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Determinants of trigonometric functions and class numbers ☆

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We study two determinants involving tangent and cotangent functions and prove two conjectures made by Zhi-Wei Sun in 2019. Both determinants are divisible by the first factor of the class numbers of associated cyclotomic fields.

1. Introduction

Zhi-Wei Sun made the following two conjectures in 2019.

Conjecture 1.1 *([11, Conjecture 5.1]). Let p be an odd prime. Then*

$$\left(\frac{-2}{p}\right)\frac{\det\left(\cot\frac{jk\pi}{p}\right)_{1\leqslant j,k\leqslant\frac{p-1}{2}}}{2^{\frac{p-3}{2}}p^{\frac{p-5}{4}}}\in\{1,2,3,\ldots\}$$

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and this number is divisible by $h(-p)$ *if* $p \equiv 3 \pmod 4$, *where* $h(-p)$ *is the class number of* $\mathbb{Q}(\sqrt{-p})$.

Conjecture 1.2 *([11, Conjecture 5.2(i)]). Let* n *be a positive integer. Then*

$$\frac{\det\left(\tan \frac{jk\pi}{2n+1}\right)_{1\leqslant j,k\leqslant n}}{(2n+1)^{\frac{n}{2}}} \in \mathbb{Z}.$$

Conjecture 1.1 is verified by Sun to be true for $p \leqslant 29$. Later, Francois Brunault extended the verification for all primes $p \leqslant 47$. In fact, $\det\left(\cot \frac{jk\pi}{p}\right)$ is closely connected with the famous Maillet's determinant which was first introduced by Maillet in [7] more than a century ago. We will prove that Conjecture 1.1 is true. We also give the exact value of $\det\left(\cot \frac{jk\pi}{p}\right)_{1\leqslant j,k\leqslant \frac{p-1}{2}}$ in Section 2.

Conjecture 1.2 is somehow related with layered networks. One can see [2] and [6] for details. Sun guessed that the sequence $s_n = \frac{\det\left(\tan \frac{jk\pi}{2n+1}\right)_{1\leqslant j,k\leqslant n}}{(2n+1)^{\frac{n}{2}}}$ is connected with t_n, the sequence A277445 in OEIS. Especially, he conjectured that $s_n = -t_n$ if $n \equiv 3 \pmod 4$ and $s_n = t_n$ otherwise. We will prove that both of Conjecture 1.2 and Sun's conjecture on t_n are true if $2n+1$ is a prime number in Section 3. We also give the exact value of the determinant in Conjecture 1.2 with $2n+1$ prime.

The determinants in these two conjectures are also very interesting for nonprime numbers. However we find that it is difficult to get the exact values of the determinants for nonprime numbers and quite different methods are needed. Hence we will only study the determinants for prime numbers. The Conjecture 5.3 of [11] is proved by Tao and Guo in [12] by different techniques.

The determinants of cotangent functions in Conjecture 1.1 are connected with Maillet's determinant by a formula of Eisenstein. Maillet's determinant has a rich history. For any integer r coprime to p, let r' be the smallest positive integer such that $rr' \equiv 1 \pmod p$. For any integer x coprime to p, let $R(x)$ be the smallest positive residue of x modulo p. Let $M_p = (R(rs'))_{1\leqslant r,s'\leqslant \frac{p-1}{2}}$. Then $D_p = \det M_p$ is called Maillet's determinant. In 1914, Malo computed $D_5 = -5, D_7 = 7^2, D_{11} = 11^4, D_{13} = -13^5$ in [8]. Malo conjectured that $D_p = (-p)^{\frac{p-3}{2}}$ based on his computation. One can see page 340–342 of [9] for details.

In 1955, Carlitz and Olson proved in [3] that Malo's conjecture was incorrect and gave the correct value up to a sign,

$$D_p = \pm p^{\frac{p-3}{2}} h_p^-, \tag{1.1}$$

where h_p^- denotes the first factor of the class number of $\mathbb{Q}(\zeta_p)$. They also mentioned that S. Chowla and A. Weil had proved the formula (1.1) several years earlier but did not publish their results. Although Maillet's determinant D_p was introduced for prime p, one can also define Maillet's determinant D_n for any integer $n > 1$. In 1984, K. Wang generalized Carlitz and Olson's formula from prime p to any positive integer m in [13] and determined the sign in (1.1). For prime p, Wang's formula is

$$D_p = -2^{\frac{3-p}{2}} \prod_{\chi \text{ odd}} \left(\sum_{a=1}^{\frac{p-3}{2}} a\chi(a) \right) (-p)^{\frac{p-3}{2}} h_p^-, \tag{1.2}$$

where χ runs over all the odd Dirichlet characters modulo p.

Let $A_p = \left(\cot \frac{jk\pi}{p}\right)_{1 \leqslant j,k \leqslant \frac{p-1}{2}}$. We will prove that

$$\det A_p = \left(\frac{-2}{p}\right) 2^{\frac{p-3}{2}} p^{\frac{p-5}{4}} h_p^-.$$

If $p \equiv 3 \pmod 4$, then the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$ is contained in $\mathbb{Q}(\zeta_p)$, and h_p^- is divisible by $h(-p)$ the class number of $\mathbb{Q}(\sqrt{-p})$.

Let $W_n = \left(\tan \frac{\pi jk}{2n+1}\right)_{1 \leqslant j,k \leqslant n}$. We assume that $p = 2n+1$ is a prime. Let $K = \mathbb{Q}(\zeta_p)$ and X_K^- the set all odd characters of conductor p. Then we will prove in Section 3 that

$$\det(W_n) = \begin{cases} 2^{n-1} p^{\frac{n}{2}-1} h_p^- \left| \prod_{\chi \in X_K^-} (1-2\chi(2)) \right|, & \text{if } p \equiv 1,3,7 \pmod 8; \\ -2^{n-1} p^{\frac{n}{2}-1} h_p^- \left| \prod_{\chi \in X_K^-} (1-2\chi(2)) \right|, & \text{if } p \equiv 5 \pmod 8), \end{cases}$$

where the product $\prod_{\chi \in X_K^-}(1-2\chi(2))$ is explicitly calculated in Lemma 3.3. In particular, $p^{-\frac{n}{2}} \det(W_n) \in \mathbb{Z}$. If $p \equiv 3 \pmod 4$, then $h(-p) | p^{-\frac{n}{2}} \det(W_n)$.

In [6], David V. Ingerman mentioned that for a prime number $p = 2n+1$, there is a matrix $T_n = (t_{jk})_{n \times n}$ with entries among ± 1 such that

$$2 \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} \begin{pmatrix} \sin \frac{\pi}{2n+1} \\ \vdots \\ \sin \frac{n\pi}{2n+1} \end{pmatrix} = \begin{pmatrix} \tan \frac{\pi}{2n+1} \\ \vdots \\ \tan \frac{n\pi}{2n+1} \end{pmatrix}. \tag{1.3}$$

One is tempted to ask that if

$$2 \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} \begin{pmatrix} \sin \frac{\pi}{2n+1} & \cdots & \sin \frac{n\pi}{2n+1} \\ \vdots & \ddots & \vdots \\ \sin \frac{n\pi}{2n+1} & \cdots & \sin \frac{n^2\pi}{2n+1} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \tan \frac{\pi}{2n+1} & \cdots & \tan \frac{n\pi}{2n+1} \\ \vdots & \ddots & \vdots \\ \tan \frac{n\pi}{2n+1} & \cdots & \tan \frac{n^2\pi}{2n+1} \end{pmatrix}.$$

Unfortunately, it is wrong even for $n = 2$,

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sin \frac{\pi}{5} & \sin \frac{2\pi}{5} \\ \sin \frac{2\pi}{5} & \sin \frac{4\pi}{5} \end{pmatrix} \neq \begin{pmatrix} \tan \frac{\pi}{5} & \tan \frac{2\pi}{5} \\ \tan \frac{2\pi}{5} & \tan \frac{4\pi}{5} \end{pmatrix}.$$

However, we can get the correct formula by a little modification,

$$
\begin{aligned}
&2\begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix}\begin{pmatrix} d_{11}\sin\frac{\pi}{2n+1} & \cdots & d_{1n}\sin\frac{n\pi}{2n+1} \\ \vdots & \ddots & \vdots \\ d_{n1}\sin\frac{n\pi}{2n+1} & \cdots & d_{nn}\sin\frac{n^2\pi}{2n+1} \end{pmatrix} \\
=&\begin{pmatrix} \tan\frac{\pi}{2n+1} & \cdots & \tan\frac{n\pi}{2n+1} \\ \vdots & \ddots & \vdots \\ \tan\frac{n\pi}{2n+1} & \cdots & \tan\frac{n^2\pi}{2n+1} \end{pmatrix}\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix},
\end{aligned} \tag{1.4}
$$

where

$$
d_{ij}=\begin{cases} -1, & \text{if } i \text{ and } j \text{ are all even;} \\ 1, & \text{otherwise} \end{cases} \qquad d_j=\begin{cases} -1, & \text{if } j \text{ is even;} \\ 1 & \text{otherwise.} \end{cases} \tag{1.5}
$$

Note that

$$
\det\left(d_{ij}\sin\frac{ij\pi}{2n+1}\right)=\begin{cases} -\det\left(\sin\frac{ij\pi}{2n+1}\right), & \text{if } n\equiv 3 \pmod 4; \\ \det\left(\sin\frac{ij\pi}{2n+1}\right), & \text{otherwise.} \end{cases} \tag{1.6}
$$

By comparing the determinants of LHS and the RHS of (1.4), we prove Sun's conjecture of s_n and t_n.

2. Determinants involving cotangent functions

Recall that for $\theta_1, \theta_2, \cdots, \theta_n \in \mathbb{R}$,

$$
\begin{aligned}
&\det\begin{pmatrix} \sin\theta_1 & \sin\theta_2 & \dots & \sin\theta_n \\ \sin 2\theta_1 & \sin 2\theta_2 & \dots & \sin 2\theta_n \\ \vdots & \vdots & & \vdots \\ \sin n\theta_1 & \sin n\theta_2 & \dots & \sin n\theta_n \end{pmatrix} \\
=&2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}\sin\theta_k\prod_{1\leqslant i<j\leqslant n}(\cos\theta_j-\cos\theta_i).
\end{aligned} \tag{2.1}
$$

Let h_p be the class number of $\mathbb{Q}(\zeta_p)$ and h_p^+ the class number of the maximal subfield $\mathbb{Q}(\zeta_p+\overline{\zeta_p})$. Recall that $h_p^- h_p^+ = h_p$.

Theorem 2.1. *Let p be an odd prime. Then*

$$
\det A_p = \left(\frac{-2}{p}\right) 2^{\frac{p-3}{2}} p^{\frac{p-5}{4}} h_p^-,
$$

where h_p^- is divisible by $h(-p)$ if $p \equiv 3 \pmod 4$.

Proof. Let $B_p = \left(\sin \frac{2jk\pi}{p}\right)_{1 \leqslant j,k \leqslant \frac{p-1}{2}}$. By Section 3 of [3] (page 268), $-B_pA_p = M_p'$, where $M_p' = \left(R(jk') - \frac{p}{2}\right)_{1 \leqslant j,k \leqslant \frac{p-1}{2}}$. By equation (2.3) of [3], $\det M_p' = -\frac{D_p}{2}$. Hence we have

$$\det A_p = \frac{(-1)^{\frac{p+1}{2}} D_p}{2 \det B_p}. \tag{2.2}$$

By (2.1), we have

$$\det B_p = (-1)^{\frac{(p-1)(p-3)}{8}} 2^{-\frac{p-1}{2}} p^{\frac{p-1}{4}}. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned} \det A_p &= (-1)^{\frac{(p-1)(p-3)}{8} + \frac{p+1}{2} + \frac{p-3}{2}} 2^{\frac{p-3}{2}} p^{\frac{p-5}{4}} h_p^- \\ &= (-1)^{\frac{(p-1)(p-3)}{8}} 2^{\frac{p-3}{2}} p^{\frac{p-5}{4}} h_p^- \\ &= \left(\frac{-2}{p}\right) 2^{\frac{p-3}{2}} p^{\frac{p-5}{4}} h_p^-. \end{aligned}$$

By Theorem 1 of [10], $4h_p^-$ is divisible by $h(-p)$ if $p \equiv 3 \pmod 4$. By Gauss genus theory, $h(-p)$ is odd for $p \equiv 3 \pmod 4$. Hence we have $h(-p) | h_p^-$ for $p \equiv 3 \pmod 4$. □

Now it is easy to see that Conjecture 1.1 follows from Theorem 2.1.

3. Determinants involving tangent functions

In this section, we will assume $p = 2n + 1$ is a prime. Let $\zeta = e^{\frac{2\pi i}{p}}$ be a primitive p-th root of unity. Let $K = \mathbb{Q}(\zeta)$, $G_K = \mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_k | 1 \leqslant k \leqslant p-1\}$, where $\sigma_k(\zeta) = \zeta^k$. Let

$$X_K = \{\chi : G_K \to \mathbb{C}^\times \mid \chi \text{ a group homomorphism}\}$$

and X_K^- the set of all odd characters modulo p. One can interpret X_K as the character group of Dirichlet characters mod p by putting $\chi(k) = \chi(\sigma_k)$. For any Dirichlet character χ, recall that the Gauss sum is defined as

$$\tau(\chi) = \sum_{k=1}^{f_\chi} \chi_f(k) e^{\frac{2\pi i k}{f_\chi}},$$

where f_χ is the conductor of χ and χ_f is the primitive Dirichlet character mod f_χ belonging to χ. For $\chi \in X_K$, we have $f_\chi = p$. For each $a \in K$ and each $\chi \in X_K$, the χ-coordinate is defined to be

$$y_K(\chi|a) = \frac{\sum\limits_{\sigma\in G_K} \overline{\chi}(\sigma)\sigma(a)}{\overline{\tau(\chi)}},$$

where $\overline{\chi}$ is the inverse of χ. One should note that Girstmair use a slightly different definition of Gauss sum with notation $T(\overline{\chi})$ in [4] and [5]. Our notation $\tau(\chi)$ is the same with [14].

Theorem 3.1 *([14, Lemma 5.26]). Let G be a finite abelian group and let f be a function on G with values in some field of characteristic 0. Then the matrix $(f(\sigma\tau^{-1}))_{\sigma,\tau\in G}$ is diagonalizable and its eigenvalues are $\sum\limits_{\sigma\in G}\chi(\sigma)f(\sigma)$, where $\chi \in \widehat{G}$ the group of characters of G.*

Corollary 3.2. *For a fixed $a \in K$, let f be the function on G_K with $f(\sigma) = \sigma(a)$. Then the matrix $(f(\sigma\tau^{-1}))_{\sigma,\tau\in G_K}$ is diagonalizable and its eigenvalues are $y_K(\overline{\chi}|a)\tau(\chi)$, $\chi \in X_K$.*

Lemma 3.3. *Let notations be as above. We assume that the order of 2 in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ is ℓ. Then*

$$\prod_{\chi\in X_K^-}(1-2\chi(2)) = \begin{cases}(1-2^\ell)^{\frac{p-1}{2\ell}}, & \text{if } \ell \text{ is odd;}\\ (1+2^{\frac{\ell}{2}})^{\frac{p-1}{\ell}}, & \text{if } \ell \text{ is even.}\end{cases}$$

In particular, p is a factor of $\prod\limits_{\chi\in X_K^-}(1-2\chi(2))$ and

$$\prod_{\chi\in X_K^-}(1-2\chi(2)) = \begin{cases}\left|\prod_{\chi\in X_K^-}(1-2\chi(2))\right|, & \text{if } p\equiv 1,3,5 \pmod 8;\\ -\left|\prod_{\chi\in X_K^-}(1-2\chi(2))\right|, & \text{if } p\equiv 7 \pmod 8.\end{cases}$$

Proof. If ℓ is odd, then $p|(1-2^\ell)$ and

$$\begin{aligned}&\prod_{\chi\in X_K^-}(1-2\chi(2))\\ =&(1-2)^{\frac{n}{l}}(1-2\zeta_\ell)^{\frac{n}{l}}\cdots(1-2\zeta_\ell^{\ell-1})^{\frac{n}{l}}\\ =&(1-2^\ell)^{\frac{p-1}{2\ell}}.\end{aligned}$$

If ℓ is even then $2^{\frac{\ell}{2}} \equiv -1 \pmod p$. Hence $p|(1+2^{\frac{\ell}{2}})$ and

$$\begin{aligned}&\prod_{\chi\in X_K^-}(1-2\chi(2))\\ =&\left(1-2\zeta_\ell\right)^{\frac{p-1}{\ell}}\left(1-2\zeta_\ell^3\right)^{\frac{p-1}{\ell}}\cdots\left(1-2\zeta_\ell^{\ell-1}\right)^{\frac{p-1}{\ell}}\\ =&(1+2^{\frac{\ell}{2}})^{\frac{p-1}{\ell}}.\end{aligned}$$

Since $\prod_{\chi\in X_K^-}(1-2\chi(2))$ is negative if and only if ℓ is odd and $\frac{p-1}{2\ell}$ is odd, one can see that $\prod_{\chi\in X_K^-}(1-2\chi(2))$ is negative if and only if $p\equiv 7 \pmod 8$, i.e.,

$$\prod_{\chi\in X_K^-}(1-2\chi(2))=\begin{cases}\left|\prod_{\chi\in X_K^-}(1-2\chi(2))\right|, & \text{if } p\equiv 1,3,5 \pmod 8;\\ -\left|\prod_{\chi\in X_K^-}(1-2\chi(2))\right|, & \text{if } p\equiv 7 \pmod 8.\end{cases}\quad\square$$

The following lemma is a direct corollary of a classical result of Gauss on the sign of the quadratic Gauss sum. A very good reference on the history of the sign of the quadratic Gauss sum is M. Baker's Math Blog [1].

Lemma 3.4 *(Gauss). Let notations be as above. Then*

$$\prod_{\chi\in X_K^-}\tau(\chi)=i^n p^{\frac{n}{2}}.$$

Conjecture 1.2 follows from the following theorem.

Theorem 3.5. *Let* $W_n=\left(\tan\frac{\pi jk}{2n+1}\right)_{1\leqslant j,k\leqslant n}$*. Then*

$$\begin{aligned}&\det(W_n)\\&=i^n(-1)^{[\frac{n}{2}]+n}\left(\prod_{\chi\in X_K^-}(1-2\chi(2))\right)\cdot\prod_{\chi\in X_K^-}B_{1,\chi}\cdot\prod_{\chi\in X_K^-}\overline{\tau(\chi)}\\&=\begin{cases}2^{n-1}p^{\frac{n}{2}-1}h_p^-\left|\prod_{\chi\in X_K^-}(1-2\chi(2))\right|, & \text{if } p\equiv 1,3,7 \pmod 8;\\ -2^{n-1}p^{\frac{n}{2}-1}h_p^-\left|\prod_{\chi\in X_K^-}(1-2\chi(2))\right|, & \text{if } p\equiv 5 \pmod 8.\end{cases}\end{aligned}$$

In particular, $p^{-\frac{n}{2}}\det(W_n)\in\mathbb{Z}$*. If* $p\equiv 3 \pmod 4$*, then* $h(-p)|p^{-\frac{n}{2}}\det(W_n)$*.*

Proof. Let $a=\frac{-1+\zeta}{1+\zeta}=i\tan\frac{\pi}{p}\in K$. Then $\sigma_k(a)=i\tan\frac{k\pi}{p}$ and $\sigma_k\sigma_{\ell'}(a)=i\tan\frac{R(k\ell')\pi}{p}$. Let f be as in Corollary 3.2 and $A=(f(\sigma\tau^{-1}))_{\sigma,\tau\in G}$. Since $\tan(\pi+x)=\tan x$, we have $\tan\frac{R(k\ell')\pi}{p}=-\tan\frac{R((p-k)\ell')\pi}{p}$ and $\tan\frac{R(k\ell')\pi}{p}=-\tan\frac{R(k(p-\ell)')\pi}{p}$. We assume that $A=\begin{pmatrix}A_1&A_2\\A_3&A_4\end{pmatrix}$. Then it is easy to see that A is similar to $\begin{pmatrix}2A_1&0\\0&0\end{pmatrix}$ by doing elementary similar transformation.

Let $\widetilde{W_n}=\left(i\tan\frac{\pi jk}{2n+1}\right)_{1\leqslant j,k\leqslant n}$. For any matrix M, let $c_k(M)$ be the k-th column of M. Note that one can get $\widetilde{W_n}$ by exchanging some columns of A_1. If $1\leqslant k_1\neq k_2\leqslant n$ satisfy $k_1k_2\equiv 1 \pmod p$, then $c_{k_1}(A_1)=c_{k_2}(\widetilde{W_n})$ and $c_{k_2}(A_1)=c_{k_1}(\widetilde{W_n})$. If $1\leqslant k_1\neq k_2\leqslant n$ satisfy $k_1k_2\equiv -1 \pmod p$, then $c_{k_1}(A_1)=-c_{k_1}(\widetilde{W_n})$ and $c_{k_2}(A_1)=-c_{k_2}(\widetilde{W_n})$. Let $[x]$ be the integral part of $x\in\mathbb{R}$. There are exactly $\left[\frac{n-1}{2}\right]$ pairs (k_1,k_2) such that $k_1k_2\equiv\pm 1$

(mod p), $1 \leqslant k_1 \neq k_2 \leqslant n$. If $p \equiv 1 \pmod 4$, then there is an extra k such that $1 \leqslant k \leqslant n$ and $k^2 \equiv -1 \pmod p$ which implies that $c_k(A_1) = -c_k(\widetilde{W}_n)$. So we have

$$\det(W_n) = (-i)^n \det(\widetilde{W}_n) = (-i)^n (-1)^{[\frac{n}{2}]} \det(A_1).$$

Since A is similar to $\begin{pmatrix} 2A_1 & 0 \\ 0 & 0 \end{pmatrix}$, $\det(2A_1)$ is the product of non-zero eigenvalues of A.

By Theorem 3 of [5], there are n nonzero eigenvalues of A,

$$y_K(\overline{\chi}|a)\overline{\tau(\chi)}, \chi \in X_K^-.$$

Hence

$$\det(2A_1) = \prod_{\chi \in X_K^-} y_K(\overline{\chi}|a)\overline{\tau(\chi)} = \prod_{\chi \in X_K^-} \left(\sum_{\sigma \in G_K} \chi(\sigma)\sigma(a) \right).$$

By Theorem 2 and Theorem 3 of [4],

$$\begin{aligned} y_K(\chi|a) &= y_K\left(\chi | i \cot \frac{\pi}{p}\right)(1 - 2\chi(2)) \\ &= 2(1 - 2\chi(2))B_{1,\chi}, \end{aligned}$$

where

$$B_{1,\chi} = \frac{1}{p} \sum_{k=1}^{2n} k\chi(k)$$

is the generalized Bernoulli number attached to χ. Hence we get

$$\begin{aligned} \det(W_n) &= (-i)^n (-1)^{[\frac{n}{2}]} 2^{-n} \det(2A_1) \\ &= (-i)^n (-1)^{[\frac{n}{2}]} \left(\prod_{\chi \in X_K^-} (1 - 2\chi(2)) \right) \cdot \prod_{\chi \in X_K^-} B_{1,\chi} \cdot \prod_{\chi \in X_K^-} \overline{\tau(\chi)}. \end{aligned}$$

By Lemma 3.3 and the analytic class number formula, Theorem 4.17 of [14], we have

$$\prod_{\chi \in X_K^-} B_{1,\chi} \prod_{\chi \in X_K^-} \overline{\tau(\chi)} = (-i)^n 2^{n-1} p^{\frac{n}{2}-1} h_p^-.$$

Hence we have

$$\det(W_n) = (-1)^{[\frac{n}{2}]+n} \left(\prod_{\chi \in X_K^-} (1 - 2\chi(2)) \right) 2^{n-1} p^{\frac{n}{2}-1} h_p^-.$$

By Lemma 3.3 and Lemma 3.4, we have the sign of $\det(W_n)$,

$$\frac{\det(W_n)}{|\det(W_n)|} = \begin{cases} 1, & \text{if } p \equiv 1,3,7 \pmod 8; \\ -1, & \text{if } p \equiv 5 \pmod 8). \end{cases}$$

In particular, $p^{-\frac{n}{2}}\det(W_n) \in \mathbb{Z}$. By Theorem 1 of [10], if $p \equiv 3 \pmod 4$, then $h(-p)|h_p^-$ which implies that $h(-p)|p^{-\frac{n}{2}}\det(W_n)$. □

Let $p = 2n+1$ be a prime number. Let $S_n = \left(\sin\frac{jk\pi}{2n+1}\right)_{1\leqslant j,k\leqslant n}$. Let t_n be the n-th term of the sequence [6], which is the determinant of a matrix $T_n = (t_{jk})_{1\leqslant j,k\leqslant n}$ with entries among $0, \pm 1$ such that

$$2\sum_{k=1}^{n} t_{jk}\sin\frac{k\pi}{2n+1} = \tan\frac{j\pi}{2n+1},\ 1 \leqslant j \leqslant n. \tag{3.1}$$

The existence of T_n is assured for any positive integer n, e.g.,

$$T_2 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, T_4 = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{3.2}$$

In fact, one can pick T_n such that its entries are among ± 1, e.g., T_4 can be replaced with the following matrix,

$$\widetilde{T_4} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

for $\sin\frac{\pi}{9} + \sin\frac{2\pi}{9} = \sin\frac{4\pi}{9}$.

Lemma 3.6. *There is an integer matrix $T_n = (t_{jk})_{1\leqslant j,k\leqslant n}$ and a diagonal matrix $D = \mathrm{diag}(1,-1,1,-1,\cdots)$ such that*

$$2T_nS_n = W_nD, \tag{3.3}$$

where $t_{jk} = \pm 1$ and $t_{nj} = t_{jn} = 1$ for any $1 \leqslant j \leqslant n$. Furthermore,

$$\det T_n = \begin{cases} p^{-\frac{n}{2}}\det W_n, & \text{if } n \equiv 3 \pmod 4; \\ -p^{-\frac{n}{2}}\det W_n, & \text{otherwise.} \end{cases}$$

Proof. Let $\zeta_{4n+2} = e^{\frac{\pi i}{2n+1}}$ be a primitive $(4n+2)$-th root of unity. Then

$$\zeta_{4n+2} + \zeta_{4n+2}^2 + \cdots + \zeta_{4n+2}^{2n} = \frac{1+\zeta_{4n+2}}{1-\zeta_{4n+2}} = i\tan\frac{n\pi}{2n+1}. \tag{3.4}$$

By changing ζ_{4n+2} to ζ_{4n+2}^k $(1 \leqslant k < 2n, 2 \nmid k)$ in (3.4), we get

$$\zeta_{4n+2}^k + \zeta_{4n+2}^{2k} + \cdots + \zeta_{4n+2}^{2kn} = \frac{1+\zeta_{4n+2}^k}{1-\zeta_{4n+2}^k} = i\tan\frac{(n-\frac{k-1}{2})\pi}{2n+1}. \tag{3.5}$$

Note that if $2|k$, then $\zeta_{4n+2}^k + \zeta_{4n+2}^{2k} + \cdots + \zeta_{4n+2}^{2kn} = -1$. For $1 \leqslant k < 2n-1$ and $2 \nmid k$, by putting $k = 1, 3, \cdots, 2n-1$ in (3.5), we have

$$\begin{aligned} \sum_{k=1}^{2n} \sin\frac{k\pi}{2n+1} &= \tan\frac{n\pi}{2n+1}, \\ \sum_{k=1}^{2n} \sin\frac{3k\pi}{2n+1} &= \tan\frac{(n-1)\pi}{2n+1}, \\ &\vdots \qquad\qquad \vdots \\ \sum_{k=1}^{2n} \sin\frac{(2n-1)k\pi}{2n+1} &= \tan\frac{\pi}{2n+1}. \end{aligned} \tag{3.6}$$

Since for any $1 \leqslant j \leqslant 2n, 1 \leqslant k \leqslant 2n$, $\sin\frac{jk\pi}{2n+1} = \pm\sin\frac{\ell\pi}{2n+1}$ for some $\ell \leqslant n$, there are integers $t'_{jk} \in \{1, -1\}$, $(1 \leqslant j, k \leqslant 2n)$ such that

$$2\begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} \begin{pmatrix} \sin\frac{1\pi}{2n+1} \\ \vdots \\ \sin\frac{n\pi}{2n+1} \end{pmatrix} = \begin{pmatrix} \tan\frac{1\pi}{2n+1} \\ \vdots \\ \tan\frac{n\pi}{2n+1} \end{pmatrix}. \tag{3.7}$$

Hence we have the following identity,

$$\sum_{k=1}^{n} t_{jk}\left(\zeta_{4n+2}^k - \zeta_{4n+2}^{-k}\right) = \tan\frac{j\pi}{2n+1}i = \frac{\zeta_{4n+2}^{2j}-1}{\zeta_{4n+2}^{2j}+1}. \tag{3.8}$$

By changing ζ_{4n+2} to ζ_{4n+2}^ℓ $(1 \leqslant \ell < 2n, 2 \nmid \ell)$ in (3.8), we get

$$2\sum_{k=1}^{n} t_{jk}\sin\frac{k\ell\pi}{2n+1} = \tan\frac{j\ell\pi}{2n+1}. \tag{3.9}$$

Let $D = \operatorname{diag}(1, -1, 1, -1\cdots, (-1)^{n-1})$ be a diagonal $n \times n$ matrix and $S'_n = \left(d_{ij}\sin\frac{ij\pi}{2n+1}\right)$, where d_{ij} is defined in (1.5). One can see that

$$2T_nS'_n = W_nD.$$

Hence we have the following identity

$$2^n \det(T_n)\det(S'_n) = \begin{cases} (-1)^{\frac{n}{2}} \det(W_n), & \text{if } \ n \text{ is even ;} \\ (-1)^{\frac{n-1}{2}} \det(W_n), & \text{if } \ n \text{ is odd .} \end{cases} \tag{3.10}$$

By (1.6), (2.1) and (3.10), we have

$$\det T_n = \begin{cases} p^{-\frac{n}{2}} \det W_n, & \text{if } \ n \equiv 3 \pmod 4; \\ -p^{-\frac{n}{2}} \det W_n, & \text{otherwise.} \quad \square \end{cases}$$

Declaration of competing interest

The author declares that there is no competing interest.

Data availability

No data was used for the research described in the article.

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