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An improvement on the parity of Schur's partition function

Yiwen Lu, Tao Wei, Xuejun Guo *,[1]

Department of Mathematics, Nanjing University, Nanjing 210093, China

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ABSTRACT

We improve S.-C. Chen's result on the parity of Schur's partition function. Let $A(n)$ be the number of Schur's partitions of n, i.e., the number of partitions of n into distinct parts congruent to 1, 2 (mod 3). S.-C. Chen [3] shows $\frac{x}{(\log x)^{\frac{47}{48}}} \ll \sharp\{0 \leq n \leq x : A(2n+1) \text{ is odd}\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}$. In this paper, we improve Chen's result to $\frac{x}{(\log x)^{\frac{11}{12}}} \ll \sharp\{0 \leq n \leq x : A(2n+1) \text{ is odd}\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}$.

1. Introduction

Let $n \geqslant 1$ be an integer. A partition of n is any non-increasing sequence of natural numbers whose sum is n. We denote by $p(n)$ the number of partitions for $n \geqslant 1$, and for convenience, let $p(0) = 1$.

Obviously, it is impossible to obtain an exact expression for $p(n)$, and one seeks instead to figure out the asymptotic behavior of $p(n)$ as n increases. The first breakthrough is attributed to G. H. Hardy and Ramanujan [7]. Twenty years later, their results were refined by Hans Rademacher [10] into the now well-known Hardy-Ramanujan-Rademacher Asymptotic Formula,

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}} \sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right]\right),$$

with

$$A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} \omega_{h,k} e^{-2\pi i hn/k}.$$

* Corresponding author.
E-mail addresses: luyw@smail.nju.edu.cn (Y. Lu), weitao@smail.nju.edu.cn (T. Wei), guoxj@nju.edu.cn (X. Guo).

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The symbol $\omega_{h,k}$ admits a representation, viz.

$$\omega_{h,k} = \exp\left(\pi i \sum_{\mu=1}^{k-1} \frac{\mu}{k}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right)\right),$$

where $[x]$ means here, as usual, the greatest integer not exceeding x. Hardy-Ramanujan-Rademacher Asymptotic Formula can even be used to calculate the number of partitions for any specific integers $n \geq 1$. This result is so accurate that it was highly praised by Hardy. We briefly introduce Rademacher's result here. Rademacher constructed some explicit functions $T_q(n)$ satisfying $p(n) = \sum_{q=1}^{\infty} T_q(n)$ for all n and then obtained the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}},$$

by $T_1(n)$. Moreover, he got the exact error such that there are explicit constants A and B making

$$\left| p(n) - \sum_{q=1}^{A\sqrt{n}} T_q(n) \right| < \frac{B}{n^{1/4}}.$$

Even if the above work has been established, we have no evidence to believe that it possesses any attractive arithmetic properties. For example, there is no evidence that would lead us to believe that $p(n)$ should have a preference to be even rather than odd. Doing a computer calculation for the first 10,000 values, we get that there are 5004 odd values and 4996 even values, which means that the ratio is almost 1:1. Then replacing 2 with 3 we find that among the first 10,000 values, there are 3,313; 3,325; and 3,362 values that are congruent to 0, 1, and 2 modulo 3 respectively, in a ratio of roughly 1:1:1 [1]. O. Kolberg [8] has proved that $p(n)$ is infinitely often even and infinitely often odd while Parkin and Shanks [9] conjectured that

$$\sharp\{0 \leq n \leq x : p(n) \text{ is even (resp. odd)}\} \sim \frac{1}{2}x, \text{ as } x \to \infty.$$

It could be seen that the above conjecture is extremely difficult to prove but there are still many special partition functions that are very attractive. The celebrated partition theorem which Schur [11] proved in 1926 is:

Theorem 1.1 *([11]). Let $A(n)$ denote the number of partitions of n into distinct parts $\equiv 1,\ 2 \pmod 3$. Let $A_1(n)$ denote the number of partitions of n with minimal difference 3 between parts and such that no two consecutive multiples of 3 occur as parts. Then*

$$A(n) = A_1(n).$$

Then in 1971 Andrews [2] found the following companion to Schur's theorem by a computer search:

Theorem 1.2 *([2]). Let $A_2(n)$ denote the number of partitions of n in the form $n = e_1 + e_2 + \cdots + e_\nu$ such that $e_l - e_{l+1} \geq 3,\ 2$ or 5 if $e_l \equiv 1,\ 2$ or $3 \pmod 3$. Then*

$$A(n) = A_1(n) = A_2(n).$$

A result by S.-C. Chen [3] tells us $\frac{x}{(\log x)^{\frac{47}{48}}} \ll \sharp\{0 \leq n \leq x : A(2n+1) \text{ is odd}\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}$. In this paper, we construct a powerful theorem by Gauss' genus theory. Then by the constructed theorem, we finally cover and generalize Chen's result. The final result is as follows:

Theorem 1.3. *We have*

$$\frac{x}{(\log x)^{\frac{11}{12}}} \ll \sharp\{0 \le n \le x : A(2n+1) \text{ is odd}\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}.$$

2. Main results

We let $R(n, ax^2+bxy+cy^2)$ be the number of the representations of n by $ax^2+bxy+cy^2$ with $x,\ y \in \mathbb{Z}$. By SageMath, we find that the reduced primitive positive definite binary quadratic forms of discriminant -216 are

$$x^2+54y^2,\ 2x^2+27y^2,\ 5x^2 \pm 2xy+11y^2,\ 7x^2 \pm 6xy+9y^2.$$

In this paper, we consider square-free integers $m \equiv 11 \pmod{24}$. Note that m can not be represented by x^2+54y^2 or $7x^2 \pm 6xy+9y^2$ from the arithmetic of modulo 3. Therefore Dirichlet's theorem on binary quadratic forms [6, Theorem 1] shows that

$$R(m, 2x^2+27y^2)+2R(m, 5x^2+2xy+11y^2) = 2\sum_{d|m}\left(\frac{-216}{d}\right) = 2\sum_{d|m}\left(\frac{-6}{d}\right),$$

where $R(m, 5x^2+2xy+11y^2) = R(m, 5x^2-2xy+11y^2)$ follows from the fact that a solution (x_0, y_0) to $m = 5x^2+2xy+11y^2$ corresponds to a solution $(x_0, -y_0)$ to $m = 5x^2-2xy+11y^2$, and $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi–Kronecker symbol (for more details about this, see [3]).

Let $m = \prod_{i=1}^{t} p_i$ be the prime factorization of m. Then

$$R(m, 2x^2+27y^2)+2R(m, 5x^2+2xy+11y^2) = 2\sum_{d|m}\left(\frac{-6}{d}\right) = 2\prod_{i=1}^{t}\left(1+\left(\frac{-6}{p_i}\right)\right).$$

It follows immediately that each arbitrary prime factor p_i of m must satisfy $\left(\frac{-6}{p_i}\right) = 1$ if $R(m, 5x^2+2xy+11y^2) \ne 0$, hence

$$p_i \equiv 1, 5, 7, 11 \pmod{24}, \quad 1 \le i \le t, \tag{1}$$

and we get

$$R(m, 2x^2+27y^2)+2R(m, 5x^2+2xy+11y^2) = 2^{t+1}. \tag{2}$$

A similar discussion for the discriminant -24 (see also [3]) tells us that

$$R(n, 2x^2+3y^2)+R(n, x^2+6y^2) = 2\sum_{d|n}\left(\frac{-6}{d}\right), \quad n \in \mathbb{N}, \tag{3}$$

and by considering modulo 3, we have

$$R(m, 2x^2+3y^2) = 2^{t+1}. \tag{4}$$

We denote by $C(D)$ the class group (see [4, p. 45-46] for definitions) of discriminant D and let $h(D)$ denote the number of classes of primitive positive definite forms of discriminant D. By [12, Table 9.1], we

know $h(-24) = 2$ and $2x^2 + 3y^2$ is the generator in $C(-24)$.
For convenience, we write

$$f(x,y) = x^2 + 6y^2, \quad g(x,y) = 2x^2 + 3y^2.$$

Thus we get the Dirichlet compositions (see [4] for definitions) as follows:

$$\begin{aligned} f(x,y)f(z,\omega) &= f(xz - 6y\omega, x\omega + yz) \\ f(x,y)g(z,\omega) &= g(xz - 3y\omega, x\omega + 2yz) \\ g(x,y)f(z,\omega) &= g(xz - 3y\omega, 2x\omega + yz) \\ g(x,y)g(z,\omega) &= f(2xz - 3y\omega, x\omega + yz). \end{aligned} \tag{5}$$

In this paper, we use Gauss' genus theory to construct a powerful theorem. Our main results are as follows:

Theorem 2.1. *Let $m \equiv 11 \pmod{24}$ be a square-free integer. Then*

$$R(m, 5x^2 + 2xy + 11y^2) \equiv 2 \pmod 4,$$

if and only if m can be written in the following two forms:

(i) $m = p_1 \cdots p_{2t_1-1} \cdot \overline{p}_1 \cdots \overline{p}_{2t_2} \cdot q_1 \cdots q_{t_3} \cdot \overline{q}_1 \cdots \overline{q}_{2t_4}$;
(ii) $m = p_1 \cdots p_{2t_1} \cdot \overline{p}_1 \cdots \overline{p}_{2t_2-1} \cdot q_1 \cdots q_{t_3} \cdot \overline{q}_1 \cdots \overline{q}_{2t_4-1}$,

where $p_1, \cdots, p_{2t_1} \in \mathscr{S}_1$, $\overline{p}_1, \cdots, \overline{p}_{2t_2} \in \mathscr{S}_2$, $q_1, \cdots, q_{t_3} \in \mathscr{S}_3$, $\overline{q}_1, \cdots, \overline{q}_{2t_4} \in \mathscr{S}_4$, $t_1, t_2, t_3, t_4 \in \mathbb{N}$ (the set of natural numbers containing 0), and $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \mathscr{S}_4$ are the following four subsets of primes:

$$\begin{aligned} \mathscr{S}_1 &= \{p : p \equiv 11 \pmod{24}, R(p, 5x^2 + 2xy + 11y^2) > 0\} \\ \mathscr{S}_2 &= \{\overline{p} : \overline{p} \equiv 5 \pmod{24}, R(\overline{p}, 5x^2 + 2xy + 11y^2) > 0\} \\ \mathscr{S}_3 &= \{q : q \equiv 1 \pmod{24}, R(q, 7x^2 + 6xy + 9y^2) > 0\} \\ \mathscr{S}_4 &= \{\overline{q} : \overline{q} \equiv 7 \pmod{24}, R(\overline{q}, 7x^2 + 6xy + 9y^2) > 0\}. \end{aligned}$$

Then by Theorem 2.1, we cover and generalize the result of S.-C. Chen [3]. The contents are as follows:

Theorem 2.2. *We have*

$$\frac{x}{(\log x)^{\frac{11}{12}}} \ll \sharp\{0 \le n \le x : A(2n+1) \text{ is odd}\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}.$$

Since $\sharp\{1 \le n \le x : A(n) \text{ is odd}\} \ge \sharp\{0 \le n \le \frac{x-1}{2} : A(2n+1) \text{ is odd}\}$, we have the following corollary.

Corollary 2.3.

$$\sharp\{1 \le n \le x : A(n) \text{ is odd}\} \gg \frac{x}{(\log x)^{\frac{11}{12}}}.$$

3. Proofs

Before starting our proofs of the main theorems, there is a lemma which will be used in the proof of Theorem 2.1.

For the Dirichlet compositions in (5) and any square-free integers $a = f(x_0, y_0) \neq 1$ or $a = g(x_0, y_0) \neq 1$, $b = f(x_1, y_1) \neq 1$ or $g(x_1, y_1) \neq 1$ where $3 \nmid x_i$ and $x_i,\ y_i \in \mathbb{N},\ i = 0, 1$. Obviously we have the following cases:

$$\begin{aligned}
&f(x_0, y_0)f(x_1, y_1) = f(x_2, y_2),\ \textit{where}\ y_2 = x_0y_1 + y_0x_1;\\
&f(x_0, y_0)g(x_1, y_1) = g(x_2, y_2),\ \textit{where}\ y_2 = x_0y_1 + 2y_0x_1;\\
&g(x_0, y_0)f(x_1, y_1) = f(x_2, y_2),\ \textit{where}\ y_2 = 2x_0y_1 + y_0x_1;\\
&g(x_0, y_0)g(x_1, y_1) = f(x_2, y_2),\ \textit{where}\ y_2 = x_0y_1 + y_0x_1,
\end{aligned}$$

or

$$\begin{aligned}
&f(x_0, y_0)f(x_1, y_1) = f(x_2', y_2'),\ \textit{where}\ y_2' = x_0y_1 - y_0x_1;\\
&f(x_0, y_0)g(x_1, y_1) = g(x_2', y_2'),\ \textit{where}\ y_2' = x_0y_1 - 2y_0x_1;\\
&g(x_0, y_0)f(x_1, y_1) = f(x_2', y_2'),\ \textit{where}\ y_2' = 2x_0y_1 - y_0x_1;\\
&g(x_0, y_0)g(x_1, y_1) = f(x_2', y_2'),\ \textit{where}\ y_2' = x_0y_1 - y_0x_1.
\end{aligned}$$

The above y_2 and y_2' are two cases of the representations of ab in the sense of absolute value.

Lemma 3.1. *For any of the above cases, we have*

(i) $3 \mid y_0,\ 3 \mid y_1 \Leftrightarrow 3 \mid y_2,\ 3 \mid y_2'$;
(ii) $3 \nmid y_0,\ 3 \nmid y_1 \Leftrightarrow 3 \mid y_2,\ 3 \nmid y_2'$ *or* $3 \nmid y_2,\ 3 \mid y_2'$;
(iii) $3 \mid y_0,\ 3 \nmid y_1$ *or* $3 \nmid y_0,\ 3 \mid y_1 \Leftrightarrow 3 \nmid y_2,\ 3 \nmid y_2'$.

Proof of Lemma 3.1. We take the Dirichlet composition of $f(x, y)$ and $f(x, y)$ as an example first, i.e.,

$$y_2 = x_0y_1 + y_0x_1,\ y_2' = x_0y_1 - y_0x_1.$$

The necessity of (i) is obvious. Conversely, we get that $3 \mid 2x_0y_1 = y_2 + y_2'$ and $3 \mid 2y_0x_1 = |y_2 - y_2'|$. Thus $3 \mid y_i$. Note that $y_2y_2' = x_0^2y_1^2 - y_0^2x_1^2$. Combining the facts $x_i^2 \equiv 1 \pmod 3$, $y_i^2 \equiv 1 \pmod 3$ and (i), the necessity of (ii) is proved. Conversely, we have $3 \mid y_2y_2' = x_0^2y_1^2 - y_0^2x_1^2$, i.e., $y_1^2 - y_0^2 \equiv 0 \pmod 3$. By (i) again, the sufficiency is proved. The necessity of (iii) is proved immediately according to the fact that $3 \nmid y_1^2 - y_0^2$. Then combining (i) and (ii), the converse holds naturally.

Note that $x_0y_1 + 2y_0x_1 \equiv x_0y_1 - y_0x_1 \pmod 3$, $x_0y_1 - 2y_0x_1 \equiv x_0y_1 + y_0x_1 \pmod 3$, $2x_0y_1 + y_0x_1 \equiv -x_0y_1 + y_0x_1 \pmod 3$ and $2x_0y_1 - y_0x_1 \equiv -x_0y_1 - y_0x_1 \pmod 3$. Then similar discussions tell us that the same conclusion holds in the remaining cases. ■

Proof of Theorem 2.1. For the case that m is a prime, we have $R(m, 2x^2+27y^2)+2R(m, 5x^2+2xy+11y^2) = 4$ by (2). Obviously

$$R(m, 5x^2 + 2xy + 11y^2) \equiv 2 \pmod 4 \Leftrightarrow R(m, 5x^2 + 2xy + 11y^2) > 0, \tag{6}$$

i.e., $m \in \mathscr{S}_1$ and can be written as the first form in Theorem 2.1.

For the case that m is not a prime, we deduce from (2) that

$$\frac{1}{4}R(m, 2x^2 + 27y^2) \equiv \frac{1}{2}R(m, 5x^2 + 2xy + 11y^2) \pmod 2,$$

which states that

$$R(m, 5x^2 + 2xy + 11y^2) \equiv 2 \pmod 4 \Leftrightarrow R(m, 2x^2 + 27y^2) = 4k, \text{ where } k \text{ is an odd integer.} \tag{7}$$

Claim 1. *$R(m, 2x^2 + 27y^2) = 4k$, where k is an odd integer in this case if and only if m is of the form (i) or (ii) in Theorem 2.1.*

We find that a solution (a, b) to $m = 2x^2 + 27y^2$ must correspond a solution $(a, 3b)$ to $m = 2x^2 + 3y^2$. Hence

$$\sharp\{(a, b) : 2a^2 + 27b^2 = m, a \in \mathbb{N}, b \in \mathbb{N}\} = \sharp\{(a', b') : 2a'^2 + 3b'^2 = m, a' \in \mathbb{N}, b' \in \mathbb{N}, 3 \mid b'\}, \tag{8}$$

which means we can explore the solutions of $m = 2x^2 + 3y^2$ that the corresponding y value is divisible by 3 for proving the claim.

Combining the facts that the primes $p \equiv 5,\ 11 \pmod{24}$ can be represented by $2x^2 + 3y^2$, the primes $q \equiv 1,\ 7 \pmod{24}$ can be represented by $x^2 + 6y^2$ [12, Table 9.1] and formula (3), we get

$$\begin{aligned} &R(p_i, 2x^2 + 3y^2) = 4, \text{ if } p_i \equiv 5,\ 11 \pmod{24}, \\ &R(p_i, x^2 + 6y^2) = 4, \text{ if } p_i \equiv 1,\ 7 \pmod{24}. \end{aligned}$$

Therefore after adjusting the order of the prime factors of m, m can be written as

$$m = p_1 \cdots p_{i_0} \cdot p_{i_0+1} \cdots p_t = (2x_1^2 + 3y_1^2) \cdots (2x_{i_0}^2 + 3y_{i_0}^2) \cdot (x_{i_0+1}^2 + 6y_{i_0+1}^2) \cdots (x_t^2 + 6y_t^2), \tag{9}$$

where

$$p_i = \begin{cases} 2x_i^2 + 3y_i^2 & 1 \le i \le i_0, \\ x_i^2 + 6y_i^2 & i_0 < i \le t, \end{cases} \tag{10}$$

where $i_0 = \#\{p_i : p_i \equiv 5,\ 11 \pmod{24}\}$.

It is easy to see that each arbitrary solution (x_0, y_0) to equation $m = 2x^2 + 3y^2$ corresponds to three other solutions $(x_0, -y_0)$, $(-x_0, -y_0)$ and $(x_0, -y_0)$. We call such four solutions a class of solutions to the equation $m = 2x^2 + 3y^2$. Therefore the equation $m = 2x^2 + 3y^2$ has $2^t - 1$ classes of solutions by (4). Similarly, $2x^2 + 3y^2 = p_i \equiv 5, 11 \pmod{24}$ and $x^2 + 6y^2 = p_i \equiv 1, 7 \pmod{24}$ both have a class of solutions. Then by (5), we know that an arbitrary class of solutions to $f(x, y) = a$ and an arbitrary class of solutions to $f(x, y) = b$ where

$$a \mid m,\ b \mid m,\ ab \mid m,\ a \ne 1,\ b \ne 1,$$

correspond to two classes of solutions to $f(x, y) = ab$ as follows:

$$\begin{aligned} &[(\pm x_0)^2 + 6(\pm y_0)^2] \cdot [(\pm x_1)^2 + 6(\pm y_1)^2] = [\pm(x_0x_1 - 6y_0y_1)]^2 + 6[\pm(x_0y_1 + y_0x_1)]^2, \\ &[(\pm x_0)^2 + 6(\pm y_0)^2] \cdot [(\pm x_1)^2 + 6(\pm y_1)^2] = [\pm(x_0x_1 + 6y_0y_1)]^2 + 6[\pm(x_0y_1 - y_0x_1)]^2. \end{aligned}$$

Similarly, we have:

$$[(\pm x_0)^2 + 6(\pm y_0)^2] \cdot [2(\pm x_1)^2 + 3(\pm y_1)^2] = 2[\pm(x_0x_1 - 3y_0y_1)]^2 + 3[\pm(x_0y_1 + 2y_0x_1)]^2,$$
$$[(\pm x_0)^2 + 6(\pm y_0)^2] \cdot [2(\pm x_1)^2 + 3(\pm y_1)^2] = 2[\pm(x_0x_1 + 3y_0y_1)]^2 + 3[\pm(x_0y_1 - 2y_0x_1)]^2,$$

which corresponds to the Dirichlet composition of $f(x, y)$ and $g(x, y)$,

$$[2(\pm x_0)^2 + 3(\pm y_0)^2] \cdot [(\pm x_1)^2 + 6(\pm y_1)^2] = 2[\pm(x_0x_1 - 3y_0y_1)]^2 + 3[\pm(2x_0y_1 + y_0x_1)]^2,$$
$$[2(\pm x_0)^2 + 3(\pm y_0)^2] \cdot [(\pm x_1)^2 + 6(\pm y_1)^2] = 2[\pm(x_0x_1 + 3y_0y_1)]^2 + 3[\pm(2x_0y_1 - y_0x_1)]^2,$$

which corresponds to the Dirichlet composition of $g(x, y)$ and $f(x, y)$, and

$$[2(\pm x_0)^2 + 3(\pm y_0)^2][2(\pm x_1)^2 + 3(\pm y_1)^2] = [\pm(2x_0x_1 - 3y_0y_1)]^2 + 6[\pm(x_0y_1 + y_0x_1)]^2,$$
$$[2(\pm x_0)^2 + 3(\pm y_0)^2][2(\pm x_1)^2 + 3(\pm y_1)^2] = [\pm(2x_0x_1 + 3y_0y_1)]^2 + 6[\pm(x_0y_1 - y_0x_1)]^2,$$

which corresponds to the Dirichlet composition of $g(x, y)$ and $g(x, y)$.

We call the corresponding discussions above the composition of solutions in the process of Dirichlet compositions. The above discussions state that each arbitrary solution to the equation $m = 2x^2 + 3y^2$ can be obtained via the composition of solutions that in the process of Dirichlet compositions in formula (5) step by step. And furthermore, the 2^{t+1} solutions of $m = 2x^2 + 3y^2$ can be viewed as a stepwise composition of 4t pairs $(\pm x_i, \pm y_i)$ $(1 \leqslant i \leqslant t)$ corresponding to the t factors p_i of m to obtain.

Claim 2.

$$R(m, 2x^2 + 27y^2) = 4k, \text{ where } k \text{ is an odd integer},$$

if and only if for all y_i in formula (10), $3 \nmid y_i$.

Because of equation (8), we will prove Claim 2 by exploring the solutions to $m = 2x^2 + 3y^2$.

We assume that there are t_0 $(t_0 \in \mathbb{N})$ pairs (x_i, y_i) in formula (9) that $3 \mid y_i$ while the rest of $t - t_0$ pairs (x_i, y_i) satisfy $3 \nmid y_i$. Then we adjust the order of p_i in (9) to make the $4t_0$ pairs $(\pm x_i, \pm y_i)$ that $3 \mid y_i$ conduct the composition of solutions first. By Lemma 3.1 (i), we finally get 2^{t_0+1} pairs (x, y) $(x, y \in \mathbb{Z})$ that satisfy $y \equiv 0 \pmod 3$ and

$$2x^2 + 3y^2 = p_1 \cdots p_{t_0} \quad \text{or} \quad x^2 + 6y^2 = p_1 \cdots p_{t_0},$$

where $p_1, \cdots, p_{t_0}$ are the corresponding primes of $4t_0$ pairs $(\pm x_i, \pm y_i)$ we mentioned above.

Next we let the right side of above equation be multiplied one by one with the remaining $t - t_0$ primes $p_{t_0+1}, p_{t_0+2}, \cdots, p_t$ (that is, let the 2^{t_0+1} pairs we have obtained with the remaining $4(t-t_0)$ pairs $(\pm x_i, \pm y_i)$ conduct the composition of solutions).

We know that each time the composition of solutions is performed, we can obtain twice as many pairs (x, y) as before. Thus, when the above process is performed n $(0 \leqslant n \leqslant t - t_0)$ times in sequence, we get 2^{t_0+1+n} pairs (x, y). Now denote the set of 2^{t_0+1+n} pairs (x, y) as S_n, then we define

$$u(n) = \sum_{(x,y)\in S_n,\ 3|y} 1, \quad v(n) = \sum_{(x,y)\in S_n,\ 3\nmid y} 1.$$

Obviously,

$$u(0) = 2^{t_0+1}, \quad v(0) = 0.$$

Since the $4(t-t_0)$ pairs $(\pm x_i, \pm y_i)$ corresponding to the remaining $t-t_0$ primes $p_{t_0+1}, p_{t_0+2}, \cdots, p_t$ all satisfy $y_i \not\equiv 0 \pmod 3$, by Lemma 3.1 (ii) and (iii) one can obtain that

$$u(1) = 0, \quad v(1) = 2^{t_0+2},$$

and moreover,

$$u(n) = v(n-1), \quad v(n) = 2u(n-1) + v(n-1), \quad 1 \le n \le t - t_0.$$

Then we immediately get that

$$\begin{aligned} u(t-t_0) &= v(t-t_0-1) = 2u(t-t_0-2) + v(t-t_0-2) \\ &= 2(v(t-t_0-3) + u(t-t_0-3)) + v(t-t_0-3) \\ &= \cdots \\ &= 2(C_1 u(1) + C_2 v(1)) + v(1) \\ &= C_2 2^{t_0+3} + 2^{t_0+2}, \end{aligned} \tag{11}$$

where $C_1,\ C_2 \in \mathbb{N}^+$ are two constants.

Through (11) we see when there are t_0 y_i in formula (10) that $3 \mid y_i$, $C_2 2^{t_0+3} + 2^{t_0+2}$ solutions (x, y) out of 2^{t+1} solutions to $m = 2x^2 + 3y^2$ satisfy $3 \mid y$. And in this case it is easy to see that $R(m, 2x^2 + 27y^2) = C_2 2^{t_0+3} + 2^{t_0+2}$ by (8). Hence $t_0 = 0$, i.e., $3 \nmid y_i$ for all y_i in formula (10) if and only if $R(m, 2x^2 + 27y^2) = 4k$ with k an odd integer.

Now Claim 2 is proved and we will prove Claim 1 by Claim 2 next.

One can find that for each arbitrary prime factor p_i of m, a solution (a_i, b_i) to $p_i = 2x^2 + 27y^2$ must correspond to a solution $(a_i, 3b_i)$ to $p_i = 2x^2 + 3y^2$ and a solution (a_i', b_i') to $p_i = 2x^2 + 3y^2$ that $3 \mid b_i'$ must correspond to a solution $(a_i', b_i'/3)$ to $p_i = 2x^2 + 27y^2$. And the case is the same for equations $p_i = x^2 + 6y^2$ and $p_i = x^2 + 54y^2$. Therefore combining the fact that $2x^2 + 3y^2 = p_i \equiv 5, 11 \pmod{24}$ and $x^2 + 6y^2 = p_i \equiv 1, 7 \pmod{24}$ both have a class of solutions, we know the condition that $3 \nmid y_i$ for all y_i in formula (10) is equivalent to

$$\begin{aligned} &R(p_i, 2x^2 + 27y^2) = 0, \text{ if } p_i \equiv 5,\ 11 \pmod{24}, \\ &R(p_i, x^2 + 54y^2) = 0, \text{ if } p_i \equiv 1,\ 7 \pmod{24}, \end{aligned} \tag{12}$$

for all the prime factors p_i of m.

Next we show that formula (12) holds for all the prime factors p_i of m if and only if m is of the form (i) or (ii) in Theorem 2.1.

First we assume all the prime factors of m satisfy formula (12). Note that the primes $p \equiv 5,\ 11 \pmod{24}$ can not be represented by $x^2 + 54y^2$ or $7x^2 \pm 6xy + 9y^2$ by considering modulo 3. Thus if a prime factor $p_i \equiv 5,\ 11 \pmod{24}$ of m satisfies $R(p_i, 2x^2 + 27y^2) = 0$, one can get the conclusion that

$$R(p_i, 5x^2 + 2xy + 11y^2) = 2,$$

by Dirichlet's theorem on binary quadratic forms [6, Theorem 1]. Note that the primes $q \equiv 1,\ 7 \pmod{24}$ can not be represented by $2x^2 + 27y^2$ and $5x^2 \pm 2xy + 11y^2$ by considering modulo 3, similarly we have

$$R(p_i, 7x^2 + 6xy + 9y^2) = 2,$$

for each arbitrary prime factor $p_i \equiv 1,\ 7 \pmod{24}$ of m. Thus

$$\forall p_i \mid m,\ p_i \in \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3 \cup \mathscr{S}_4.$$

Moreover, the fact $m \equiv 11 \pmod{24}$ ensures that in this case m can only be of the two forms in Theorem 2.1.

Conversely, let m be of the form (i) or (ii) in Theorem 2.1. Then once again using Dirichlet's theorem on binary quadratic forms [6, Theorem 1] and our discussions above, for the prime factors p_i of m, we get

$$R(p_i, 5x^2 + 2xy + 11y^2) = 2,\ \text{if } p_i \equiv 5,\ 11 \pmod{24},$$
$$R(p_i, 7x^2 + 6xy + 9y^2) = 2,\ \text{if } p_i \equiv 1,\ 7 \pmod{24},$$

and furthermore, formula (12) holds for all the prime factors of m.

Finally, based on our discussions above and Claim 2, Claim 1 has been proved.

If $R(m, 5x^2+2xy+11y^2) \equiv 2 \pmod 4$, we have (1) and equation (2). Hence by (6), (7) and Claim 1, m must be of the form (i) or (ii) in Theorem 2.1. Conversely, equation (2) also holds. Putting (6), (7) and Claim 1 together, we obtain $R(m, 5x^2 + 2xy + 11y^2) \equiv 2 \pmod 4$. This completes the proof of Theorem 2.1. ■

Proof of Theorem 2.2. Define $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_3$ and $\mathscr{S}_4$ as the subsets of primes in Theorem 2.1, respectively. We let $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3 \cup \mathscr{S}_4$ and

$$\mathscr{C} = \{x : x \in \mathbb{N},\ x \text{ is square-free },\ p \mid x \Rightarrow p \in \mathscr{S}\}.$$

For simplicity, we define several functions next. If n is a square-free integer, then we let

$$\mu_i(n) := (-1)^{\sum_{p|n, p\in\mathscr{S}_i} 1},$$

where $1 \leqslant i \leqslant 4$. By [3, (8)], we have

$$A(2n+1) \equiv \frac{1}{2} R(24n + 11, 5x^2 + 2xy + 11y^2) \pmod 2. \tag{13}$$

Combining (13) with Theorem 2.1, we get

$$A\left(\frac{m+1}{12}\right) \equiv 1 \pmod 2,$$

for any square-free integers m of the form (i) or (ii) in Theorem 2.1. Thus

$$\begin{aligned} \sum_{\substack{0\le n\le x \\ A(2n+1) \text{ odd}}} 1 &\ge \sum_{\substack{m\le x \\ m \text{ is the form (i)}}} 1 + \sum_{\substack{m\le x \\ m \text{ is the form (ii)}}} 1 \\ &= \sum_{\substack{m\in\mathscr{C},\, m\le x \\ \mu_1(m)=-1, \mu_2(m)=1, \mu_4(m)=1}} 1 + \sum_{\substack{m\in\mathscr{C},\, m\le x \\ \mu_1(m)=1, \mu_2(m)=-1, \mu_4(m)=-1}} 1. \end{aligned} \tag{14}$$

Since the number of classes of discriminant -216 is 6, the Chebotarev density theorem [4, Theorem 9.12] shows that the Dirichlet density of the set of primes represented by $5x^2 + 2xy + 11y^2$ is $\frac{1}{6}$. Applying the orthogonality of Dirichlet character modulo 24, we see that the Dirichlet density of $\mathscr{S}_1$ is $\frac{1}{6} \cdot \frac{1}{\phi(24)} = \frac{1}{48}$, where $\phi(\cdot)$ is Euler's totient function. Similarly, the Dirichlet density of $\mathscr{S}_2$, $\mathscr{S}_3$ and $\mathscr{S}_4$ are all $\frac{1}{6} \cdot \frac{1}{\phi(24)} = \frac{1}{48}$. Thus the Dirichlet density of $\mathscr{S}$ is $4 \cdot \frac{1}{48} = \frac{1}{12}$.

We define

$$h(x) = \sum_{m \in \mathscr{C},\, m \le x} 1,$$

and

$$h_1(x) = \sum_{\substack{m \in \mathscr{C},\, m \le x \\ \mu_1(m)=1}} 1, \qquad h_{-1}(x) = \sum_{\substack{m \in \mathscr{C},\, m \le x \\ \mu_1(m)=-1}} 1,$$

where $h(x) = h_1(x) + h_{-1}(x)$ obviously. A classical result of Wirsing [13] on multiplicative functions (see also [5, Proposition 4]) tells us that

$$h(x) = C_h \frac{x}{(\log x)^{\frac{11}{12}}} + o\left(\frac{x}{(\log x)^{\frac{11}{12}}}\right), \tag{15}$$

where constant $C_h > 0$, as $x \to \infty$.
Note that $h_1(x) \ge 0$, $h_{-1}(x) \ge 0$. We deduce from (15) that

$$h_1(x) = C_{h_1} \frac{x}{(\log x)^{\frac{11}{12}}} + o\left(\frac{x}{(\log x)^{\frac{11}{12}}}\right), \qquad h_{-1}(x) = C_{h_{-1}} \frac{x}{(\log x)^{\frac{11}{12}}} + o\left(\frac{x}{(\log x)^{\frac{11}{12}}}\right),$$

where there are three possible cases of C_{h_1} and $C_{h_{-1}}$ as follows:

(1) $C_{h_1} > 0$, $C_{h_{-1}} > 0$ and $C_{h_1} + C_{h_{-1}} = C_h$;
(2) $C_{h_1} = C_h$, $C_{h_{-1}} = 0$;
(3) $C_{h_1} = 0$, $C_{h_{-1}} = C_h$.

We find that

$$\begin{aligned} h_1(x) &= \sum_{\substack{m \in \mathscr{C},\, m \le x \\ 11 \mid m,\, \mu_1(m)=1}} 1 + \sum_{\substack{m \in \mathscr{C},\, m \le x \\ 11 \nmid m,\, \mu_1(m)=1}} 1 \\ &= \sum_{\substack{m \in \mathscr{C},\, m \le \frac{x}{11} \\ 11 \nmid m,\, \mu_1(m)=-1}} 1 + \sum_{\substack{m \in \mathscr{C},\, m \le 11x \\ 11 \mid m,\, \mu_1(m)=-1}} 1 \\ &\le \sum_{\substack{m \in \mathscr{C},\, m \le \frac{x}{11} \\ \mu_1(m)=-1}} 1 + \sum_{\substack{m \in \mathscr{C},\, m \le 11x \\ \mu_1(m)=-1}} 1 \\ &= h_{-1}\left(\frac{x}{11}\right) + h_{-1}(11x). \end{aligned} \tag{16}$$

Similarly, we have

$$h_{-1}(x) \le h_1\left(\frac{x}{11}\right) + h_1(11x). \tag{17}$$

Therefore case (1) holds because case (2) and case (3) are absurd to the inequality (16) and inequality (17), respectively, i.e.,

$$\sum_{\substack{m \in \mathscr{C},\, m \le x \\ \mu_1(m)=-1}} 1 = h_{-1}(x) = C_{h_{-1}} \frac{x}{(\log x)^{\frac{11}{12}}} + o\left(\frac{x}{(\log x)^{\frac{11}{12}}}\right),$$

where constant $C_{h_{-1}} > 0$, as $x \to \infty$. Then we write $h_1(x)$ above still as $h(x)$. In a similar way, we construct the new functions $h_1(x)$ and $h_{-1}(x)$ as follows:

$$h_1(x) = \sum_{\substack{m\in\mathscr{C},\, m\le x\\ \mu_1(m)=-1,\, \mu_2(m)=1}} 1, \qquad h_{-1}(x) = \sum_{\substack{m\in\mathscr{C},\, m\le x\\ \mu_1(m)=-1,\, \mu_2(m)=-1}} 1,$$

where $h_1(x) + h_{-1}(x) = h(x)$ and

$$h_1(x) \le h_{-1}(\frac{x}{5}) + h_{-1}(5x),$$
$$h_{-1}(x) \le h_1(\frac{x}{5}) + h_1(5x).$$

Similarly, we get

$$\sum_{\substack{m\in\mathscr{C},\, m\le x\\ \mu_1(m)=-1,\, \mu_2(m)=1}} 1 = h_1(x) = C_0\frac{x}{(\log x)^{\frac{11}{12}}} + o\left(\frac{x}{(\log x)^{\frac{11}{12}}}\right),$$

where constant $C_0 > 0$, as $x \to \infty$. Continuously through similar discussions, we can get

$$\sum_{\substack{m\in\mathscr{C},\, m\le x\\ \mu_1(m)=-1,\mu_2(m)=1,\mu_4(m)=1}} 1 = C_1\frac{x}{(\log x)^{\frac{11}{12}}} + o\left(\frac{x}{(\log x)^{\frac{11}{12}}}\right),$$

and

$$\sum_{\substack{m\in\mathscr{C},\, m\le x\\ \mu_1(m)=1,\mu_2(m)=-1,\mu_4(m)=-1}} 1 = C_2\frac{x}{(\log x)^{\frac{11}{12}}} + o\left(\frac{x}{(\log x)^{\frac{11}{12}}}\right),$$

where constants $C_1 > 0$, $C_2 > 0$, as $x \to \infty$. Then by (14) we finally get

$$\sum_{\substack{0\le n\le x\\ A(2n+1)\text{ odd}}} 1 \gg \frac{x}{(\log x)^{\frac{11}{12}}}.$$

Combining the upper bound [3, Theorem 1.1] with the above inequality, we complete the proof of Theorem 2.2. ■

Remark 3.2. Obviously, Theorem 2.1 can also be used to discuss the case that m is not a square-free integer, i.e., the primes that appear in form (i) and (ii) in Theorem 2.1 can be the same. This allows one to obtain more integers $n \in \mathbb{N}$ that satisfy $A(2n+1)$ is odd. Therefore one may be able to further raise the lower bound of $\sharp\{0 \le n \le x : A(2n+1) \text{ is odd }\}$ and obtain a more accurate asymptotic.

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