J. Math. Anal. Appl. 487 (2020) 123971

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Non-vanishing theta values of characters with special prime conductors

Xuejun Guo [a,*], Yuzhen Peng [b]

[a] Department of Mathematics, Nanjing University, Nanjing 210093, China
[b] School of Mathematics and Statistics, Nanning Normal University, Nanning 530001, China

ARTICLE INFO

Article history:
Received 10 September 2019
Available online 22 February 2020
Submitted by B.C. Berndt

Keywords:
Theta function
Dirichlet characters
Mean values

ABSTRACT

Let p be a prime of the form $2\ell+1$ or $4\ell+1$, where ℓ is also a prime. We prove $\theta(\chi, i) \neq 0$ for all primitive Dirichlet characters χ with conductor p except for the quadratic and the quartic ones. Our results generalize a theorem of Bengoechea, which asserts $\theta(\chi, i) \neq 0$ for non-quadratic χ with "large" prime conductor $p = 2\ell+1$, where ℓ is also a prime.

1. Introduction

Let χ be a primitive Dirichlet character of conductor N. The *theta function* associated to χ, denoted $\theta(\chi, \tau)$, is defined on the upper half plane as

$$\theta(\chi, \tau) = \begin{cases} \sum_{n=1}^{\infty} \chi(n) e^{\frac{n^2}{N}\pi i \tau}, & \text{if } \chi \text{ is even,} \\ \sum_{n=1}^{\infty} n\chi(n) e^{\frac{n^2}{N}\pi i \tau}, & \text{if } \chi \text{ is odd.} \end{cases} \tag{1.1}$$

If $\theta(\chi, i) \neq 0$ then the normalized Gauss sum can be expressed as

$$W(\chi) = \frac{\theta(\chi, i)}{\theta(\overline{\chi}, i)}. \tag{1.2}$$

One can use (1.1) to compute efficiently the numerical value of $W(\chi)$. So it is hoped that $\theta(\chi, i)$ does not vanish for "many" χ.

* Corresponding author.
E-mail addresses: guoxj@nju.edu.cn (X. Guo), yzpeng@mail.ustc.edu.cn (Y. Peng).

https://doi.org/10.1016/j.jmaa.2020.123971
0022-247X/

Cohen and Zagier [2] showed that $\theta(\chi, i)$ vanishes for only two characters and their complex conjugates and for no other primitive characters of conductor ≤ 52100. Louboutin [3] proved that there is a positive constant c such that for prime p, at least $cp/\log(p)$ of the characters χ of conductor p have $\theta(\chi, i) \neq 0$. He raised the question whether $\theta(\chi, i)$ is always nonzero.

Bengoechea [1] calculated the Galois action on the special values of theta functions and proved:

Theorem 1.1 *([1], Theorem 4.7). There is a constant $c > 0$ such that for all non-quadratic χ with prime conductor $p = 2\ell + 1$, where ℓ is prime, satisfying $p > c$, we have $\theta(\chi, i) \neq 0$.*

Bengoechea's proof uses the following theorem, which is due to Louboutin for odd characters and Louboutin-Munsch for even characters.

Theorem 1.2. *There is a constant $c > 0$ such that $\theta(\chi, i) \neq 0$ for at least $cp/\log(p)$ characters of the $(p-1)/2$ odd characters with conductor p and of the $(p-1)/2$ even ones.*

For a fixed prime p, denote by X the group of Dirichlet characters modulo p, which is a cyclic group of order $p-1$. For any nonempty subset T of X, the second and fourth mean value of theta functions at i for characters ranging in T, are defined respectively to be

$$S_2(T) := \sum_{\chi \in T} |\theta(\chi, i)|^2, \qquad S_4(T) := \sum_{\chi \in T} |\theta(\chi, i)|^4. \tag{1.3}$$

Let $N(T)$ be the number of characters χ in T such that $\theta(\chi, i) \neq 0$. Then Cauchy-Schwarz inequality yields

$$N(T) \geqslant S_2^2(T)/S_4(T). \tag{1.4}$$

Louboutin and Munsch [4] gave the following asymptotic estimations for the set X^+ of the $\frac{p-1}{2}$ even characters as well as the set X^- of the $\frac{p-1}{2}$ odd ones:

$$S_2(X^+) \sim \frac{p^{\frac{3}{2}}}{4\sqrt{2}}, \quad S_2(X^-) \sim \frac{p^{\frac{5}{2}}}{16\pi\sqrt{2}}, \qquad S_4(X^+) \sim \frac{3p^2 \log p}{16\pi}, \quad S_4(X^-) \sim \frac{3p^4 \log p}{512\pi^3}. \tag{1.5}$$

Theorem 1.2 follows from (1.4) and (1.5).

In this paper we take a more direct approach to prove the non-vanishing of theta values. We consider the first moment of the theta values

$$S_1(T) := \sum_{\chi \in T} \theta(\chi, i). \tag{1.6}$$

Louboutin and Munsch [4] have studied $S_1(X^+)$ and $S_1(X^-)$. They showed that

$$S_1(X^+) \sim \frac{p}{2}, \quad S_1(X^-) \sim \frac{p}{2}. \tag{1.7}$$

We shall give lower and upper bounds of $S_1(T)$ for some specific subsets T. Since $N(T) = 0$ implies $S_1(T) = 0$, we can get $N(T) \neq 0$ as long as we prove $S_1(T) \neq 0$. In particular, when T is a Galois orbit and $N(T) \neq 0$, then a theorem of Bengoechea will force all the characters in T to have non-vanishing theta values.

The prime p in Theorem 1.1 is called a *safe prime*, and the prime ℓ is called a *Sophie Germain prime*. The heuristic estimate for the number of Sophie Germain primes less than x is

$$C\frac{x}{(\log x)^2}, \tag{1.8}$$

where

$$C = 2\prod_{p>2}\frac{p(p-2)}{(p-1)^2} \approx 1.32032.$$

We shall prove that Theorem 1.1 holds not just for large enough safe primes, it is actually valid for *all* safe primes:

Theorem 1.3. $\theta(\chi, i) \neq 0$ *for all non-quadratic* χ *with prime conductor* $p = 2\ell + 1$*, where* ℓ *is a prime.*

In this paper, we will also generalize Theorem 1.1 to the characters χ with prime conductor $p = 4\ell + 1$, where ℓ is a prime.

Theorem 1.4. *Let* p *be a prime of the form* $4\ell + 1$*, where* ℓ *is also a prime. If a primitive Dirichlet character* χ *with conductor* p *is neither quadratic nor quartic, then* $\theta(\chi, i) \neq 0$.

The heuristic estimate for the number of primes $\ell < x$ such that $p = 4\ell + 1$ is also prime is the same as (1.8). One can see Conjecture 5.24 and 5.25 of [5].

2. Main results

Let χ be a primitive character with odd conductor N and order m. Let $M = 24mN^2$. Consider the order $\mathcal{O} = \mathbb{Z}[iN]$ in $K = \mathbb{Q}(i)$. Let $H_{\mathcal{O}} = K(j(iN))$ be the ring class field of $\mathcal{O}$ and $H_{M,\mathcal{O}}$ be the ray class field with conductor M over $H_{\mathcal{O}}$. As in [1], we define

$$A_\chi(\tau) = \frac{\theta(\chi, \tau/N)}{\eta(\tau/N)^{1+2\epsilon}}, \quad B_\chi(\tau) = |A_\chi(\tau)|^2 = A_\chi(\tau)A_{\overline{\chi}}(\tau),$$

where

$$\epsilon = \begin{cases} 0, & \text{if } \chi \text{ is even;} \\ 1, & \text{if } \chi \text{ is odd;} \end{cases}$$

and η is the classical Dedekind η-function. In particular, if the conductor N is a prime, say p, we denote by $X(p, m)$ the set of characters with conductor p and order m up to complex conjugation. With these notations Bengoechea proved the following theorem.

Theorem 2.1 *([1], Theorem 4.2 (i)). The set*

$$\left\{B_\chi(ip)^2 \mid \chi \in X(p, m)\right\}$$

is an orbit for the action of the group $\mathrm{Gal}(H_{M,\mathcal{O}}/H_{\mathcal{O}})$.

This theorem leads to a straightforward result, which plays a crucial role in our argument:

Proposition 2.2. *Once there exists a* $\chi \in X(p,m)$ *such that* $\theta(\chi, i) \neq 0$*, it is so for each* $\chi \in X(p,m)$*.*

We will make a frequent use of a simple fact from elementary calculus: Let $f(x)$ be a nonnegative descending continuous real function on the interval $[n_0, +\infty)$, where n_0 is an integer. If the integral $\int_{n_0}^{+\infty} f(x)\,\mathrm{d}x$ converges, then it is an upper bound of the series $\sum_{n=n_0+1}^{\infty} f(n)$.

In what follows p is always an odd prime, X is the character group modulo p, X^+ is the subgroup of X consisting of all even characters modulo p, and χ_0 is the trivial character modulo p. For convenience we establish a lemma concerning S_1 of a general subgroup of X^+.

Lemma 2.3. *Let* G *be a subgroup of* X^+ *with order* d*. Then*

$$S_1(G) = d \sum_{\substack{n=1 \\ n^{\frac{p-1}{d}} \equiv 1 \pmod p}}^{\infty} e^{-\frac{\pi}{p} n^2},$$

which has the obvious lower bound $d\, e^{-\frac{\pi}{p}}$*.*

Proof. The orthogonality relation on G reads

$$\sum_{\chi \in G} \chi(n) = \begin{cases} d, & \text{if } n^{\frac{p-1}{d}} \equiv 1 \pmod p, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$S_1(G) = \sum_{\chi \in G} \theta(\chi, i) = \sum_{\chi \in G} \sum_{n=1}^{\infty} \chi(n)\, e^{-\frac{\pi}{p} n^2} = \sum_{n=1}^{\infty} \left(\sum_{\chi \in G} \chi(n) \right) e^{-\frac{\pi}{p} n^2}$$
$$= d \sum_{\substack{n=1 \\ n^{\frac{p-1}{d}} \equiv 1 \pmod p}}^{\infty} e^{-\frac{\pi}{p} n^2} > d\, e^{-\frac{\pi}{p}}. \quad \square$$

2.1. Even characters and odd characters

By definition,

$$\theta(\chi_0, i) = \sum_{\substack{n=1 \\ n \not\equiv 0 \pmod p}}^{\infty} e^{-\frac{\pi}{p} n^2} < \sum_{n=1}^{\infty} e^{-\frac{\pi}{p} n^2} \leqslant \int_0^{+\infty} e^{-\frac{\pi}{p} x^2}\,\mathrm{d}x = \frac{\sqrt{p}}{2}, \tag{2.1}$$

and by Lemma 2.3,

$$S_1(X^+) \geqslant \frac{p-1}{2}\, e^{-\frac{\pi}{p}}. \tag{2.2}$$

Thus

$$S_1(X^+ \backslash \{\chi_0\}) = S_1(X^+) - \theta(\chi_0, i) \geqslant \frac{p-1}{2}\, e^{-\frac{\pi}{p}} - \frac{\sqrt{p}}{2}. \tag{2.3}$$

One sees the RHS of the above inequality stays positive when $p \geqslant 7$. Hence we obtain

Proposition 2.4. *For any prime $p \geqslant 7$ there exists a nontrivial $\chi \in X^+$ such that $\theta(\chi, i) \neq 0$.*

Then consider the set X^- of odd characters modulo p, that is, $X^- = X \backslash X^+$. Combining the orthogonality relations on X and on X^+, we have

$$\sum_{\chi \in X^-} \chi(n) = \begin{cases} (p-1)/2, & \text{if } n \equiv 1 \pmod p, \\ -(p-1)/2, & \text{if } n \equiv -1 \pmod p, \\ 0, & \text{if } n \not\equiv \pm 1 \pmod p. \end{cases}$$

Hence

$$\begin{aligned} S_1(X^-) &= \sum_{\chi \in X^-} \theta_\chi(i) = \sum_{\chi \in X^-} \sum_{n=1}^{\infty} n\,\chi(n)\, e^{-\frac{\pi}{p}n^2} = \sum_{n=1}^{\infty} \left(\sum_{\chi \in X^-} \chi(n) \right) n\, e^{-\frac{\pi}{p}n^2} \\ &= \frac{p-1}{2} \left(\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1 \\ n \equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} \right). \end{aligned} \tag{2.4}$$

It is clear that

$$\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} > e^{-\frac{\pi}{p}} > 1 - \frac{\pi}{p}.$$

For the series $\sum_{n=1,\ n\equiv -1 \ (\mathrm{mod}\ p)}^{\infty} n\, e^{-\frac{\pi}{p}n^2}$, notice that the function $f(x) = x\, e^{-\frac{\pi}{p}x^2}$ descends on the interval $[\sqrt{\frac{p}{2\pi}}, +\infty)$, and particularly on $[p-2, +\infty)$, we have

$$\sum_{\substack{n=1 \\ n \equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} < \sum_{n=p-1}^{\infty} n\, e^{-\frac{\pi}{p}n^2} \leqslant \int_{p-2}^{+\infty} x\, e^{-\frac{\pi}{p}x^2}\, \mathrm{d}x = \frac{p}{2\pi}\, e^{-\frac{\pi(p-2)^2}{p}}.$$

So

$$\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1 \\ n \equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} > 1 - \frac{\pi}{p} - \frac{p}{2\pi}\, e^{-\frac{\pi(p-2)^2}{p}}, \tag{2.5}$$

and

$$S_1(X^-) > \frac{p-1}{2} \left(1 - \frac{\pi}{p} - \frac{p}{2\pi}\, e^{-\frac{\pi(p-2)^2}{p}} \right). \tag{2.6}$$

Note the RHS of the above inequality is positive when $p \geqslant 5$. Thus for any prime $p \geqslant 5$, there exists a $\chi \in X^-$ such that $\theta(\chi, i) \neq 0$.

2.2. Quadratic character

Let χ_1 be the unique quadratic character in X, which is even when $p \equiv 1 \pmod 4$ and odd when $p \equiv 3 \pmod 4$. Note $\chi_1(n) = \pm 1$ for any $n \geqslant 1$.

Case $p \equiv 1 \pmod 4$**.** The theta value of χ_1 is

$$\theta(\chi_1, i) = \sum_{\substack{n=1 \\ n\equiv \pm 1 \pmod p}}^{\infty} e^{-\frac{\pi}{p}n^2} + \sum_{\substack{n=1 \\ n\not\equiv 0,\pm 1 \pmod p}}^{\infty} \chi_1(n)\, e^{-\frac{\pi}{p}n^2}, \tag{2.7}$$

hence

$$\theta(\chi_1, i) < \sum_{n=1}^{\infty} e^{-\frac{\pi}{p}n^2} \leqslant \int_0^{+\infty} e^{-\frac{\pi}{p}x^2}\,\mathrm{d}x = \frac{\sqrt{p}}{2}. \tag{2.8}$$

Hence by (2.6) and (2.8),

$$S_1(X^-\backslash\{\chi_1\}) > \frac{p-1}{2}\left(1 - \frac{\pi}{p} - \frac{p}{2\pi}\, e^{-\frac{\pi(p-2)^2}{p}}\right) - \frac{\sqrt{p}}{2} \tag{2.9}$$

and the RHS of the above inequality is positive when $p \geqslant 5$.

Case $p \equiv 3 \pmod 4$**.** The theta value of χ_1 is

$$\theta(\chi_1, i) = \sum_{\substack{n=1 \\ n\equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1 \\ n\equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} + \sum_{\substack{n=1 \\ n\not\equiv 0,\pm 1 \pmod p}}^{\infty} n\, \chi_1(n)\, e^{-\frac{\pi}{p}n^2}. \tag{2.10}$$

Note that

$$\sum_{\substack{n=1 \\ n\not\equiv 0,\pm 1 \pmod p}}^{\infty} n\, \chi_1(n)\, e^{-\frac{\pi}{p}n^2} < \sum_{\substack{n=1 \\ n\not\equiv 0,\pm 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} < \sum_{n=1}^{\infty} n\, e^{-\frac{\pi}{p}n^2}.$$

Since the function $f(x) = x\, e^{-\frac{\pi}{p}x^2}$ ascends on the interval $[0, \sqrt{\frac{p}{2\pi}}]$ and descends on the interval $[\sqrt{\frac{p}{2\pi}}, +\infty)$, it takes its maximum value $\sqrt{\frac{p}{2\pi}}\, e^{-\frac{1}{2}}$ at $x = \sqrt{\frac{p}{2\pi}}$. Therefore,

$$\sum_{n=1}^{\infty} n\, e^{-\frac{\pi}{p}n^2} < \sqrt{\frac{p}{2\pi}} \cdot \sqrt{\frac{p}{2\pi}}\, e^{-\frac{1}{2}} + \int_{\sqrt{\frac{p}{2\pi}}}^{+\infty} x\, e^{-\frac{\pi}{p}x^2}\,\mathrm{d}x = \frac{p}{\pi}\, e^{-\frac{1}{2}}.$$

So

$$\begin{aligned} & S_1(X^-\backslash\{\chi_1\}) = S_1(X^-) - \theta(\chi_1, i) \\ =& \frac{p-3}{2}\left(\sum_{\substack{n=1 \\ n\equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1 \\ n\equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2}\right) - \sum_{\substack{n=1 \\ n\not\equiv \pm 1 \pmod p}}^{\infty} n\, \chi_1(n)\, e^{-\frac{\pi}{p}n^2} \\ >& \frac{p-3}{2}\left(1 - \frac{\pi}{p} - \frac{p}{2\pi}\, e^{-\frac{\pi(p-2)^2}{p}}\right) - \frac{p}{\pi}\, e^{-\frac{1}{2}}. \end{aligned} \tag{2.11}$$

One sees the RHS of the above inequality (2.11) is positive when $p \geqslant 11$. By (2.9) and (2.11), we have proved the following proposition.

Proposition 2.5. *For any prime $p \geqslant 11$, there exists a non-quadratic $\chi \in X^-$ such that $\theta(\chi, i) \neq 0$.*

Table 1
Table of primitive characters with conductor $p = 2\ell + 1$.

order d	$\#X(p, d)$	parity
1	1	even
2	1	odd
ℓ	$\ell - 1$	even
2ℓ	$\ell - 1$	odd

2.3. Quartic characters

Let χ_2 and $\overline{\chi_2}$ be the two quartic characters in X, which are even when $p \equiv 1 \pmod 8$ and odd otherwise. To deduce our main result we only need to provide an upper bound for $\theta(\chi_2, i) + \theta(\overline{\chi_2}, i)$. Note that $\chi_2(n) + \overline{\chi_2}(n) = 0, \pm 2$ for any $n \geqslant 1$.

Case $p \equiv 1 \pmod 8$**.** By definition, $\theta(\chi_2, i) + \theta(\overline{\chi_2}, i)$ is

$$\sum_{\substack{n=1 \\ n \not\equiv 0 \pmod p}}^{\infty} \left(\chi_2(n) + \overline{\chi_2}(n)\right) e^{-\frac{\pi}{p} n^2},$$

which is less than $2 \sum_{n=1}^{\infty} e^{-\frac{\pi}{p} n^2}$. So

$$\theta(\chi_2, i) + \theta(\overline{\chi_2}, i) < 2 \sum_{n=1}^{\infty} e^{-\frac{\pi}{p} n^2} \leqslant 2 \int_0^{+\infty} e^{-\frac{\pi}{p} x^2} \, \mathrm{d}x = \sqrt{p}.$$

Case $p \not\equiv 1 \pmod 8$**.** In this case $\theta(\chi_2, i) + \theta(\overline{\chi_2}, i)$ is

$$2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} - 2 \sum_{\substack{n=1 \\ n \equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} + \sum_{\substack{n=1 \\ n \not\equiv 0, \pm 1 \pmod p}}^{\infty} n \left(\chi_2(n) + \overline{\chi_2}(n)\right) e^{-\frac{\pi}{p} n^2}, \tag{2.12}$$

wherein

$$\sum_{\substack{n=1 \\ n \not\equiv 0, \pm 1 \pmod p}}^{\infty} n \left(\chi_2(n) + \overline{\chi_2}(n)\right) e^{-\frac{\pi}{p} n^2} < 2 \sum_{n=1}^{\infty} n\, e^{-\frac{\pi}{p} n^2} = \frac{2p}{\pi} e^{-\frac{1}{2}}.$$

Therefore,

$$\theta(\chi_2, i) + \theta(\overline{\chi_2}, i) < 2 \left(\sum_{\substack{n=1 \\ n \equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} - \sum_{\substack{n=1 \\ n \equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} + \frac{p}{\pi} e^{-\frac{1}{2}} \right). \tag{2.13}$$

2.4. Primes of type $2\ell + 1$

Let $p = 2\ell + 1$ be a prime, where ℓ is also a prime. Then the character group X is a cyclic group of order 2ℓ. The sizes and parity of each orbit $X(p, d)$ for $d \mid 2\ell$ are listed in Table 1.

Theorem 2.6. *$\theta(\chi, i) \neq 0$ for all non-quadratic characters χ with prime conductor $p = 2\ell + 1$, where ℓ is also a prime.*

Table 2
Table of primitive characters with conductor $p = 4\ell + 1$.

order d	$\#X(p,d)$	parity
1	1	even
2	1	even
4	2	odd
ℓ	$\ell - 1$	even
2ℓ	$\ell - 1$	even
4ℓ	$2(\ell - 1)$	odd

Proof. Since $X^+ = X(p,\ell) \cup \{\chi_0\}$, it follows from Proposition 2.4 that, when $p \geqslant 7$ there exists a $\chi \in X(p,\ell)$ such that $\theta(\chi, i) \neq 0$, and then by Proposition 2.2, $\theta(\chi, i) \neq 0$ for all $\chi \in X(p,\ell)$. Similarly, $X^- = X(p, 2\ell) \cup \{\chi_1\}$ and it follows from Propositions 2.2 and 2.5 that $\theta(\chi, i) \neq 0$ for all $\chi \in X(p, 2\ell)$ when $p \geqslant 11$. Since Cohen and Zagier proved $\theta(\chi, i) = 0$ for only two characters of conductors 300 and 600 and their complex conjugates and for no other primitive characters of conductor ≤ 52100, the theorem holds also for $p < 11$. □

2.5. Primes of type $4\ell + 1$

Now let $p = 4\ell + 1$, where ℓ is also a prime. The least such prime is 13. In this situation the sizes and parity of each orbit $X(p,d)$ for $d \,|\, 4\ell$ are listed in Table 2.

Note

$$X^+ = \{\chi_0\} \cup \{\chi_1\} \cup X(p,\ell) \cup X(p, 2\ell), \quad X^- = \{\chi_2, \overline{\chi_2}\} \cup X(p, 4\ell).$$

Lemma 2.7. *$S_1(X(p,\ell)) > 0$ for $p \geqslant 13$, and $S_1(X(p, 2\ell)) > 0$ for $p \geqslant 29$.*

Proof. We assume that $p \geqslant 13$. Since $X(p,\ell)$ together with χ_0 constitutes a subgroup of X^+, it follows from Lemma 2.3 that $S_1(X(p,\ell) \cup \{\chi_0\}) > \ell\, e^{-\frac{\pi}{p}}$. Combining this with (2.1) yields

$$S_1(X(p,\ell)) = S_1(X(p,\ell) \cup \{\chi_0\}) - \theta(\chi_0, i) > \ell\, e^{-\frac{\pi}{p}} - \frac{\sqrt{p}}{2},$$

wherein $\ell\, e^{-\frac{\pi}{p}} - \frac{\sqrt{p}}{2}$ is positive for any $p \geqslant 13$.

To show $S_1(X(p, 2\ell))$ is also positive an upper bound of $S_1(X(p,\ell) \cup \{\chi_0\})$ is needed. In view of Lemma 2.3,

$$S_1(X(p,\ell) \cup \{\chi_0\}) = \ell \sum_{\substack{n=1 \\ n^4 \equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} e^{-\frac{\pi}{p} n^2}.$$

Let b be the solution of the congruence equation $x^2 \equiv -1 \pmod{p}$ with $2 \leqslant b \leqslant \frac{p-1}{2}$. Since $b^2 \equiv -1 \pmod{p}$ and the least positive integer satisfying this congruence condition is $p - 1$, we have

$$b^2 \geqslant p - 1. \tag{2.14}$$

Similarly,

$$(p - b)^2 \geqslant 2p - 1. \tag{2.15}$$

Note that $1, b, p - b$ and $p - 1$ are the four solutions of the congruence equation $x^4 \equiv 1 \pmod{p}$ between 1 and $p - 1$.

Thus

$$\sum_{\substack{n=1\\ n^4\equiv 1 \pmod p}}^{\infty} e^{-\frac{\pi}{p}n^2} = \sum_{k=0}^{\infty}\left(e^{-\frac{\pi(kp+1)^2}{p}} + e^{-\frac{\pi(kp+b)^2}{p}} + e^{-\frac{\pi(kp+p-b)^2}{p}} + e^{-\frac{\pi(kp+p-1)^2}{p}}\right)$$
$$< e^{-\frac{\pi}{p}} + e^{-\frac{\pi b^2}{p}} + e^{-\frac{\pi(p-b)^2}{p}} + e^{-\frac{\pi(p-1)^2}{p}} + 4\sum_{k=1}^{\infty} e^{-\frac{\pi(kp+1)^2}{p}}.$$

By (2.14) and (2.15),

$$e^{-\frac{\pi b^2}{p}} + e^{-\frac{\pi(p-b)^2}{p}} \leqslant 2\, e^{-\frac{\pi(p-1)}{p}} < 2\, e^{-\frac{12\pi}{13}} < 0.12.$$

For the item $e^{-\frac{\pi(p-1)^2}{p}}$, it is no greater than $e^{-\frac{12^2\pi}{13}} < 0.01$. Meanwhile, the sum $\sum_{k=1}^{\infty} e^{-\frac{\pi(kp+1)^2}{p}}$ has an upper bound

$$\int_0^{+\infty} e^{-\frac{\pi(px+1)^2}{p}}\,\mathrm{d}x < \frac{1}{\sqrt{\pi p}}\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t = \frac{1}{2\sqrt{p}}.$$

Summing up all the above estimates, we obtain

$$S_1(X(p,\ell)\cup\{\chi_0\}) = \ell \sum_{\substack{n=1\\ n^4\equiv 1 \pmod p}}^{\infty} e^{-\frac{\pi}{p}n^2} < \ell\left(e^{-\frac{\pi}{p}} + 0.13 + \frac{2}{\sqrt{p}}\right). \tag{2.16}$$

Finally, from (2.2), (2.8) and (2.16) it follows that

$$\begin{aligned} S_1(X(p,2\ell)) &= S_1(X^+) - S_1(X(p,\ell)\cup\{\chi_0\}) - \theta(\chi_1, i)\\ &> 2\ell\, e^{-\frac{\pi}{p}} - \ell\left(e^{-\frac{\pi}{p}} + 0.13 + \frac{2}{\sqrt{p}}\right) - \frac{\sqrt{p}}{2}\\ &= \ell\left(e^{-\frac{\pi}{p}} - 0.13 - \frac{2}{\sqrt{p}}\right) - \frac{\sqrt{p}}{2}. \end{aligned} \tag{2.17}$$

One checks that $\ell\left(e^{-\frac{\pi}{p}} - 0.13 - \frac{2}{\sqrt{p}}\right) - \frac{\sqrt{p}}{2} > 0$ for any $p \geqslant 29$. $\square$

Lemma 2.8. *$S_1(X(p,4\ell))$ is positive for $p \geqslant 53$.*

Proof. By (2.4) and (2.13),

$$\begin{aligned} S_1(X(p,4\ell)) =& S_1(X^-) - \theta(\chi_2, i) - \theta(\overline{\chi_2}, i)\\ >& 2\ell\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1\\ n\equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2}\right)\\ & - 2\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1\\ n\equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} + \frac{p}{\pi}\, e^{-\frac{1}{2}}\right) \end{aligned}$$

$$=(2\ell-2)\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod p}}^{\infty} n\,e^{-\frac{\pi}{p}n^2}-\sum_{\substack{n=1\\ n\equiv -1 \pmod p}}^{\infty} n\,e^{-\frac{\pi}{p}n^2}\right)-\frac{2p}{\pi}\,e^{-\frac{1}{2}}.$$

In view of (2.5), we have

$$S_1(X(p,4\ell))>(2\ell-2)\left(1-\frac{\pi}{p}-\frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}\right)-\frac{2p}{\pi}\,e^{-\frac{1}{2}}.$$

It suffices to show

$$\frac{p-5}{4}\left(1-\frac{\pi}{p}-\frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}\right)-\frac{p}{\pi}\,e^{-\frac{1}{2}}>0,$$

or equivalently,

$$\left(1-\frac{5}{p}\right)\left(1-\frac{\pi}{p}-\frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}\right)-\frac{4}{\pi\sqrt{e}}>0.$$

One easily checks that the above inequality holds for $p \geqslant 53$. □

Theorem 2.9. *$\theta(\chi,i)\neq 0$ for all non-quadratic, non-quartic characters χ with prime conductor $p=4\ell+1$, where ℓ is also a prime.*

Proof. This theorem follows from Proposition 2.2, Lemma 2.7, Lemma 2.8 and the fact that Cohen and Zagier proved $\theta(\chi,i)=0$ for only two characters of conductors 300 and 600 and their complex conjugates and for no other primitive characters of conductor ≤ 52100. □

Acknowledgments

The authors are deeply grateful to the referees for very helpful suggestions to improve the paper. The authors are supported by National Nature Science Foundation of China (Nos. 11971226, 11631009) and the Scientific Research Foundation of Guangxi Educational Committee (No. KY2016YB287).

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