J. K-Theory **1** (2008), 185–192
doi: 10.1017/is007011017jkt018

Uniqueness of Moore's higher reciprocity law at the prime 2 for real number fields

by

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Abstract

Let F be a real number field with r_1 real embeddings. In this paper, we prove that the sequence

$$K_{2i}(F)\{2\}\longrightarrow \bigoplus_{\wp \text{ noncomplex}} \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i))\longrightarrow H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(i))\longrightarrow 0$$

is a complex, where $\widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i))$ are the Tate cohomology groups. Moreover if $i \equiv 0,\ 1,$ or $2 \pmod 4$, then it is exact; if $i \equiv 3 \pmod 4$, then the homology group at the second term of this complex is isomorphic to $\bigoplus_{r_1}\mathbb{Z}/2\mathbb{Z}$.

Key Words: Higher Moore reciprocity law, Bloch-Lichtenbaum spectral sequence, Tate-Poitou duality theorem

Mathematics Subject Classification 2000: Primary 19D50; Secondary 11R70, 11S70, 14F20, 19F27

1. Introduction

Let F be a number field and $\mathcal{O}_F$ its ring of integers. The Moore reciprocity uniqueness theorem states that the sequence

$$K_2(F)\longrightarrow \bigoplus_{\wp \text{ noncomplex}} \mu(F_\wp)\longrightarrow \mu(F)\longrightarrow 0$$

is exact, where $F_\wp$ denotes the completion of F at noncomplex primes $\wp$, and $\mu(F)$ ($\mu(F_\wp)$ resp.) denotes the group of roots of unity in F ($F_\wp$ resp.). This theorem follows from Moore ([5]), see also the appendix of [4]. In [1], Banaszak generalized the exact sequence above to the odd torsion in higher K-groups. For any group G and natural number l, let $G\{l\} = \{g \in G | g^{l^n} = 1 \text{ for some natural number } n\}$. Kahn proved ([2], Theorem 4.3) the following theorem.

Theorem 1.1 (Moore [5] for $i = 1$, Banaszak [1] for l odd) *Let $m_i = m_i(F)$ and, for any noncomplex prime $\wp$ of F, $m_i(\wp) = m_i(F_\wp)$. Then, at least in the following cases*

• l is odd, or
• $l = 2$ and F is totally imaginary
the sequence

$$K_{2i}(F)\{l\} \xrightarrow{(h_{\wp}^{(i)})} \bigoplus_{\wp \text{ noncomplex}} H^0(F_{\wp};\mathbb{Q}_l/\mathbb{Z}_l(i)) \xrightarrow{(\frac{m_i(\wp)}{m_i})} H^0(F;\mathbb{Q}_l/\mathbb{Z}_l(i)) \longrightarrow 0 \tag{1.1}$$

is exact, where $\frac{m_i(\wp)}{m_i}$ *is interpreted as* 0 *if* $m_i(\wp)$ *is infinite (this happens exactly when* $\wp$ *is real and* i *is even.)*

See [2] for the notation used in Theorem 1.1.

Let F be a real number field with r_1 real embeddings. For any abelian group G, let G^* denote $Hom(G,\mathbb{Q}/\mathbb{Z})$. When E is a local field let $\widehat{H}^n(E;M)$ be the Tate cohomology groups of the absolute Galois group G_E with coefficients in M. Recall that by the Tate-Poitou duality theorem,

$$\begin{aligned} H^2(F_{\wp};\mathbb{Z}_2(i+1)) &\simeq \widehat{H}^0(F_{\wp};\mathbb{Q}_2/\mathbb{Z}_2(-i))^* \\ &\simeq \begin{cases} H^0(F_{\wp};\mathbb{Q}_2/\mathbb{Z}_2(i)) & \text{for finite } \wp, \\ \mathbb{Z}/2 & \text{for } F_{\wp} = \mathbb{R} \text{ and } i \text{ odd}, \\ 0 & \text{for } F_{\wp} = \mathbb{R} \text{ and } i \text{ even}, \\ 0 & \text{for } F_{\wp} = \mathbb{C}. \end{cases} \end{aligned} \tag{1.2}$$

See §4 of [6] for details.

The natural map $F \longrightarrow \prod_{\wp \text{ noncomplex}} F_{\wp}$ induces the following natural homomorphisms of the cohomology groups:

$$H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i)) \longrightarrow \prod_{\wp \text{ noncomplex}} \widehat{H}^0(F_{\wp};\mathbb{Q}_2/\mathbb{Z}_2(-i)) .$$

By duality at each factor of the product, we have maps

$$\bigoplus_{\wp \text{ noncomplex}} \widehat{H}^0(F_{\wp};\mathbb{Q}_2/\mathbb{Z}_2(-i))^* \longrightarrow H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^*.$$

Since $\widehat{H}^0(F_{\wp};\mathbb{Q}_2/\mathbb{Z}_2(-i))^* \simeq \widehat{H}^0(F_{\wp};\mathbb{Q}_2/\mathbb{Z}_2(i))$ and $H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^* \simeq H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(i))$, we obtain maps

$$\beta_2^{(i)} : \bigoplus_{\wp \text{ noncomplex}} \widehat{H}^0(F_{\wp};\mathbb{Q}_2/\mathbb{Z}_2(i)) \longrightarrow H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(i)) .$$

Obviously $\beta_2^{(i)}$ is surjective.

In this paper, we will prove (Theorem 2.4) that there is a homomorphism $(h_\wp^{(i)})$ such that the sequence

$$K_{2i}(F)\{2\} \xrightarrow{(h_\wp^{(i)})} \bigoplus_{\wp \text{ noncomplex}} \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i)) \xrightarrow{\beta_2^{(i)}} H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(i)) \longrightarrow 0 \tag{1.3}$$

is a complex. Moreover if $i \equiv 0,\ 1,$ or $2 \pmod 4$, then the complex is exact; if $i \equiv 3 \pmod 4$, then

$$\text{kernel}(\beta_2^{(i)})/\text{image}((h_\wp^{(i)})) \simeq \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z}\ .$$

This result can be seen as the uniqueness of Moore's higher reciprocity law at the prime 2 for real number fields.

2. Main Results

Let F be a real number field with r_1 real embeddings and S a finite set of places of F, including all the dyadic and archimedean ones. Let $\mathcal{O}_S$ be the ring of S-integers in F.

By the universal coefficient theorem for algebraic K-groups with coefficients, there is a short split exact sequence

$$0 \longrightarrow K_{2i+1}(F)\bigotimes\mathbb{Z}/2^\infty \longrightarrow K_{2i+1}(F;\mathbb{Z}/2^\infty) \longrightarrow K_{2i}(F)\{2\} \longrightarrow 0\ .$$

Note that $K_{2i+1}(F)\bigotimes\mathbb{Z}/2^\infty \simeq \bigoplus_r \mathbb{Z}/2^\infty$, where $r = \text{rank}_\mathbb{Q}(K_{2i+1}(F)\bigotimes_\mathbb{Z}\mathbb{Q})$. Hence $K_{2i+1}(F)\bigotimes\mathbb{Z}/2^\infty$ is the maximal divisible subgroup of $K_{2i+1}(F;\mathbb{Z}/2^\infty)$. We use $f_1^{(i)}$ for a split inverse

$$f_1^{(i)}:\ K_{2i}(F)\{2\} \longrightarrow K_{2i+1}(F;\mathbb{Z}/2^\infty).$$

Similarly we can define

$$f_1^{(i,S)}:\ K_{2i}(\mathcal{O}_S)\{2\} \longrightarrow K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty).$$

Although $f_1^{(i)}$ ($f_1^{(i,S)}$ resp.) is not unique, we will see (Lemma 2.1) that $h_\wp^{(i)}$ ($h_\wp^{(i,S)}$ resp.) does not depend on the choice of $f_1^{(i)}$ ($f_1^{(i,S)}$ resp.). Let the homomorphism

$$f_2^{(i)}:\ K_{2i+1}(F;\mathbb{Z}/2^\infty) \longrightarrow H^1(F;\mathbb{Q}_2/\mathbb{Z}_2(i+1))$$

be the edge map in the Bloch-Lichtenbaum spectral sequence with coefficients for F, see Theorem 6.7 of [6] for details. By (6.10) of [6], $f_2^{(i)}$ induces a homomorphism

$$f_2^{(i,S)}:\ K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \longrightarrow H^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)).$$

Note that (6.10) is proved for the special case of the ring of 2-integers in F, not for general $\mathcal{O}_S$. However the arguments of Rognes and Weibel work also for general $\mathcal{O}_S$.

The short exact sequence of the G_F-modules

$$0 \longrightarrow \mathbb{Z}_2(i+1) \xrightarrow{2^v} \mathbb{Z}_2(i+1) \longrightarrow \mathbb{Z}/2^v(i+1) \longrightarrow 0$$

induces a long exact sequence in etale cohomology, which leads to the following short exact sequence:

$$0 \longrightarrow H^1(F;\mathbb{Z}_2(i+1))/2^v \longrightarrow H^1(F;\mathbb{Z}/2^v(i+1)) \longrightarrow {}_{2^v}H^2(F;\mathbb{Z}_2(i+1)) \longrightarrow 0\,.$$

Passing to direct limits, we have

$$0 \longrightarrow H^1(F;\mathbb{Z}_2(i+1))/2^\infty \longrightarrow H^1(F;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) \longrightarrow H^2(F;\mathbb{Z}_2(i+1))\{2\} \longrightarrow 0\,.$$

We use $f_3^{(i)}$ for the composition of natural homomorphisms

$$f_3^{(i)}:\ H^1(F;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) \longrightarrow H^2(F;\mathbb{Z}_2(i+1))\{2\} \longrightarrow H^2(F;\mathbb{Z}_2(i+1))\,.$$

Similarly, we define

$$f_3^{(i,S)}:\ H^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) \longrightarrow H^2(\mathcal{O}_S;\mathbb{Z}_2(i+1))\{2\} \longrightarrow H^2(\mathcal{O}_S;\mathbb{Z}_2(i+1))\,.$$

Let $f_4^{(i)}$ be the composition

$$H^2(F;\mathbb{Z}_2(i+1)) \longrightarrow H^2(F_\wp;\mathbb{Z}_2(i+1)) \xrightarrow{\sim} \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i)).$$

Similarly we can define

$$f_4^{(i,S)}:\ H^2(\mathcal{O}_S;\mathbb{Z}_2(i+1)) \longrightarrow \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i)).$$

Let

$$h_\wp^{(i)} = f_4^{(i)} \circ f_3^{(i)} \circ f_2^{(i)} \circ f_1^{(i)}:\quad K_{2i}(F)\{2\} \longrightarrow \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i))\,;$$
$$h_\wp^{(i,S)} = f_4^{(i,S)} \circ f_3^{(i,S)} \circ f_2^{(i,S)} \circ f_1^{(i,S)}:\quad K_{2i}(\mathcal{O}_S)\{2\} \longrightarrow \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i))\,.$$

Lemma 2.1 *The homomorphism $h_\wp^{(i)}$ ($h_\wp^{(i,S)}$ resp.) does not depend on the choice of $f_1^{(i)}$ ($f_1^{(i,S)}$ resp.).*

Proof: Let $f_1^{(i)}$ and $f_1^{(i)'}$ be two split inverses from $K_{2n}(F)\{2\}$ to $K_{2n+1}(F;\mathbb{Z}/2^\infty)$. And let $h_\wp^{(i)}$, $h_\wp^{(i)'}$ be the corresponding compositions. For any $a \in K_{2n}(F)\{2\}$, $f_1^{(i)}(a)(f_1^{(i)'}(a))^{-1}$ is contained in the maximal divisible subgroup of $K_{2n+1}(F;\mathbb{Z}/2^\infty)$. The target of $h_\wp^{(i)}$ and $h_\wp^{(i)'}$ is $\widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i))$ which is finite. Since the images of divisible elements are divisible too, $h_\wp^{(i)}(a)(h_\wp^{(i)'(a)})^{-1}$ must be trivial. Hence $h_\wp^{(i)} = h_\wp^{(i)'}$.

Similarly, one can prove that $h_\wp^{(i,S)}$ does not depend on the choice of $f_1^{(i,S)}$. □

By Theorem 4.9 of [6], we have the following exact sequence

$$H^2(\mathcal{O}_S;\mathbb{Z}_2(i+1)) \longrightarrow \bigoplus_{\wp\in S} H^2(F_\wp;\mathbb{Z}_2(i+1)) \longrightarrow H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^* \longrightarrow 0\,. \quad (2.1)$$

We replace $H^0(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(-i))^*$ in Theorem 4.9 of [6] by $H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^*$ since these two groups are naturally isomorphic.

Let

$$\widetilde{H}^n(R;M) = \ker(H^n(R;M) \longrightarrow \bigoplus_{r_1} H^n(\mathbb{R};M)),$$

where R is a subring of F.

Lemma 2.2 *Let F be a real number field and S a finite set of places of F, including all the dyadic and archimedean ones. Let $\mathcal{O}_S$ be the ring of S-integers of F. The odd degree mod 2^∞ algebraic K-theory of R is given as follows:*

$$K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \simeq \begin{cases} H^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) & \text{for } i \equiv 0,\ 1 \pmod 4, \\ (\mathbb{Z}/2)^{r_1-1} \rtimes H^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) & \text{for } i \equiv 2 \pmod 4, \\ \widetilde{H}^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) & \text{for } i \equiv 3 \pmod 4. \end{cases}$$

Proof: This lemma follows from the proof of Theorem 6.9 of [6]. It can be proved also by Theorem 14.5 and 14.10 of [3]. □

Lemma 2.3 *Let F be a real number field and S a finite set of places of F, including all the dyadic and archimedean ones. Let $\mathcal{O}_S$ be the ring of S-integers of F. The sequence*

$$K_{2i}(\mathcal{O}_S)\{2\} \xrightarrow{(h_\wp^{(i,S)})} \bigoplus_{\wp\in S} \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i)) \xrightarrow{\beta_2^{(i)}} H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(i)) \longrightarrow 0$$

is a complex. Moreover if $i \equiv 0,\ 1,$ or $2 \pmod 4$, then the complex is exact; if $i \equiv 3 \pmod 4$, then

$$kernel(\beta_2^{(i,S)})/image((h_\wp^{(i,S)})) \simeq \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z}\,.$$

Proof: (1) Suppose $i \equiv 0,\ 1$, or $2 \pmod 4$. By (2.1) and (1.2), it suffices to prove $f_3^{(i,S)} \circ f_2^{(i,S)} \circ f_1^{(i,S)}$ is surjective. By Theorem 14.5 of [3], $H^2(\mathcal{O}_{F,S};\mathbb{Z}_2(i+1))$ is a finite 2-group. So $f_3^{(i,S)}$ is surjective. By Lemma 2.2, $f_2^{(i,S)}$ is surjective.

For any abelian group G, let DivSub(G) be the maximal divisible subgroup of G. Then DivSub(G) is a summand of G. By the universal coefficient theorem for algebraic K-groups with coefficients, the composition

$$f_1^{(i,S)}:\ K_{2i}(\mathcal{O}_S)\{2\} \longrightarrow K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \longrightarrow \frac{K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty)}{\mathrm{DivSub}(K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty))}$$

is an isomorphism. Since the image of a divisible element is also divisible, we have a surjective homomorphism

$$\frac{K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty)}{\mathrm{DivSub}(K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty))} \longrightarrow \frac{H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i+1))}{\mathrm{DivSub}(H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i+1)))}\,.$$

Since DivSub$(H^1(\mathcal{O}_S;\ \mathbb{Z}/2^\infty(i+1)))$ is contained in the kernel of $f_3^{(i,S)}$, the induced homomorphism

$$\frac{H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i+1))}{\mathrm{DivSub}(H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i+1)))} \longrightarrow H^2(\mathcal{O}_S;\mathbb{Z}_2(i+1))$$

is surjective. Hence the composition $f_3^{(i,S)} \circ f_2^{(i,S)} \circ f_1^{(i,S)}$ is surjective.

(2) Now suppose $i \equiv 3 \pmod 4$. We have the following commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) & \xrightarrow{\ \tau_1\ } & \bigoplus_{r_1} H^1(\mathbb{R};\mathbb{Q}_2/\mathbb{Z}_2(i+1)) \\ \downarrow f_3^{(i,S)} & & \downarrow \delta \\ H^2(\mathcal{O}_S;\mathbb{Z}_2(i+1)) & \xrightarrow[\ \tau_2\ ]{} & \bigoplus_{r_1} H^2(\mathbb{R};\mathbb{Z}_2(i+1)) \end{array}\ .$$

By (1.2) and Lemma 4.3 of [6],

$$\bigoplus_{r_1} H^1(\mathbb{R};\mathbb{Q}_2/\mathbb{Z}_2(i+1)) \simeq \bigoplus_{r_1} H^2(\mathbb{R};\mathbb{Z}_2(i+1)) \simeq \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z}\,.$$

So the right vertical arrows δ are isomorphisms. We have shown in (1) that $f_3^{(i,S)}$ is an isomorphism, and the kernel of τ_1 is $\widetilde{H}^1(\mathcal{O}_S;\ \mathbb{Q}_2/\mathbb{Z}_2(i+1))$. The kernel of τ_2 is $\widetilde{H}^2(\mathcal{O}_S;\ \mathbb{Q}_2/\mathbb{Z}_2(i+1))$. By the snake lemma, the induced homomorphism

$$\widetilde{H}^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) \longrightarrow \widetilde{H}^2(\mathcal{O}_S;\ \mathbb{Z}_2(i+1))$$

is surjective.

By Lemma 2.2, the homomorphism

$$K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \longrightarrow \widetilde{H}^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1))$$

is surjective. Hence the composition of homomorphisms

$$K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \longrightarrow \widetilde{H}^1(\mathcal{O}_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) \longrightarrow \widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}_2(i+1))$$

is surjective.

By the same arguments as in (1), we know that the homomorphism

$$K_{2i}(\mathcal{O}_S)\{2\}\longrightarrow\widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}_2(i+1))$$

is surjective.

So it suffices to prove the homology group at the middle term of the complex

$$\widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}_2(i+1))\longrightarrow\bigoplus_{\wp\in S} H^2(F_\wp;\mathbb{Z}_2(i+1))\longrightarrow H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^* \quad (2.2)$$

is isomorphic to $\bigoplus_{r_1}\mathbb{Z}/2\mathbb{Z}$.

Recall that $\bigoplus_{\wp\in S} H^2(F_\wp;\mathbb{Z}_2(i+1))\simeq(\bigoplus_{\wp\in S\text{ finite}} H^2(F_\wp;\mathbb{Z}_2(i+1)))\bigoplus(\bigoplus_{r_1}\mathbb{Z}/2\mathbb{Z})$ and $H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^*\simeq\mathbb{Z}/2\mathbb{Z}$. Let G be the kernel of the homomorphism

$$\bigoplus_{\wp\in S\text{ finite}} H^2(F_\wp;\mathbb{Z}_2(i+1))\longrightarrow H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^* .$$

By (2.1), G is isomorphic to the image of the map

$$\widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}_2(i+1)) \longrightarrow \bigoplus_{\wp\in S\text{ finite}} H^2(F_\wp;\mathbb{Z}_2(i+1)) .$$

Hence the homology group at the middle term of the complex is isomorphic to $\bigoplus_{r_1}\mathbb{Z}/2\mathbb{Z}$. □

Theorem 2.4 *Let F be a real number field with r_1 real embeddings. The sequence*

$$K_{2i}(F)\{2\} \xrightarrow{(h_\wp^{(i)})} \bigoplus_{\wp\text{ noncomplex}} \widehat{H}^0(F_\wp;\mathbb{Q}_2/\mathbb{Z}_2(i)) \xrightarrow{\beta_2^{(i)}} H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(i)) \longrightarrow 0$$

is a complex. Moreover if $i\equiv 0,\ 1,$ or $2\ (mod\ 4)$, then the complex is exact; if $i\equiv 3\ (mod\ 4)$, then

$$kernel(\beta_2^{(i)})/image((h_\wp^{(i)}))\simeq\bigoplus_{r_1}\mathbb{Z}/2\mathbb{Z} .$$

Proof: Obviously any element $a \in \text{kernel}(\beta_2^{(i)})$ belongs to $\bigoplus_{\wp \in S} \widehat{H}^0(F_\wp; \mathbb{Q}_2/\mathbb{Z}_2(i))$ for some finite set S of places of F, including all dyadic and archimedean ones. By Lemma 2.3, a comes from $K_{2i}(\mathcal{O}_S)\{2\}$ which is a subgroup of $K_{2i}(F)\{2\}$. So if $i \equiv 0,\ 1,$ or $2 \pmod 4$, then the complex is exact.

For $i \equiv 3 \pmod 4$, the proof is similar. □

Acknowledgments: Supported by NSFC, NSF(Jiangsu) and SRFDP.

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Received: August, 2005