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# The extended Bloch groups of biquadratic and dihedral number fields [☆]

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ABSTRACT

In this paper, we study the Galois action on the extended Bloch groups of biquadratic and dihedral number fields. We prove that if $F$ is a biquadratic number field, then the index $Q_2(F)$ in Browkin and Gangl's formulas on the Brauer–Kuroda relation can only be 1 or 2. This is exactly what Browkin and Gangl predicted in their paper. Moreover we give the explicit criteria for $Q_2(F) = 1$ or 2 in terms of the Tate kernels. We also prove that $Q_2(F) = 1$ or $p$ for any dihedral extension $F/\mathbb{Q}$ whose Galois group is the dihedral group of order $2p$, where $p$ is an odd prime.

## 1. Introduction

There are several different definitions of the Bloch groups in the literature. In this paper, we will use Suslin's definition of the Bloch group in [17]. Although Browkin and Gangl use in [5] a different definition of the Bloch group from Suslin's, these two definitions differ only in the torsion elements. Note that the Dilogarithm function is trivial on torsion elements. So even if we change the Bloch groups in Browkin and Gangl's paper to Suslin's Bloch groups, their results on the regulators and the Brauer–Kuroda relations still hold. One can see Section 2 of this paper for details.

Let $E/F$ be a finite Galois extension of fields with the Galois group $G$. In 2004, Neumann introduced in [10] the extended Bloch groups. In 2013, Zickert defined in [20] the extended Bloch group $\widehat{\mathcal{B}}$ for "free" fields which include number fields. He proved that there is a natural isomorphism

$$\widehat{\mathcal{B}}(E) \simeq K_3^{\text{ind}} E,$$

where $K_3^{\text{ind}} E$ is the indecomposable part of $K_3 E$. Note that this isomorphism respects the Galois action.

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Levine proved that the indecomposable $K_3$ satisfies Galois descent (Theorem 18.4 of [6])

$$K_3^{\text{ind}}F \simeq (K_3^{\text{ind}}E)^G,$$

where $E/F$ is a finite Galois extension of fields with the Galois group $G$. One can see [8] and [9] for details. Hence the extended Bloch groups also satisfy Galois descent.

We prove that if $F$ is a biquadratic number field, then the index $Q_2(F)$ in Browkin and Gangl's formulas (Theorem 1 of [5]) on the Brauer–Kuroda relation for $K_2$ can only be 1 or 2 by considering the norm index of the extended Bloch groups for biquadratic number fields. This is exactly what Browkin and Gangl predicted in their paper [5].

We give an explicit method to compute the exact value of $Q_2(F)$. In particular, we prove that if $F = F_1F_2$, where $F_1$ and $F_2$ are imaginary quadratic number fields, then $Q_2(F) = 1$ or 2. Our method is based on the study of the Tate kernels of $F$, $F_1$, $F_2$. A method of determining explicit Tate kernels of imaginary quadratic number fields has been developed in [14], together with [15] and [16]. The method can also be used to determine the Tate kernel of a number field containing an imaginary quadratic field. We give several examples to show the strength of this method. In Section 3 of this paper, we will show how to determine the Tate kernels of certain biquadratic number fields by applying the results on the Tate kernels of two imaginary quadratic number fields.

We also prove that $Q_2(F) = 1$ or $p$ for any dihedral extension $F/\mathbb{Q}$ whose Galois group is the dihedral group of order $2p$, where $p$ is an odd prime. In [21], Zhou proved that $Q_2(F)|4$ for bi-quadratic number fields and proved that $Q_2(F)|3^3$ for any dihedral extension $F/\mathbb{Q}$ whose Galois group is the dihedral group of order 6.

## 2. The extended Bloch group and dilogarithm

Let $E$ be a number field, $E^\times$ the multiplicative group of $E$. Let $\mathbb{Z}[E^\times\backslash\{1\}]$ be the free abelian group generated by

$$[a],\ a \in E^\times,\ a \neq 1.$$

The Suslin (or Dupont–Sah) scissors congruence group $\mathcal{PS}(E)$ is the group $\mathbb{Z}[E^\times\backslash\{1\}]$ modulo the subgroup $\mathcal{F}(E)$ generated by

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}],\ x \neq y \in \mathbb{Z}[E^\times\backslash\{1\}].$$

Let

$$S(E) := (E^\times \otimes E^\times)/(x \otimes y + y \otimes x)$$

with generators $a \circ b$. Then Suslin's Bloch group is defined as

$$\mathcal{BS}(E) := \ker(\lambda:\ \mathcal{PS}(E) \longrightarrow S(E)),\ \lambda([x]) = x \circ (1-x).$$

Suslin proved in [17] that the order of $[x] + [x^{-1}]$ is at most 2, and the elements $[x] + [1-x]$ are independent of $x$ and have order dividing 6.

For the discussion on relations between Suslin's definition of the Bloch group and Dupont–Sah's definition of the Bloch group, one can see [11], [12] and [13] for details.

In [5], Browkin and Gangl use Zagier's definition of the Bloch group. In their paper, the "pre-Bloch" group $\mathcal{PZ}(E)$ is defined as the group $\mathbb{Z}[E^\times\backslash\{1\}]$ modulo the subgroup $\mathcal{F}(E)$ generated by

$$[x]+[y]+[\frac{1-x}{1-xy}]+[1-xy]+[\frac{1-y}{1-xy}],\ x,\ y,\ xy\in\mathbb{Z}[E^\times\backslash\{1\}]$$

and the elements

$$[x]+[x^{-1}]\ \text{ and }\ [x]+[1-x],\ x\in E^\times\backslash\{1\}.$$

Let

$$\Lambda(E):=(E^\times\otimes E^\times)/(x\otimes x,\ y\otimes -1),$$

with generators $x\wedge y$. Then Zagier's Bloch group is defined as

$$\mathcal{BZ}(E):=\ker(\partial:\ \mathcal{PZ}(E)\longrightarrow\Lambda(E)),\ \partial([z])=z\wedge(1-z).$$

Note that there is a natural surjective map

$$\begin{aligned}\mathcal{PS}(E)&\longrightarrow\mathcal{PZ}(E),\\ [x]&\longrightarrow\qquad [x],\end{aligned}$$

whose kernel is annihilated by 6 by Lemma 5.1 of [17]. So there is a natural surjective map from $\mathcal{BS}(E)$ to $\mathcal{BZ}(E)$, whose kernel is also annihilated by 6. This map induces an isomorphism from $\mathcal{BS}(E)/\mathrm{tor}$ to $\mathcal{BZ}(E)/\mathrm{tor}$, where "tor" means the torsion part.

There is an exact sequence

$$0\longrightarrow\mathcal{BS}(E)\longrightarrow\mathcal{PS}(E)\longrightarrow S(E)\longrightarrow K_2E\longrightarrow 0,$$

where $x\circ y\in S(E)$ maps to $\{x,\ y\}\in K_2E$.

For abbreviation, we will use $\mathcal{B}(E)$ for $\mathcal{BS}(E)$. In [5], Browkin and Gangl use the Bloch groups in the Dilogarithm function. Note that the Dilogarithm function is trivial on the torsion elements. So if we change their definition to Suslin's definition of the Bloch groups, then their results on the regulators still hold. We will not use the exact definition of the extended Bloch group $\widehat{\mathcal{B}}(F)$. One can see the details of the definition in [20].

**Theorem 2.1** *(Zickert, [20]). For every number field $F$, there is a natural isomorphism*

$$\widehat{\lambda}:\ K_3^{\mathrm{ind}}F\simeq\widehat{\mathcal{B}}(F)$$

*respecting the Galois actions.*

Let $\mu_F$ be the group of roots of unity of $F$, and $\widetilde{\mu_F}$ the unique non-trivial $\mathbb{Z}/2\mathbb{Z}$ extension of $\mu_F$. Suslin proved that there is a short exact sequence

$$0\longrightarrow\widetilde{\mu_F}\longrightarrow K_3^{\mathrm{ind}}F\longrightarrow\mathcal{B}(F)\longrightarrow 0.$$

Theorem 2.1 implies the following short exact sequence

$$0\longrightarrow\widetilde{\mu_F}\longrightarrow\widehat{\mathcal{B}}(F)\longrightarrow\mathcal{B}(F)\longrightarrow 0. \tag{2.1}$$

Recall that the standard Bloch–Wigner function is defined as

$$D(z) = -\mathrm{Im}\int_0^z \log(1-t)\frac{dt}{t} + \arg(1-z)\log|z|,$$

and the normalized Bloch–Wigner function is defined as

$$\widetilde{D}(z) = \frac{1}{\pi}D(z).$$

In this paper, when we talk about the Bloch–Wigner function, we always mean the normalized one. Let $\overline{z}$ be the complex conjugate of $z$. Then the Bloch–Wigner function is a real analytic function satisfying the identity

$$\widetilde{D}(\overline{z}) = -\widetilde{D}(z).$$

For a number field $F$, the Bloch–Wigner function can be extended to a linear map

$$\begin{aligned}\widetilde{D}:\ \mathcal{B}(F) &\longrightarrow \qquad\qquad \mathbb{R},\\ [a_1]+\cdots+[a_n] &\longmapsto \widetilde{D}(a_1)+\cdots+\widetilde{D}(a_n).\end{aligned}$$

Let $\sigma_1,\ \ldots,\ \sigma_{r_2}$ be the complex places of $F$ and $\widetilde{D}_i := \widetilde{D}\circ\sigma_i$. Let $\mathbb{D} = (\widetilde{D}_1,\ \ldots,\ \widetilde{D}_{r_2})$. Then we get a dilogarithm

$$\mathbb{D}:\ \mathcal{B}(F)\longrightarrow \mathbb{R}^{r_2}.$$

The image of $\mathbb{D}$ is a lattice of rank $r_2$ in $\mathbb{R}^{r_2}$. Let $\widetilde{R}_2(F)$ be the covolume of this lattice. Let $b_1,\ \ldots,\ b_{r_2}\in\mathcal{B}(F)$ such that

$$\mathbb{D}(b_1),\ \ldots,\ \mathbb{D}(b_{r_2})$$

is a basis of $\mathbb{D}(\mathcal{B}(F))$. Then

$$\widetilde{R}_2(F) = |\det(\widetilde{D}(\sigma_i(b_j)))|.$$

By (2.1), the homomorphism $\mathbb{D}$ can be pulled back to $\widehat{\mathcal{B}}(F)$, i.e.,

$$\mathbb{D}:\ \widehat{\mathcal{B}}(F)\longrightarrow\mathbb{R}^{r_2}.$$

By Theorem 2.1, there is a natural Galois action on $\widehat{\mathcal{B}}(F)$.

## 3. Galois descent of the extended Bloch groups of biquadratic number fields

Let $F_1,\ F_2$ be two imaginary quadratic number fields and $F = F_1F_2$ be their composite. Suppose that $\mathrm{Gal}(F/\mathbb{Q}) = \{1,\ \sigma_1,\ \sigma_2,\ \sigma_1\sigma_2\}$, where $\sigma_1$ is the identity on $F_2$ and $\sigma_2$ is the identity on $F_1$. Since $\sigma_i(F_i) = F_i$ $(i = 1,\ 2)$, the restriction $\overline{\sigma_i}\in\mathrm{Gal}(F_i/\mathbb{Q})$. By abuse of notations, we will also use the symbol $\sigma_i$ to denote $\overline{\sigma_i}$, i.e., $\mathrm{Gal}(F_i/\mathbb{Q}) = \{1,\ \sigma_i\}$ for $i = 1,\ 2$.

Since the indecomposable $K_3$ satisfies Galois descent, so does the extended Bloch group $\widehat{\mathcal{B}}$, i.e.,

$$\widehat{\mathcal{B}}(F) = \widehat{\mathcal{B}}(E)^G$$

for any Galois extension $F \subset E$ with $\mathrm{Gal}(E/F) = G$. Hence if $F$ is a subfield of $E$, then $\widehat{\mathcal{B}}(F)$ is a subgroup of $\widehat{\mathcal{B}}(E)$. By Theorem 18.1 of [6], for any prime $\ell$, the subgroup of the $\ell$-torsion elements of $K_3^{\mathrm{ind}}F$ is cyclic. In particular, the subgroup of the 2-torsion elements of $\widehat{\mathcal{B}}(F_i)$ is cyclic for $i = 1,\ 2$.

For a number field $E$ with $r_2$ pairs of complex embeddings, the rank of $\widehat{\mathcal{B}}(E)$ is $r_2$. Hence

$$\widehat{\mathcal{B}}(E) \simeq \text{cyclic group of even order} \oplus \mathbb{Z}^{r_2}. \tag{3.1}$$

So

$$\begin{aligned} \widehat{\mathcal{B}}(F_1)/2\widehat{\mathcal{B}}(F_1) &\simeq (\mathbb{Z}/2\mathbb{Z})^2, \\ \widehat{\mathcal{B}}(F_2)/2\widehat{\mathcal{B}}(F_2) &\simeq (\mathbb{Z}/2\mathbb{Z})^2, \\ \widehat{\mathcal{B}}(F)/2\widehat{\mathcal{B}}(F) &\simeq (\mathbb{Z}/2\mathbb{Z})^3. \end{aligned} \tag{3.2}$$

We have

$$\mathbb{D} : \widehat{\mathcal{B}}(F) \longrightarrow \mathbb{R}^2,\ \mathbb{D}(b) = (\widetilde{D}(b),\ \widetilde{D}(\sigma_1(b)),$$

because $\sigma_1\sigma_2$ is the complex conjugation in $F$.

By Galois descent:

$$\widehat{\mathcal{B}}(F)^{\langle\sigma_2\rangle} = \widehat{\mathcal{B}}(F_1),\ \widehat{\mathcal{B}}(F)^{\langle\sigma_1\rangle} = \widehat{\mathcal{B}}(F_2).$$

For $j = 1,\ 2$, we fix $b_j \in \widehat{\mathcal{B}}(F_j)$ such that $\mathbb{D}(b_j)$ generates the lattice $\mathbb{D}(\widehat{\mathcal{B}}(F_j))$. Similarly, let $e_1,\ e_2 \in \widehat{\mathcal{B}}(F)$ satisfy: $\mathbb{D}(e_1),\ \mathbb{D}(e_2)$ generate the lattice $\mathbb{D}(\widehat{\mathcal{B}}(F))$.

Since for $j = 1,\ 2$, $\mathbb{D}(\widehat{\mathcal{B}}(F_j))$ are sublattices of $\mathbb{D}(\widehat{\mathcal{B}}(F))$, and they span a sublattice of rank 2, we get

$$\begin{aligned} \mathbb{D}(b_1) &= a\mathbb{D}(e_1) + b\mathbb{D}(e_2), \\ \mathbb{D}(b_2) &= c\mathbb{D}(e_1) + d\mathbb{D}(e_2), \end{aligned} \tag{3.3}$$

where $a,\ b,\ c,\ d \in \mathbb{Z}$. Hence by linear algebra,

$$(\mathbb{D}(\widehat{\mathcal{B}}(F))) : (\mathbb{D}(\widehat{\mathcal{B}}(F_1)) + \mathbb{D}(\widehat{\mathcal{B}}(F_2))) = |ad - bc|.$$

Obviously,

$$e_j + \sigma_k(e_j) \in \widehat{\mathcal{B}}(F)^{\langle\sigma_k\rangle} \quad \text{for} \quad j,\ k \in \{1,\ 2\}.$$

Hence

$$\begin{aligned} \mathbb{D}(e_j) + \mathbb{D}(\sigma_2(e_j)) &= \alpha_j\mathbb{D}(b_1) \quad \text{for some} \quad \alpha_j \in \mathbb{Z},\ j = 1,\ 2; \\ \mathbb{D}(e_j) + \mathbb{D}(\sigma_1(e_j)) &= \beta_j\mathbb{D}(b_2) \quad \text{for some} \quad \beta_j \in \mathbb{Z},\ j = 1,\ 2. \end{aligned} \tag{3.4}$$

Here $\sigma_2(e_j) = \sigma_1\sigma_2 \cdot \sigma_1(e_j) = \overline{\sigma_1(e_j)}$. Hence $\mathbb{D}(\sigma_2(e_j)) = -\mathbb{D}(\sigma_1(e_j))$. Adding two equalities of (3.4), we get

$$2\mathbb{D}(e_j) = \alpha_j\mathbb{D}(b_1) + \beta_j(\mathbb{D}(b_2)),\ j = 1,\ 2. \tag{3.5}$$

Since $\mathbb{D}(b_1)$, $\mathbb{D}(b_2)$ and $\mathbb{D}(e_1)$, $\mathbb{D}(e_2)$ are linear bases of $\mathbb{R}^2$, (3.3) and (3.5) imply that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_1/2 & \beta_1/2 \\ \alpha_2/2 & \beta_2/2 \end{pmatrix}^{-1},$$

hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Consequently

$$ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4.$$

Hence we just proved the following theorem.

**Theorem 3.1.** *Let the notations be as above. Let* $\overline{\mathcal{B}}(F)$, $\overline{\mathcal{B}}(F_1)$, $\overline{\mathcal{B}}(F_2)$ *denote the torsion free quotients* $\widehat{\mathcal{B}}(F)/\widehat{\mathcal{B}}(F)_{\mathrm{tor}}$, $\widehat{\mathcal{B}}(F_1)/\widehat{\mathcal{B}}(F_1)_{\mathrm{tor}}$, $\widehat{\mathcal{B}}(F_2)/\widehat{\mathcal{B}}(F_2)_{\mathrm{tor}}$ *respectively. And let* $\overline{b_1}$, $\overline{b_2}$, $\overline{e_1}$, $\overline{e_2}$ *be the corresponding images. Then* $\mathbb{Z}\overline{b_1} + \mathbb{Z}\overline{b_2}$ *is a sublattice of* $\mathbb{Z}\overline{e_1} + \mathbb{Z}\overline{e_2}$ *and the cardinality of*

$$(\mathbb{Z}\overline{e_1} + \mathbb{Z}\overline{e_2})/(\mathbb{Z}\overline{b_1} + \mathbb{Z}\overline{b_2})$$

*divides* 4.

Define

$$Q_2(F) = |(\mathbb{Z}\overline{e_1} + \mathbb{Z}\overline{e_2})/(\mathbb{Z}\overline{b_1} + \mathbb{Z}\overline{b_2})| = |ad - bc|.$$

Note that in [21], Zhou proved a result which is equivalent to the above Theorem in the language of motivic cohomology.

Based on numerical computation, Browkin and Gangl conjectured in [5] that the absolute value of the determinant $ad - bc$ can only be 1 or 2. We will show that Browkin and Gangl's conjecture is true. Let $E$ be a number field. Recall that the Tate kernel of $E$ is defined as

$$\Delta_E = \{x \in E^\times | \{-1, x\} = 1 \in K_2E\}/(E^\times)^2.$$

By Theorem 6.3 of [18],

$$\Delta_E \simeq (\mathbb{Z}/2\mathbb{Z})^{r_2+1},$$

where $r_2$ is the number of complex places of $E$.

Let $E = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field, where $d < -2$ is a squarefree integer. We use $NE$ for the set of norms from $E$ over $\mathbb{Q}$. Note that if $2 \in NE$, then $d \in N\mathbb{Q}(\sqrt{2})$ by the reciprocity law for the norm residue symbols. And since $\mathbb{Z}[\sqrt{2}]$ is a principal ideal domain, we have $d \in N\mathbb{Z}(\sqrt{2})$. Hence we can assume that

$$d = u^2 - 2w^2, \ u, \ w \in \mathbb{Z}.$$

Let

$$\gamma = \begin{cases} 1, & \text{if } 2 \notin NE; \\ 1 \text{ or } u + \sqrt{d}, & \text{if } 2 \in NE, \ d = u^2 - 2w^2, \ d \equiv 2 \text{ or } 3 \pmod 4; \\ 1 \text{ or } \frac{1}{2}(u + \sqrt{d}), & \text{if } 2 \in NE, \ d = u^2 - 2w^2, \ d \equiv 1 \pmod 4. \end{cases}$$

**Lemma 3.2** *(J. Browkin and A. Schinzel, Theorem 4 of [4]). In the notations above, the 2-torsion part of* $_2K_2\mathcal{O}_E$ *is generated by the Steinberg symbols*

$$\{-1,\ \gamma\delta\},\ \delta|d.$$

This lemma is contained in the proof of Theorem 4 of [4]. Also by proof of Theorem 4 of [4], we know that there is always a $\gamma\delta$ such that $\Delta_E$ is generated by $2,\ \gamma\delta$.

On page 258 of [9], there is a short exact sequence

$$1 \longrightarrow K_3^{\mathrm{ind}}E/(K_3^{\mathrm{ind}}E)^n \longrightarrow H^1_{et}(E,\ {\mu_n}^{\otimes 2}) \xrightarrow{f} {}_nK_2E \longrightarrow 1$$

for any field $E$ with characteristic not dividing $n$. Note that we only quote the middle part of Levine's exact sequence.

In the following context of this section, we will consider the above short exact sequence for quadratic and biquadratic number fields and $n = 2$. By (3.3) of [18]

$$H^1_{et}(E,\ {\mu_2}^{\otimes 2}) \simeq E^\times/(E^\times)^2.$$

Hence the above short sequence can be rewritten as

$$1 \longrightarrow K_3^{\mathrm{ind}}E/(K_3^{\mathrm{ind}}E)^2 \longrightarrow E^\times/(E^\times)^2 \xrightarrow{f} {}_2K_2E \longrightarrow 1,$$

where $f(\bar{a}) = \{-1,\ a\}$ for $a \in E^\times$. Note that $\ker(f)$ is just the Tate kernel of $E$. Hence we get the following

$$K_3^{\mathrm{ind}}E/(K_3^{\mathrm{ind}}E)^2 \simeq \Delta_E,$$

which implies the isomorphism

$$\widehat{\mathcal{B}}(E)/2\widehat{\mathcal{B}}(E) \simeq \Delta_E. \tag{3.6}$$

For any subfield $F$ of $E$, there is a natural commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{B}}(E)/2\widehat{\mathcal{B}}(E) \arrow[r] & \Delta_E \\
\widehat{\mathcal{B}}(F)/2\widehat{\mathcal{B}}(F) \arrow[r] \arrow[u] & \Delta_F \arrow[u]
\end{tikzcd} \tag{3.7}$$

by the functoriality of cohomology groups and the extended Bloch groups.

**Lemma 3.3.** *Let $F$ be a field with characteristic $\neq 2$. If $\sqrt{2} \in F$, then in $K_2F$,*

$$\{-1,\ 2+\sqrt{2}\} = 1.$$

*In particular, if $2+\sqrt{2} \notin F^2$, then $2+\sqrt{2}$ is a non-trivial element in the Tate kernel of $F$.*

**Proof.** In $K_2F$, we have

$$\{\sqrt{2},\ 2+\sqrt{2}\} = \{\sqrt{2},\ 2+\sqrt{2}\}\{\sqrt{2},\ 1-\sqrt{2}\} = \{\sqrt{2},\ -\sqrt{2}\} = 1.$$

Hence

$$\begin{aligned}\{-1,\ 2+\sqrt{2}\} &= \{-1,\ \sqrt{2}\}\{-1,\ 1+\sqrt{2}\}\\ &= \{\sqrt{2},\ \sqrt{2}\}\{\sqrt{2},\ 1+\sqrt{2}\}\\ &= \{\sqrt{2},\ 2+\sqrt{2}\}\\ &= 1.\end{aligned}$$

So if $2+\sqrt{2} \notin F^2$, then $2+\sqrt{2}$ is a non-trivial element in the Tate kernel of $F$. □

**Theorem 3.4.** *Let $F_1,\ F_2$ be two imaginary quadratic number fields, $F = F_1F_2$, $F_0$ the real quadratic subfield of $F$. Then*

$$Q_2(F) = 1 \quad or \quad 2.$$

*We assume that the Tate kernel of $F_i$ is generated by 2 and $\delta_i$ for $i = 1,\ 2$. Then $Q_2(F) = 1$ if and only if none of $\delta_1,\ \delta_2$ and $\delta_1\delta_2$ is trivial in the Tate kernel $\Delta_F$. Equivalently,*

$$Q_2(F) = 1 \iff \begin{cases}\Delta_F \ \textit{is generated by}\ 2,\ \delta_1 \ \textit{and}\ \delta_2, & \textit{if}\ \sqrt{2} \notin F\\ \Delta_F \ \textit{is generated by}\ 2+\sqrt{2},\ \delta_1 \ \textit{and}\ \delta_2, & \textit{if}\ \sqrt{2} \in F.\end{cases}$$

**Proof.** Since the extended Bloch group $\widehat{\mathcal{B}}$ satisfies Galois descent for the biquadratic field $F = F_1F_2$, where $F_1,\ F_2$ are imaginary quadratic number fields, $\widehat{\mathcal{B}}(F_1)$ and $\widehat{\mathcal{B}}(F_2)$ are subgroups of $\widehat{\mathcal{B}}(F)$.

By No. 5 on page 256 of [9] and Proposition 22 of [19], we know that if $\sqrt{2} \notin F$, then

$$\widehat{\mathcal{B}}(F)_{\mathrm{tor}} = \widehat{\mathcal{B}}(F_1)_{\mathrm{tor}} = \widehat{\mathcal{B}}(F_2)_{\mathrm{tor}} = \mathbb{Z}/16\mathbb{Z} \bigoplus \ (\text{odd part})$$

and if $\sqrt{2} \in F$, then $\widehat{\mathcal{B}}(F)_{\mathrm{tor}} = \mathbb{Z}/32\mathbb{Z} \bigoplus$ (odd part) and

$$Tr_{F/F_i}(\widehat{\mathcal{B}}(F)_{\mathrm{tor}}) = \widehat{\mathcal{B}}(F_i)_{\mathrm{tor}} = \mathbb{Z}/16\mathbb{Z} \bigoplus \ (\text{odd part}) \ \ (i = 1,\ 2),$$

where “$Tr$” is the trace map.

We use the same notations as in the proof of Theorem 3.1. By (3.5), the sublattice of $\mathbb{D}(\widehat{\mathcal{B}}(F))$ generated by $\mathbb{D}(b_1)$ and $\mathbb{D}(b_2)$ contains the sublattice generated by $\mathbb{D}(2e_1)$ and $\mathbb{D}(2e_2)$. If $|ad - bc| = 4$, then these two sublattices have the same covolume which implies that they are equal to each other. Hence

$$b_1 = 2x_1 + t_1,\ b_2 = 2x_2 + t_2$$

for some $x_1,\ x_2 \in \widehat{\mathcal{B}}(F)$, and $t_1,\ t_2 \in \widehat{\mathcal{B}}(F)_{\mathrm{tor}}$. In fact, one can see that $t_1 \in \widehat{\mathcal{B}}(F_1)_{\mathrm{tor}}$ and $t_2 \in \widehat{\mathcal{B}}(F_2)_{\mathrm{tor}}$. Hence we can replace $b_i$ by $b_i - t_i$ so that $b_i = 2x_i$ for $i = 1,\ 2$. Note that this replacement will not change $\mathbb{D}(b_1)$ or $\mathbb{D}(b_2)$.

We assume further that

$$b_1 = 2e_1,\ b_2 = 2e_2.$$

It follows that

$$\frac{\widehat{\mathcal{B}}(F)}{\widehat{\mathcal{B}}(F_1) + \widehat{\mathcal{B}}(F_2) + 2\widehat{\mathcal{B}}(F)} \simeq \begin{cases}(\mathbb{Z}/2\mathbb{Z})^2, & \text{if}\ \sqrt{2} \notin F;\\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{if}\ \sqrt{2} \in F.\end{cases} \tag{3.8}$$

By (3.6), (3.7) and (3.8), we have

$$\frac{\Delta_F}{\overline{\Delta}_{F_1}+\overline{\Delta}_{F_2}} \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } \sqrt{2} \notin F; \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{if } \sqrt{2} \in F, \end{cases} \tag{3.9}$$

where $\overline{\Delta}_{F_1}$, $\overline{\Delta}_{F_2}$ are the images in $\Delta_F$.

Recall that the Tate kernel $\Delta(F_i)$ is generated by $\{2,\ \delta_i\}$ for $i = 1,\ 2$, and

$$\Delta_F \simeq (\mathbb{Z}/2\mathbb{Z})^3,\ \Delta_{F_1} \simeq \Delta_{F_2} \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

Hence the above isomorphism of (3.9) implies that $\{2,\ \delta_1,\ \delta_2\}$ generates a cyclic subgroup, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, of $\Delta(F)$ if $\sqrt{2} \notin F$, and $\{2,\ \delta_1,\ \delta_2\}$ generates the trivial subgroup of $\Delta(F)$ if $\sqrt{2} \in F$.

If $\sqrt{2} \in F$, then we can assume that $F_1 = \mathbb{Q}(\sqrt{d})$ and $F_2 = \mathbb{Q}(\sqrt{2d})$, where $d$ is a square free, odd and negative integer. In this case, both of the torsion parts $T_1 = \widehat{\mathcal{B}}(F_1)_{\mathrm{tor}}$ and $T_2 = \widehat{\mathcal{B}}(F_2)_{\mathrm{tor}}$ are equal to $2T = 2\widehat{\mathcal{B}}(F)_{\mathrm{tor}}$. So if $b_1 = 2e_1$ and $b_2 = 2e_2$, then $\delta_1$ and $\delta_2$ are squares of $F$. If $\delta_1|d$, then $\mathbb{Q}(\sqrt{\delta_1})$ is also a subfield of $F$ which implies that $\delta_1 = d$. This is impossible. If $\delta_1 = (u + \sqrt{d})\delta$ for $d = u^2 - 2w^2$ and $\delta|d$, then $F$ is a cyclic quartic field which contradicts the assumption that $F$ is biquadratic.

If $\sqrt{2} \notin F$, then we can assume that $F_1 = \mathbb{Q}(\sqrt{d_1})$ and $F_2 = \mathbb{Q}(\sqrt{d_2})$, where $d_1,\ d_2$ are negative integers. The assumption that $b_1 = 2e_1$ and $b_2 = 2e_2$ implies that there are $x,\ y \in F$ such that

$$\delta_1 = 2^k x^2 \text{ and } \delta_2 = 2^l y^2,$$

where $k,\ l = 0$ or 1. This assertion can be easily proved by similar arguments as in the above paragraph. We just prove one case as an example. If $k = l = 0$, then

$$\delta_1 = x^2,\ \delta_2 = y^2.$$

Hence $\delta_2 = d_1$ and $\delta_1 = d_2$, which implies that $d_1|d_2$ and $d_2|d_1$. This is impossible.

We have proven that

$$Q_2(F) = 1 \quad \text{or} \quad 2.$$

By the above argument, we can see that if $\sqrt{2} \notin F$, then $Q_2(F) = 1$ if and only if

$$\widehat{\mathcal{B}}(F) = \widehat{\mathcal{B}}(F_1) + \widehat{\mathcal{B}}(F_2) + 2\widehat{\mathcal{B}}(F),$$

which means that the Tate kernel $\Delta_F$ is generated by $2,\ \delta_1$ and $\delta_2$. Similarly if $\sqrt{2} \in F$, then $Q_2(F) = 1$ if and only if $\Delta_F$ is generated by $2 + \sqrt{2},\ \delta_1$ and $\delta_2$. □

For a number field $E$, let $k_2(E) = |K_2\mathcal{O}_E|$.

**Theorem 3.5.** *Let $F_1,\ F_2$ be two imaginary quadratic number fields, $F = F_1F_2$, $F_0$ the real quadratic subfield of $F$. Assuming the Lichtenbaum Conjecture holds for these fields, we have*

$$k_2(F) = \frac{Q_2(F)}{8} k_2(F_0)k_2(F_1)k_2(F_2),$$

*where $Q_2(F) = 1$ or 2. The sufficient and necessary condition for $Q_2(F) = 1$ is given in Theorem 3.4.*

**Proof.** This follows from Theorems 3.4 and Corollary 1 of [5]. □

In [5], Browkin and Gangl conjectured that if both of $k_2(F_1)$ and $k_2(F_2)$ are odd, then the index $Q_2(F)$ in the above Theorem is 2, and otherwise it is 1. By Theorem 3.4, we can prove that there are infinitely many biquadratic number fields such that $Q_2(F) = 2$ while both of $k_2(F_1)$ and $k_2(F_2)$ are even.

**Corollary 3.6.** *The index $Q_2(F) = 2$ in the following cases*

(1) *$k_2(F_1)$ and $k_2(F_2)$ are odd.*
(2) *$F_1 = \mathbb{Q}(\sqrt{-p})$, $F_2 = \mathbb{Q}(\sqrt{-q})$, where $p$ and $q$ are distinct primes such that*

$$p \equiv q \equiv 9 \pmod{16}.$$

(3) *$F_1 = \mathbb{Q}(\sqrt{-p_1q_1})$, $F_2 = \mathbb{Q}(\sqrt{-p_2q_2})$, where $p_i$ and $q_i$ are distinct primes such that*

$$p_1 \equiv q_1 \equiv -p_2 \equiv -q_2 \equiv 3 \pmod{8}.$$

(4) *$F_1 = \mathbb{Q}(\sqrt{-pq_1})$, $F_2 = \mathbb{Q}(\sqrt{-pq_2})$, where $p$ and $q_i$ are distinct primes such that*

$$p \equiv -q_1 \equiv -q_2 \equiv 3 \pmod{8}.$$

**Proof.** By Table 1 of [15] and Table 6 of [16], we know that in the first three cases, both of the Tate kernels of $F_1$ and $F_2$ are generated by 2 and $-1$, and in the fourth case, both of the Tate kernels of $F_1$ and $F_2$ are generated by 2 and $-p$. Hence $\delta_1 = \delta_2$ in all four cases. By Theorem 3.4, we find that $Q_2(F) = 2$. □

**Corollary 3.7.** *The index $Q_2(F) = 1$ in the following cases*

(1) *$F_1 = \mathbb{Q}(\sqrt{-2p_1q_1})$, $F_2 = \mathbb{Q}(\sqrt{-p_2q_2})$, where $p_i$ and $q_i$ are distinct primes such that*

$$p_1 \equiv q_1 \equiv p_2 \equiv -q_2 \equiv 3 \pmod{8}.$$

(2) *$F_1 = \mathbb{Q}(\sqrt{-2p_1q_1})$, $F_2 = \mathbb{Q}(\sqrt{-2p_2q_2})$, where $p_i$ and $q_i$ are distinct primes such that*

$$p_1 \equiv q_1 \equiv q_2 \equiv -p_2 \equiv 5 \pmod{8}.$$

**Proof.** By Table 6 of [16], we know that in the first case $\delta_1 = -1$, $\delta_2 = -p_2$ and in the second case $\delta_1 = -1$, $\delta_2 = p_2$. Hence in both cases, we have

$$\Delta_F =< 2,\ \delta_1,\ \delta_2 > .$$

Hence the index $Q_2(F) = 1$. □

**Theorem 3.8.** *Let $F_1 = \mathbb{Q}(\sqrt{-d_1})$, $F_2 = \mathbb{Q}(\sqrt{-d_2})$ be two distinct imaginary quadratic number fields such that $d_1/d_2 \neq 2$ or $1/2$. Assume that*

$$\Delta_{F_1} =< 2,\ \delta_1(u + \sqrt{-d_1}) >,\ \Delta_{F_2} =< 2,\ \delta_2 >,$$

*where $|\delta_i|$ is a divisor of $d_i$ for $i = 1,\ 2$. If $\delta_2 \neq -d_1$ or $-d_1/2$, then*

$$\Delta_F =< 2,\ \delta_1(u + \sqrt{-d_1}),\ \delta_2 >,$$

*which implies that the index $Q_2(F) = 1$.*

**Proof.** By our assumption $\sqrt{2} \notin F = F_1F_2$ and $\delta_2 \neq -d_1$ or $-d_1/2$, hence $2,\ \delta_1(u+\sqrt{-d_1})$ and $\delta_2$ are linearly independent in $\Delta_F$. Hence $\Delta_F =< 2,\ \delta_1(u+\sqrt{-d_1}),\ \delta_2 >$ which implies that the index $Q_2(F) = 1$. □

**Corollary 3.9.** *Let $F_1 = \mathbb{Q}(\sqrt{-p})$, $F_2 = \mathbb{Q}(\sqrt{-q})$, where $p$ and $q$ are distinct primes such that*

$$p \equiv 9 \pmod{16},\ q \equiv 7 \pmod 8,$$

*and $F = F_1F_2$. Then*

$$\Delta_F =< 2,\ -1,\ u+\sqrt{-q} >$$

*and the index $Q_2(F) = 1$.*

**Proof.** This result follows from the above Theorem and Table 1 of [15]. □

One should note that there are many other cases such that $Q_2(F) = 1$ or 2 by the tables of [14], [15] and [16].

**Example 3.10.**

(1) $F = \mathbb{Q}(\sqrt{-6},\ \sqrt{-15})$, $F_1 = \mathbb{Q}(\sqrt{-6})$, $F_2 = \mathbb{Q}(\sqrt{-15})$. In this case, $\Delta_{F_1} =< 2,\ -1 >$ and $\Delta_{F_2} =< 2,\ -3 >$ by Table 6 of [16]. Since $2 \times (-3)$ is a square in $F$, $< 2,\ -1,\ -3 >=< 2,\ -1 >\neq \Delta_F$. So $Q_2(F) = 2$. In Example 9 of [5], Browkin and Gangl proved that $Q_2(F) \geq 2$.

(2) $F = \mathbb{Q}(\sqrt{-1},\ \sqrt{-33})$, $F_1 = \mathbb{Q}(\sqrt{-1})$, $F_2 = \mathbb{Q}(\sqrt{-33})$. In this case, $\Delta_{F_1} =< 2,\ \sqrt{-1} >$ and $\Delta_{F_2} =< 2,\ -1 >$ by Table 6 of [16]. Since $-1$ is a square in $F$, $< 2,\ \sqrt{-1},\ -1 >=< 2,\ \sqrt{-1} >\neq \Delta_F$. So $Q_2(F) = 2$. In Example 9 of [5], Browkin and Gangl also suggested that $Q_2(F) = 2$.

(3) $F = \mathbb{Q}(\sqrt{-1},\ \sqrt{-123})$, $F_1 = \mathbb{Q}(\sqrt{-1})$, $F_2 = \mathbb{Q}(\sqrt{-123})$. We have $\Delta_{F_1} =< 2,\ \sqrt{-1} >$ and $\Delta_{F_2} =< 2,\ -1 >$ by Table 6 of [16]. This is also an example discussed in [5]. By the same argument, we can show that $Q_2(F) = 2$.

## 4. Galois descent of the extended Bloch groups of dihedral number fields

In this section, we will use the same notations as in [5]. Let $F$ be any Galois extension of $\mathbb{Q}$ with the Galois group $G = D_{2p}$, where $p$ is an odd prime and $D_{2p}$ is the dihedral group of order $2p$. We assume that $F$ is not real. Then $F$ has a unique quadratic subfield $F_0$, and $p$ subfields $F_1,\ \ldots,\ F_p$ of degree $p$. We assume furthermore that $F_p$ is fixed by the complex conjugation.

Let $\widehat{\mathcal{B}}_F = Z + T$, $\widehat{\mathcal{B}}_{F_0} = Z_0 + T_0$, $\widehat{\mathcal{B}}_{F_1} = Z_1 + T_1$ and $\widehat{\mathcal{B}}_{F_p} = Z_p + T_p$, where $Z_0,\ Z_1,\ Z_p,\ Z$ are free $\mathbb{Z}$-modules and $T_0,\ T_1,\ T_p,\ T$ are finite cyclic groups. Let $\overline{Z_0},\ \overline{Z_1},\ \overline{Z_p},\ \overline{Z}$ be the images of $Z_0,\ Z_1,\ Z_p,\ Z$ in the quotients $\widehat{\mathcal{B}}_{F_0}/T_0$, $\widehat{\mathcal{B}}_{F_1}/T_1$, $\widehat{\mathcal{B}}_{F_p}/T_p$, $\widehat{\mathcal{B}}_F/T$ respectively. By Section 10 of [5], the sub-lattice

$$\overline{Z_0} + \overline{Z_1} + \overline{Z_p}$$

is of maximal rank in $\overline{Z}$. Let

$$Q_2(F) = \#\overline{Z}/(\overline{Z_0} + \overline{Z_1} + \overline{Z_p}).$$

**Lemma 4.1.** *Let notations be as above. Then*

$$Q_2(F) | 2^{p-1}p.$$

**Proof.** It is not hard to prove this lemma using the same argument as in the proof of Theorem 3.1. We omit details of the proof. □

**Lemma 4.2.** *Let notations be as above. Then*

$$Q_2(F)|p^p.$$

**Proof.** Suppose that $\mathrm{Gal}(F/F_0) = \{1,\ \tau,\ ...,\ \tau^{p-1}\}$, $\mathrm{Gal}(F/F_1) = \{1,\ \sigma\}$, $\mathrm{Gal}(F/F_2) = \{1,\ \sigma\tau^{p-1}\}$, ..., $\mathrm{Gal}(F/F_p) = \{1,\ \sigma\tau\}$. Then we have the following identity

$$\begin{aligned}&(1+\sigma)(-(p-1)\tau-(p-2)\tau^2-\cdots-\tau^{p-1})\\&+(1+\sigma\tau)(p+(p-1)\tau+\cdots+\tau^{p-1})\\=&p+\sigma+\sigma\tau+\cdots+\sigma\tau^{p-1}.\end{aligned}\tag{4.1}$$

For any $x \in \widehat{\mathcal{B}}(F)$, put

$$\begin{aligned}x_1 =&(-(p-1)\tau-(p-2)\tau^2-\cdots-\tau^{p-1})(x),\\x_2 =&(p+(p-1)\tau+\cdots+\tau^{p-1})(x),\\x_3 =&(1+\tau+\cdots+\tau^{p-1})(x).\end{aligned}$$

Then

$$\begin{aligned}Tr_{F/F_0}(x) &= x_3 = (1+\tau+\cdots+\tau^{p-1})(x),\\Tr_{F/F_1}(x_1) &= (1+\sigma)(x_1) = (1+\sigma)(-(p-1)\tau-(p-2)\tau^2-\cdots-\tau^{p-1})(x),\\Tr_{F/F_p}(x_2) &= (1+\sigma\tau)(x_2) = (1+\sigma\tau)(p+(p-1)\tau+\cdots+\tau^{p-1})(x).\end{aligned}$$

By the equality (4.1), we have

$$px + Tr_{F/\mathbb{Q}}(x) = Tr_{F/F_1}(x_1) + Tr_{F/F_p}(x_2) + Tr_{F/F_0}(x).\tag{4.2}$$

Let $\overline{x}$ be the image of $x$ in $\overline{Z}$. Then (4.2) shows that

$$p\overline{x} \in \overline{Z_0} + \overline{Z_1} + \overline{Z_p} \subset \overline{Z},$$

which implies that the exponent of

$$\overline{Z}/(\overline{Z_0} + \overline{Z_1} + \overline{Z_p})$$

is 1 or $p$. □

Combining Lemmas 4.1 and 4.2, we have the following theorem.

**Theorem 4.3.** *Let notations be as above. Then*

$$Q_2(F)|p.$$

By Corollary 2 of Section 10 of [5] and using the same argument as in the proof of Theorem 3.4, we have the following Theorem.

**Theorem 4.4.** *With notations as above. We assume that $w_2(F) = 24$ and the Lichtenbaum Conjecture holds for the fields in this section. Then we have*

$$k_2(F) = \frac{Q_2(F)}{4p} k_2(F_0) k_2(F_1)^2,$$

*where $Q_2(F)|p$.*

In [21], Zhou proved that for $p = 3$, $Q_2(F)|3^3$.

**Theorem 4.5.** *Let $F_0 = \mathbb{Q}(\sqrt{-3})$, $F_3 = \mathbb{Q}(\sqrt[3]{n})$ a pure cubic field and $F = F_0F_3$. Then there are infinitely many $n$ such that the index $Q_2(F) = 1$.*

**Proof.** It suffices to prove that there are infinitely many $n$ such that

$$K_3^{\text{ind}} F_0 / ((K_3^{\text{ind}} F_0)^3 Tr_{F/F_0}(K_3^{\text{ind}} F)) \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

By Theorem 4.1 and Corollary 4.3 of [7],

$$K_3^{\text{ind}} F_0 / ((K_3^{\text{ind}} F_0)^3 Tr_{F/F_0}(K_3^{\text{ind}} F)) \simeq D_{F_0}^{(2)} / (N_{F/F_0} D_F^{(2)} (F_0^\times)^3),$$

where $D_{F_0}^{(2)}$, $D_F^{(2)}$ are the classical Tate kernels,

$$D_F^{(2)} = \{x \in F^\times | \{\zeta_3, x\} = 1 \in K_2F\} / (F^\times)^3.$$

Hutchinson pointed out in [7] that Corollary 4.3 of [7] is firstly proven by Assim and Movahhedi in [3]. One can also see the generalization of this result in [1] and [2].

Note that the capitulation kernel $H_{F_0}$ is trivial since $K_2(\mathcal{O}_{F_0})$ has no 3-primary part. Hence Corollary 3.11 of [3] applies. One can see the details of capitulation kernel $H_{F_0}$ in [3]. Now the Theorem follows from Corollary 3.11 of [3] and Chebotarev's density theorem. □

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