*(M_2))\rangle \end{pmatrix}\right| \\ &= \left|\det\begin{pmatrix} \langle\gamma_E, M_1\rangle & \langle\gamma_{E'}, M_2\rangle \\ \langle\phi_*\gamma_E, M_2\rangle & \langle({}^\sigma\phi)_*\gamma_{E'}, M_1\rangle \end{pmatrix}\right| \end{aligned}$$

because $\langle\gamma_{E'}, {}^\sigma(\phi^*(M_2))\rangle = \langle\gamma_{E'}, ({}^\sigma\phi)^*(M_1)\rangle = \langle({}^\sigma\phi)_*\gamma_{E'}, M_1\rangle$.

Now we only need to determine $\phi_*\gamma_E$ and $({}^\sigma\phi)_*\gamma_{E'}$. Since ϕ and ${}^\sigma\phi$ are defined over the real field K, ϕ_* maps $H_1(E;\mathbb{Z})^-$ to $H_1(E';\mathbb{Z})^-$ and ${}^\sigma\phi_*$ maps $H_1(E';\mathbb{Z})^-$ to $H_1(E;\mathbb{Z})^-$. So we have

$$\phi_*\gamma_E = a\gamma_{E'}, \quad ({}^\sigma\phi)_*\gamma_{E'} = b\gamma_E,$$

where $a, b \in \mathbb{Z}$ and $ab = -4$ since ${}^\sigma\phi \circ \phi = [-4]$. We need to determine a and b. Lalin and Ramamonjisoa [18] proved relations between homology classes on certain elliptic curve by comparing the integral of the holomorphic differential form on these loops. But we should be able to show this by numerical calculation since $a, b \in \mathbb{Z}$. In fact, numerical computation by PARI/GP [22] gives $\int_{\gamma_{E'}} \omega \approx \pm 6.58227$ and $\int_{\phi_*\gamma_E} \omega = \int_{\gamma_E} \phi^*\omega \approx \pm 6.58227$ where $\omega = \frac{dx}{y}$ on the Weierstrass form of E' given by

$$y^2 = x^3 + ((4 - 4\sqrt{2})^2/4 - 2)x^2 + x.$$

Hence we should have $a = \pm 1, b = \mp 4$ and

$$\begin{aligned} R &= \left|\det\begin{pmatrix} \langle\gamma_E, M_1\rangle & \langle\gamma_{E'}, M_2\rangle \\ \langle\phi_*\gamma_E, M_2\rangle & \langle({}^\sigma\phi)_*\gamma_{E'}, M_1\rangle \end{pmatrix}\right| \\ &= \left|\det\begin{pmatrix} \langle\gamma_E, M_1\rangle & \langle\gamma_{E'}, M_2\rangle \\ \langle\gamma_{E'}, M_2\rangle & -4\langle\gamma_E, M_1\rangle \end{pmatrix}\right| \\ &= \left|\det\begin{pmatrix} m(P) & 2m({}^\sigma P) \\ 2m({}^\sigma P) & -4m(P) \end{pmatrix}\right| \\ &= 4(m(P)^2 + m({}^\sigma P)^2) \\ &= 8L'(g_{64}, 0)L'(\bar{g}_{64}, 0) \\ &= 4L''(E, 0). \end{aligned}$$

By the function equations of $g_{64}, \bar{g}_{64}$ (3.4), (3.5) and Lemma 4.2, we have

$$L''(E, 0) = \frac{512L(E, 2)}{\pi^4} \neq 0.$$

This shows M_1, M_2 are linearly independent.

In sum, we have the following theorem.

Theorem 4.3. *Let notations be as above. The Beilinson regulator R of two linearly independent elements $M_1, M_2 \in K_2^T(E)_{\text{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and the second order derivative of the L-function of E at 0 have the following relation in accordance with Beilinson's conjecture*

$$R = 4L''(E, 0).$$

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