Uniqueness of Moore’s higher reciprocity law at the prime 2 for real number fields

by

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Abstract

Let $F$ be a real number field with $r_1$ real embeddings. In this paper, we prove that the sequence

\[ K_{2i}(F) \to \bigoplus_{\varphi \text{ noncomplex}} \tilde{H}^0(F_{\varphi}; \mathbb{Q}_2/\mathbb{Z}_2(i)) \to H^0(F; \mathbb{Q}_2/\mathbb{Z}_2(i)) \to 0 \]

is a complex, where $\tilde{H}^0(F_{\varphi}; \mathbb{Q}_2/\mathbb{Z}_2(i))$ are the Tate cohomology groups. Moreover if $i \equiv 0, 1, \text{ or } 2 \pmod{4}$, then it is exact; if $i \equiv 3 \pmod{4}$, then the homology group at the second term of this complex is isomorphic to $\bigoplus \mathbb{Z}/2\mathbb{Z}$. 

Key Words: Higher Moore reciprocity law, Bloch-Lichtenbaum spectral sequence, Tate-Poitou duality theorem

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1. Introduction

Let $F$ be a number field and $\mathcal{O}_F$ its ring of integers. The Moore reciprocity uniqueness theorem states that the sequence

\[ K_2(F) \to \bigoplus_{\varphi \text{ noncomplex}} \mu(F_{\varphi}) \to \mu(F) \to 0 \]

is exact, where $F_{\varphi}$ denotes the completion of $F$ at noncomplex primes $\varphi$, and $\mu(F)$ ($\mu(F_{\varphi})$ resp.) denotes the group of roots of unity in $F$ ($F_{\varphi}$ resp.). This theorem follows from Moore ([5]), see also the appendix of [4]. In [1], Banaszak generalized the exact sequence above to the odd torsion in higher $K$-groups. For any group $G$ and natural number $l$, let $G\{l\} = \{g \in G | g^{ln} = 1 \}$ for some natural number $n$. Kahn proved ([2], Theorem 4.3) the following theorem.

**Theorem 1.1** (Moore [5] for $i = 1$, Banaszak [1] for $l$ odd) Let $m_i = m_i(F)$ and, for any noncomplex prime $\varphi$ of $F$, $m_i(\varphi) = m_i(F_{\varphi})$. Then, at least in the following cases
\[ l \text{ is odd, or} \]
\[ l = 2 \text{ and } F \text{ is totally imaginary} \]

the sequence

\[ K_{2i}(F)\{l\} \xrightarrow{(h_{(i)}^{(l)})} \bigoplus_{\varphi \text{ noncomplex}} H^0(F_\varphi; \mathbb{Q}_L/Z_L(i)) \xrightarrow{(\frac{m_i(\varphi)}{m_i})} H^0(F; \mathbb{Q}_L/Z_L(i)) \to 0 \]

is exact, where \( \frac{m_i(\varphi)}{m_i} \) is interpreted as 0 if \( m_i(\varphi) \) is infinite (this happens exactly when \( \varphi \) is real and \( i \) is even.)

See [2] for the notation used in Theorem 1.1.

Let \( F \) be a real number field with \( r_1 \) real embeddings. For any abelian group \( G \), let \( G^* \) denote \( \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \). When \( E \) is a local field let \( \hat{H}^n(E; M) \) be the Tate cohomology groups of the absolute Galois group \( G_E \) with coefficients in \( M \). Recall that by the Tate-Poitou duality theorem,

\[ H^2(F_\varphi; \mathbb{Z}_L(i + 1)) \cong \hat{H}^0(F_\varphi; \mathbb{Q}_L/Z_L(\varphi)) \]

\[ \cong \begin{cases} 
H^0(F_\varphi; \mathbb{Q}_L/Z_L(i)) & \text{for finite } \varphi, \\
\mathbb{Z}/2 & \text{for } F_\varphi = \mathbb{R} \text{ and } i \text{ odd,} \\
0 & \text{for } F_\varphi = \mathbb{R} \text{ and } i \text{ even,} \\
0 & \text{for } F_\varphi = \mathbb{C}. 
\end{cases} \]


The natural map \( F \to \prod_{\varphi \text{ noncomplex}} F_\varphi \) induces the following natural homomorphisms of the cohomology groups:

\[ H^0(F; \mathbb{Q}_L/Z_L(-i)) \to \prod_{\varphi \text{ noncomplex}} \hat{H}^0(F_\varphi; \mathbb{Q}_L/Z_L(-i)). \]

By duality at each factor of the product, we have maps

\[ \bigoplus_{\varphi \text{ noncomplex}} \hat{H}^0(F_\varphi; \mathbb{Q}_L/Z_L(-i))^* \to H^0(F; \mathbb{Q}_L/Z_L(-i))^*. \]

Since \( \hat{H}^0(F_\varphi; \mathbb{Q}_L/Z_L(-i))^* \cong \hat{H}^0(F_\varphi; \mathbb{Q}_L/Z_L(i)) \) and \( H^0(F; \mathbb{Q}_L/Z_L(-i))^* \cong H^0(F; \mathbb{Q}_L/Z_L(i)) \), we obtain maps

\[ \beta_2^{(i)} : \bigoplus_{\varphi \text{ noncomplex}} \hat{H}^0(F_\varphi; \mathbb{Q}_L/Z_L(i)) \to H^0(F; \mathbb{Q}_L/Z_L(i)). \]
Obviously $\beta_2(i)\_{\phi}$ is surjective.

In this paper, we will prove (Theorem 2.4) that there is a homomorphism $(h_\phi^{(i)})$ such that the sequence

$$K_{2i}(F\{2\}) \overset{(h_\phi^{(i)})}{\longrightarrow} \bigoplus_{\phi \text{ noncomplex}} \hat{H}^0(F\phi; \mathbb{Q}/\mathbb{Z}(i)) \overset{\beta_2(i)}{\longrightarrow} H^0(F; \mathbb{Q}/\mathbb{Z}(i)) \longrightarrow 0$$

is a complex. Moreover if $i \equiv 0, 1, \text{ or } 2 \pmod{4}$, then the complex is exact; if $i \equiv 3 \pmod{4}$, then

$$\text{kernel} (\beta_2(i)) / \text{image} ((h_\phi^{(i)})) \cong \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z}.$$ 

This result can be seen as the uniqueness of Moore’s higher reciprocity law at the prime 2 for real number fields.

2. Main Results

Let $F$ be a real number field with $r_1$ real embeddings and $S$ a finite set of places of $F$, including all the dyadic and archimedean ones. Let $\mathcal{O}_S$ be the ring of $S$-integers in $F$.

By the universal coefficient theorem for algebraic $K$-groups with coefficients, there is a short split exact sequence

$$0 \longrightarrow K_{2i+1}(F) \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow K_{2i+1}(F; \mathbb{Z}/2\mathbb{Z}) \longrightarrow K_{2i}(F\{2\}) \longrightarrow 0.$$ 

Note that $K_{2i+1}(F) \otimes \mathbb{Z}/2\mathbb{Z} \cong \bigoplus_r \mathbb{Z}/2\mathbb{Z}$, where $r = \text{rank}_\mathbb{Q}(K_{2i+1}(F) \otimes \mathbb{Z}/\mathbb{Q})$.

Hence $K_{2i+1}(F) \otimes \mathbb{Z}/2\mathbb{Z}$ is the maximal divisible subgroup of $K_{2i+1}(F; \mathbb{Z}/2\mathbb{Z})$.

We use $f_1^{(i)}$ for a split inverse

$$f_1^{(i)} : K_{2i}(F\{2\}) \longrightarrow K_{2i+1}(F; \mathbb{Z}/2\mathbb{Z}).$$

Similarly we can define

$$f_1^{(i,S)} : K_{2i}(\mathcal{O}_S\{2\}) \longrightarrow K_{2i+1}(\mathcal{O}_S; \mathbb{Z}/2\mathbb{Z}).$$

Although $f_1^{(i)}$ ($f_1^{(i,S)}$ resp.) is not unique, we will see (Lemma 2.1) that $h_\phi^{(i)}$ ($h_\phi^{(i,S)}$ resp.) does not depend on the choice of $f_1^{(i)}$ ($f_1^{(i,S)}$ resp.). Let the homomorphism

$$f_2^{(i)} : K_{2i+1}(F; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(F; \mathbb{Q}/\mathbb{Z}(i + 1))$$
be the edge map in the Bloch-Lichtenbaum spectral sequence with coefficients for $F$, see Theorem 6.7 of [6] for details. By (6.10) of [6], $f_2^{(i)}$ induces a homomorphism

$$f_2^{(i,S)}: K_{2i+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \to H^1(\mathcal{O}_S;\mathbb{Q}/\mathbb{Z}_2(i + 1)).$$

Note that (6.10) is proved for the special case of the ring of 2-integers in $F$, not for general $\mathcal{O}_S$. However the arguments of Rognes and Weibel work also for general $\mathcal{O}_S$.

The short exact sequence of the $G_F$-modules

$$0 \to \mathbb{Z}_2(i + 1) \to \mathbb{Z}_2(i + 1) \to \mathbb{Z}/2^v(i + 1) \to 0$$

induces a long exact sequence in etale cohomology, which leads to the following short exact sequence:

$$0 \to H^1(F;\mathbb{Z}_2(i + 1))/2^v \to H^1(F;\mathbb{Z}/2^v(i + 1)) \to 2^vH^2(F;\mathbb{Z}_2(i + 1)) \to 0.$$ 

Passing to direct limits, we have

$$0 \to H^1(F;\mathbb{Z}_2(i + 1))/2^\infty \to H^1(F;\mathbb{Q}/\mathbb{Z}_2(i + 1)) \to H^2(F;\mathbb{Z}_2(i + 1)) \to H^2(F;\mathbb{Z}_2(i + 1)) \to 0.$$ 

We use $f_3^{(i)}$ for the composition of natural homomorphisms

$$f_3^{(i)}: H^1(F;\mathbb{Q}/\mathbb{Z}_2(i + 1)) \to H^2(F;\mathbb{Z}_2(i + 1)) \to H^2(F;\mathbb{Z}_2(i + 1)).$$

Similarly, we define

$$f_3^{(i,S)}: H^1(\mathcal{O}_S;\mathbb{Q}/\mathbb{Z}_2(i + 1)) \to H^2(\mathcal{O}_S;\mathbb{Z}_2(i + 1)) \to H^2(\mathcal{O}_S;\mathbb{Z}_2(i + 1)).$$

Let $f_4^{(i)}$ be the composition

$$H^2(F;\mathbb{Z}_2(i + 1)) \to H^2(F;\mathbb{Q}/\mathbb{Z}_2(i + 1)) \to \hat{H}^0(F;\mathbb{Q}/\mathbb{Z}_2(i)).$$

Similarly we can define

$$f_4^{(i,S)}: H^2(\mathcal{O}_S;\mathbb{Z}_2(i + 1)) \to \hat{H}^0(F;\mathbb{Q}/\mathbb{Z}_2(i)).$$

Let

$$h_\varphi^{(i)} = f_4^{(i)} \circ f_3^{(i)} \circ f_2^{(i)} \circ f_1^{(i)}: K_{2i}(F;\mathbb{Q}2/\mathbb{Z}_2(i)) \to \hat{H}^0(F;\mathbb{Q}/\mathbb{Z}_2(i));$$

$$h_\varphi^{(i,S)} = f_4^{(i,S)} \circ f_3^{(i,S)} \circ f_2^{(i,S)} \circ f_1^{(i,S)}: K_{2i}(\mathcal{O}_S;\mathbb{Q}) \to \hat{H}^0(F;\mathbb{Q}/\mathbb{Z}_2(i)).$$

**Lemma 2.1** The homomorphism $h_\varphi^{(i)}$ ($h_\varphi^{(i,S)}$ resp.) does not depend on the choice of $f_1^{(i)}$ ($f_1^{(i,S)}$ resp.).
Proof: Let $f_1^{(i)}$ and $f_1^{(i)'}$ be two split inverses from $K_{2n}(F;\mathbb{Z}/2\infty)$ to $K_{2n+1}(F;\mathbb{Z}/2\infty)$. And let $h_\varphi^{(i)}$ and $h_\varphi^{(i)'}$ be the corresponding compositions. For any $a \in K_{2n}(F;\mathbb{Z}/2\infty)$, $f_1^{(i)}(a)(f_1^{(i)'}(a))^{-1}$ is contained in the maximal divisible subgroup of $K_{2n+1}(F;\mathbb{Z}/2\infty)$. The target of $h_\varphi^{(i)}$ and $h_\varphi^{(i)'}$ is $\tilde{H}^0(F_\varphi; \mathbb{Q}_2/\mathbb{Z}_2(i))$ which is finite. Since the images of divisible elements are divisible too, $h_\varphi^{(i)}(a)(h_\varphi^{(i)'}(a))^{-1}$ must be trivial. Hence $h_\varphi^{(i)} = h_\varphi^{(i)'}$.

Similarly, one can prove that $h_\varphi^{(i,S)}$ does not depend on the choice of $f_1^{(i,S)}$. 

By Theorem 4.9 of [6], we have the following exact sequence
\[
H^2(O_S;\mathbb{Z}_2(i+1)) \to \bigoplus_{\varphi \in S} H^2(F_\varphi;\mathbb{Z}_2(i+1)) \to H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^* \to 0 . (2.1)
\]
We replace $H^0(O_S;\mathbb{Q}_2/\mathbb{Z}_2(-i))^*$ in Theorem 4.9 of [6] by $H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(-i))^*$ since these two groups are naturally isomorphic.

Let
\[
\tilde{H}^n(R;M) = \ker(H^n(R;M)) \to \bigoplus_{r_1} H^n(\mathbb{R};M),
\]
where $R$ is a subring of $F$.

**Lemma 2.2** Let $F$ be a real number field and $S$ a finite set of places of $F$, including all the dyadic and archimedean ones. Let $O_S$ be the ring of $S$-integers of $F$. The odd degree mod $2\infty$ algebraic K-theory of $R$ is given as follows:
\[
K_{2i+1}(O_S;\mathbb{Z}/2\infty) \cong \begin{cases} 
H^1(O_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) & \text{for } i \equiv 0, 1 \pmod{4}, \\
(\mathbb{Z}/2)^{r_1} \times H^1(O_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) & \text{for } i \equiv 2 \pmod{4}, \\
\tilde{H}^1(O_S;\mathbb{Q}_2/\mathbb{Z}_2(i+1)) & \text{for } i \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof:** This lemma follows from the proof of Theorem 6.9 of [6]. It can be proved also by Theorem 14.5 and 14.10 of [3].

**Lemma 2.3** Let $F$ be a real number field and $S$ a finite set of places of $F$, including all the dyadic and archimedean ones. Let $O_S$ be the ring of $S$-integers of $F$. The sequence
\[
K_{2i}(O_S;\{2\}) \xrightarrow{(h_\varphi^{(i,S)})} \bigoplus_{\varphi \in S} \tilde{H}^0(F_\varphi;\mathbb{Q}_2/\mathbb{Z}_2(i)) \xrightarrow{\beta_2^{(i)}} H^0(F;\mathbb{Q}_2/\mathbb{Z}_2(i)) \to 0
\]
is a complex. Moreover if $i \equiv 0, 1, \text{ or } 2 \pmod{4}$, then the complex is exact; if $i \equiv 3 \pmod{4}$, then
\[
\ker(\beta_2^{(i,S)})/\text{image}(\beta_2^{(i,S)})) \cong \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z}.
\]
Proof: (1) Suppose \( i \equiv 0, 1, \text{ or } 2 \mod 4 \). By (2.1) and (1.2), it suffices to prove \( f_3^{(i,S)} \circ f_2^{(i,S)} \circ f_1^{(i,S)} \) is surjective. By Theorem 14.5 of [3], \( H^2(O_F:S;\mathbb{Z}_2(i + 1)) \) is a finite 2-group. So \( f_3^{(i,S)} \) is surjective. By Lemma 2.2, \( f_2^{(i,S)} \) is surjective.

For any abelian group \( G \), let \( \text{DivSub}(G) \) be the maximal divisible subgroup of \( G \). Then \( \text{DivSub}(G) \) is a summand of \( G \). By the universal coefficient theorem for algebraic K-groups with coefficients, the composition

\[
  f_1^{(i,S)} : K_2i(O_S)\{2\} \to K_2i+1(O_S;\mathbb{Z}/2^{\infty}) \to \frac{K_2i+1(O_S;\mathbb{Z}/2^{\infty})}{\text{DivSub}(K_2i+1(O_S;\mathbb{Z}/2^{\infty}))}
\]

is an isomorphism. Since the image of a divisible element is also divisible, we have a surjective homomorphism

\[
  \frac{K_2i+1(O_S;\mathbb{Z}/2^{\infty})}{\text{DivSub}(K_2i+1(O_S;\mathbb{Z}/2^{\infty}))} \to \frac{H^1(O_S;\mathbb{Z}/2^{\infty}(i + 1))}{\text{DivSub}(H^1(O_S;\mathbb{Z}/2^{\infty}(i + 1)))}.
\]

Since \( \text{DivSub}(H^1(O_S;\mathbb{Z}/2^{\infty}(i + 1))) \) is contained in the kernel of \( f_3^{(i,S)} \), the induced homomorphism

\[
  \frac{H^1(O_S;\mathbb{Z}/2^{\infty}(i + 1))}{\text{DivSub}(H^1(O_S;\mathbb{Z}/2^{\infty}(i + 1)))} \to H^2(O_S;\mathbb{Z}_2(i + 1))
\]

is surjective. Hence the composition \( f_3^{(i,S)} \circ f_2^{(i,S)} \circ f_1^{(i,S)} \) is surjective.

(2) Now suppose \( i \equiv 3 \mod 4 \). We have the following commutative diagram

\[
\begin{array}{ccc}
H^1(O_S;\mathbb{Q}/\mathbb{Z}_2(i + 1)) & \to & \bigoplus_{r_1} H^1(\mathbb{R};\mathbb{Q}/\mathbb{Z}_2(i + 1)) \\
\text{f}_3^{(i,S)} & & \downarrow \tau_1 \\
H^2(O_S;\mathbb{Z}_2(i + 1)) & \to & \bigoplus_{r_1} H^2(\mathbb{R};\mathbb{Z}_2(i + 1)) \\
\text{f}_2^{(i,S)} & & \downarrow \tau_2 \\
\end{array}
\]

By (1.2) and Lemma 4.3 of [6],

\[
\bigoplus_{r_1} H^1(\mathbb{R};\mathbb{Q}/\mathbb{Z}_2(i + 1)) \cong \bigoplus_{r_1} H^2(\mathbb{R};\mathbb{Z}_2(i + 1)) \cong \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z}.
\]

So the right vertical arrows \( \delta \) are isomorphisms. We have shown in (1) that \( f_3^{(i,S)} \) is an isomorphism, and the kernel of \( \tau_1 \) is \( \bar{H}^1(O_S;\mathbb{Q}_2/\mathbb{Z}_2(i + 1)) \). The kernel of \( \tau_2 \) is \( \bar{H}^2(O_S;\mathbb{Q}_2/\mathbb{Z}_2(i + 1)) \). By the snake lemma, the induced homomorphism

\[
\bar{H}^1(O_S;\mathbb{Q}_2/\mathbb{Z}_2(i + 1)) \to \bar{H}^2(O_S;\mathbb{Z}_2(i + 1))
\]
is surjective.

By Lemma 2.2, the homomorphism

\[ K_{2i+1}(O_S; \mathbb{Z}/2^{\infty}) \rightarrow \tilde{H}^1(O_S; \mathbb{Q}_2/\mathbb{Z}_2(i + 1)) \]

is surjective. Hence the composition of homomorphisms

\[ K_{2i+1}(O_S; \mathbb{Z}/2^{\infty}) \rightarrow \tilde{H}^1(O_S; \mathbb{Q}_2/\mathbb{Z}_2(i + 1)) \rightarrow \tilde{H}^2(O_S; \mathbb{Z}_2(i + 1)) \]

is surjective.

By the same arguments as in (1), we know that the homomorphism

\[ K_{2i}(O_S) \{2\} \rightarrow \tilde{H}^2(O_S; \mathbb{Z}_2(i + 1)) \]

is surjective.

So it suffices to prove the homology group at the middle term of the complex

\[ \tilde{H}^2(O_S; \mathbb{Z}_2(i + 1)) \rightarrow \bigoplus_{\varphi \in S} H^2(F_{\varphi}; \mathbb{Z}_2(i + 1)) \rightarrow H^0(F; \mathbb{Q}_2/\mathbb{Z}_2(-i))^* \quad (2.2) \]

is isomorphic to \( \bigoplus \mathbb{Z}/2\mathbb{Z} \).

Recall that

\[ \bigoplus_{\varphi \in S} H^2(F_{\varphi}; \mathbb{Z}_2(i + 1)) \simeq ( \bigoplus_{\varphi \in S} H^2(F_{\varphi}; \mathbb{Z}_2(i + 1))) \bigoplus (\bigoplus_{r_1 \text{ finite}} \mathbb{Z}/2\mathbb{Z}) \]

and \( H^0(F; \mathbb{Q}_2/\mathbb{Z}_2(-i))^* \simeq \mathbb{Z}/2\mathbb{Z} \). Let \( G \) be the kernel of the homomorphism

\[ \bigoplus_{\varphi \in S} H^2(F_{\varphi}; \mathbb{Z}_2(i + 1)) \rightarrow H^0(F; \mathbb{Q}_2/\mathbb{Z}_2(-i))^* \]

By (2.1), \( G \) is isomorphic to the image of the map

\[ \tilde{H}^2(O_S; \mathbb{Z}_2(i + 1)) \rightarrow \bigoplus_{\varphi \in S \text{ finite}} H^2(F_{\varphi}; \mathbb{Z}_2(i + 1)) \]

Hence the homology group at the middle term of the complex is isomorphic to \( \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z} \).

**Theorem 2.4** Let \( F \) be a real number field with \( r_1 \) real embeddings. The sequence

\[ K_{2i}(F) \{2\} \rightarrow \bigoplus_{\varphi \text{ noncomplex}} \tilde{H}^0(F_{\varphi}; \mathbb{Q}_2/\mathbb{Z}_2(i)) \rightarrow H^0(F; \mathbb{Q}_2/\mathbb{Z}_2(i)) \rightarrow 0 \]

is a complex. Moreover if \( i \equiv 0, 1, \text{ or } 2 \pmod{4} \), then the complex is exact; if \( i \equiv 3 \pmod{4} \), then

\[ \text{kernel}(\beta_2^{(i)})/\text{image}((h_{\varphi}^{(i)})) \simeq \bigoplus_{r_1} \mathbb{Z}/2\mathbb{Z} \].
**Proof:** Obviously any element $a \in \ker(\tilde{\beta}_2^{(i)})$ belongs to $\bigoplus_{\varphi \in S} \widehat{H}^0(F_{\varphi}; \mathbb{Q}_2/\mathbb{Z}_2(i))$ for some finite set $S$ of places of $F$, including all dyadic and archimedean ones. By Lemma 2.3, $a$ comes from $K_{2i}(\mathcal{O}_S)[2]$ which is a subgroup of $K_{2i}(F)[2]$. So if $i \equiv 0, 1, \text{ or } 2 \pmod{4}$, then the complex is exact.

For $i \equiv 3 \pmod{4}$, the proof is similar. \qed

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