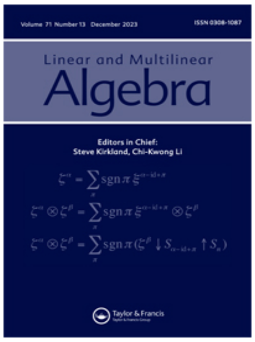

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On determinants involving tangent functions

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ABSTRACT
In this paper, we study certain determinants involving tangent functions. We prove that for any odd number $n \geqslant 3$ and $s \in \mathbb{Z}^+$, the determinant $D(n,s) = \det\left(\tan^s \pi \frac{j+k}{n}\right)_{1 \leqslant j,k \leqslant n-1} \in n^{n-2}\mathbb{Z}$. In the special case $s = 2$, we confirm a conjecture raised by Zhi-Wei Sun in 2019.

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1. Introduction

In [1,2], Zhi-Wei Sun did systematic research on determinants and permanents of matrices involving trigonometric functions. Sun found many very interesting formulas on determinants and permanents. He also made a lot of conjectures.

Some of his conjectures are connected with arithmetic objects, e.g. class numbers, Bernoulli numbers, etc. For example, Sun's Conjecture 5.1 in [1] states that if p is an odd prime, then

$$\left(\frac{-2}{p}\right) \frac{\det\left(\cot \frac{jk\pi}{p}\right)_{1 \leqslant j,k \leqslant \frac{p-1}{2}}}{2^{\frac{p-3}{2}} p^{\frac{p-5}{4}}} \in \{1, 2, 3, \ldots\},$$

and this number is divisible by $h(-p)$ if $p \equiv 3 \pmod 4$, where $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$. This conjecture is proved by the second author in [3]. Some of Sun's determinants have closed form, such as (cf. [2])

$$\det\left(\tan^2 \pi \frac{j-k}{n}\right)_{1 \leqslant j,k \leqslant n} = (n-1)n^{n-2} \cdot (n!!)^2. \tag{1}$$

However most of Sun's determinants are difficult to calculate.

In this paper, we will study the following conjecture.

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Conjecture 1.1 ([1], Conjecture 5.3): *For any odd number $n \geqslant 3$, the determinant*

$$\det\left(\tan^2 \pi \frac{j+k}{n}\right)_{1\leqslant j,k\leqslant n-1} \in n^{n-2}\mathbb{Z}.$$

We will prove that Conjecture 1.1 is true and generalize it to any power of the tangent function. For any odd $n \geqslant 3$ and $s \in \mathbb{Z}^+$, we define

$$D(n,s) = \det\left(\tan^s \pi \frac{j+k}{n}\right)_{1\leqslant j,k\leqslant n-1}.$$

The main result of this paper is the following theorem.

Theorem 1.2: *For any odd number $n \geqslant 3$ and $s \in \mathbb{Z}^+$, we have $D(n,s) \in n^{n-2}\mathbb{Z}$.*

Remark 1.3: It is calculated in [1] that $D(n,1) = (-1)^{\frac{n-1}{2}} n^{n-2}$. But for $s \geqslant 2$, there does not seem to be a similar simple formula. Since, for example, calculated by a computer

$$\begin{aligned}
&D(3,2) = 3^2, \quad D(5,2) = 3\cdot 5^3\cdot 19, \quad D(7,2) = 3^2\cdot 5\cdot 7^5\cdot 47,\\
&D(9,2) = 3^{15}\cdot 5\cdot 7\cdot 1321, \quad D(11,2) = 3^4\cdot 5^2\cdot 7\cdot 11^9\cdot 23\cdot 43,\\
&D(13,2) = 3^5\cdot 5\cdot 7\cdot 11\cdot 13^{11}\cdot 29\cdot 751,\\
&D(15,2) = 3^{17}\cdot 5^{14}\cdot 7^2\cdot 11\cdot 13\cdot 141481,\\
&\qquad\cdots\\
&D(41,2) = 3^{17}\cdot 5^9\cdot 7^5\cdot 11^3\cdot 13^3\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41^{39}\\
&\qquad\cdot 2098185082863029,\\
&D(41,3) = 5^{20}\cdot 17^4\cdot 19^4\cdot 23^2\cdot 31^4\cdot 41^{41}\cdot 47^2\cdot 59^2\cdot 79^2\cdot 139^2\cdot 151^2\\
&\qquad\cdot 163^2\cdot 167^2\cdot 227^2\cdot 359^2\cdot 419^2\cdot 619^2\cdot 659^2\cdot 727^2\cdot 827^2\cdot 839^2,\\
&D(41,4) = 3^{15}\cdot 5^{16}\cdot 7^{18}\cdot 11^3\cdot 13^9\cdot 17\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37^3\cdot 41^{40}\cdot 47\\
&\qquad\cdot 97\cdot 113\cdot 127\cdot 167\cdot 233\cdot 251\cdot 277\cdot 359\cdot 751\cdot 811\cdot 839\\
&\qquad\cdot 1973\cdot 2153\cdot 2297\cdot 2477\cdot 9203\cdot 245547995728549357\\
&\qquad\cdot 150728723003686382004465019251988977507 1.
\end{aligned}$$

There seems to be a random large prime factor as n increases.

In the following sections, we will introduce the group determinant and study the eigenvalues associated with the above determinant.

In addition to Theorem 1.2, we also make the following conjecture.

Conjecture 1.4: *If s is even and n has at least two distinct prime factors, then $D(n,s) \in n^{n-1}\mathbb{Z}$.*

Remark 1.5: We have verified the above conjecture for $s = 2, n < 400$ and $s = 4, n < 200$.

2. Group determinant

Let G be a finite abelian group and $f : G \to \mathbb{C}$ be a function from G to $\mathbb{C}$. There is a classical method to calculate the group determinant $\det(f(ab^{-1}))_{a,b\in G}$ (cf. [4, Lemma 5.26]). Let ζ be the primitive nth root of unity $e^{\frac{2\pi \mathrm{i}}{n}}$ and consider the multiplicative subgroup of $\mathbb{C} \setminus \{0\}$, $G = \{1, \zeta, \dots, \zeta^{n-1}\}$. For $s \in \mathbb{Z}^+$, define $f_s : G \to \mathbb{C}$ by $f_s(\zeta^k) = \tan^s(\frac{k\pi}{n}) = (-\mathrm{i}\frac{\zeta^k-1}{\zeta^k+1})^s$. The n-dimensional $\mathbb{C}$-linear space

$$V = \{h | h : G \to \mathbb{C} \text{ is function}\}$$

has two bases $\alpha_0, \dots, \alpha_{n-1}$ and $\chi_0, \dots, \chi_{n-1}$, where α_j is the characteristic function of $\{\zeta^j\} \subset G$, and $\chi_k : G \to \mathbb{C}^*$ is the group homomorphism defined by $\chi_k(\zeta) = \zeta^k$. Consider the $\mathbb{C}$-linear map

$$\Phi = \sum_{k=0}^{n-1} f_s(\zeta^k)\sigma_k : V \to V, \tag{2}$$

where $\sigma_k \in \mathrm{End}_{\mathbb{C}}(V)$ is defined by

$$\begin{aligned} \sigma_k : V &\longrightarrow V, \\ g(x) &\longmapsto g(\zeta^k x). \end{aligned}$$

By Lemma 5.26 of [4], the matrices of Φ under those two bases are

$$A = (f_s(\zeta^{j-k}))_{0\leqslant j,k\leqslant n-1}$$

and

$$B = \mathrm{diag}\left\{\sum_{k=0}^{n-1} \chi_0(\zeta^k)f_s(\zeta^k), \dots, \sum_{k=0}^{n-1} \chi_{n-1}(\zeta^k)f_s(\zeta^k)\right\}.$$

In other words, the eigenvalues of A are

$$\lambda_{s,j} = \sum_{k=0}^{n-1} \chi_j(\zeta^k)f_s(\zeta^k), \quad j = 0, \dots, n-1.$$

Since the transition matrix from $\alpha_0, \dots, \alpha_{n-1}$ to $\chi_0, \dots, \chi_{n-1}$ is the Vandermonde matrix

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{n-1} & \cdots & \zeta^{(n-1)^2} \end{pmatrix},$$

we have $A = PBP^{-1}$.

It is easy to see that $D(n,s)$ is equal to $\pm A(\begin{smallmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{smallmatrix})$, where

$$A\begin{pmatrix} j_1 & \cdots & j_r \\ k_1 & \cdots & k_r \end{pmatrix}$$

is the determinant of the $r \times r$ submatrix of A obtained by choosing the j_1th, $\dots, j_r$th rows and the k_1th, $\dots, k_r$th columns of A. Also, we will use

$$A\begin{bmatrix} j_1 & \cdots & j_r \\ k_1 & \cdots & k_r \end{bmatrix}$$

to denote the $r \times r$ submatrix of A obtained by choosing the j_1th, $\dots, j_r$th rows and the k_1th, $\dots, k_r$th columns of A.

3. The eigenvalues $\lambda_{s,j}$ and the proof of Theorem 1.2

In this section, we study the eigenvalues $\lambda_{s,j}$. Assume that $h(x)$ is a monic polynomial of degree n and $t_1, \dots, t_n$ are the roots of $h(x)$, define

$$h(x) = (x - t_1)\cdots(x - t_n) := x^n - a_1 x^{n-1} + a_2 x^{n-2} + \cdots + (-1)^n a_n.$$

For $k \geqslant 0$, define

$$s_k = \begin{cases} n, & \text{if } k = 0, \\ \sum_{j=1}^{n} t_j^k, & \text{if } k \geqslant 1. \end{cases}$$

We can recursively compute s_k via the a_j's by the well-known

Theorem 3.1 (Newton's identities): *With notations as above, if* $1 \leqslant k \leqslant n-1$, *then*

$$s_k - s_{k-1}a_1 + s_{k-2}a_2 - \cdots + (-1)^{k-1}s_1 a_{k-1} + (-1)^k k a_k = 0;$$

if $k \geqslant n$, *then*

$$s_k - s_{k-1}a_1 + s_{k-2}a_2 - \cdots + (-1)^n s_{k-n}a_n = 0.$$

Proofs of this theorem can be found in linear algebra textbooks. Now, we can prove

Lemma 3.2: *The first eigenvalue* $\lambda_{s,0} = \sum_{k=0}^{n-1} \tan^s(\frac{k\pi}{n})$ *of* A *in Section 2 is an integer divisible by* n.

Proof: Since

$$\tan nx = \frac{\tan^n x - \binom{n}{n-2}\tan^{n-2} x + \cdots + (-1)^{\frac{n-1}{2}} n \tan x}{\binom{n}{n-1}\tan^{n-1} x - \binom{n}{n-3}\tan^{n-3} x + \cdots + (-1)^{\frac{n-1}{2}}}$$

and $0, \frac{\pi}{n}, \dots, \frac{(n-1)\pi}{n}$ are the roots of $\tan nx = 0$, denoting $\tan\frac{k\pi}{n}$ by t_k, it is easy to see that $t_0, t_1, \dots, t_{n-1}$ are the distinct roots of the polynomial

$$x^n - \binom{n}{n-2}x^{n-2} + \cdots + (-1)^{\frac{n-3}{2}}\binom{n}{3}x^3 + (-1)^{\frac{n-1}{2}} n x$$

$$:= x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n.$$

Since $\lambda_{s,0} = \sum_{k=0}^{n-1} t_k^s$, by Theorem 3.1, we have

$$\lambda_{s,0} - a_1\lambda_{s-1,0} + a_2\lambda_{s-2,0} - \cdots + (-1)^{s-1}a_{s-1}\lambda_{1,0} + (-1)^s s a_s = 0, \quad \text{if } s \leqslant n-1,$$
$$\lambda_{s,0} - a_1\lambda_{s-1,0} + a_2\lambda_{s-2,0} - \cdots + (-1)^n a_n\lambda_{s-n,0} = 0, \quad \text{if } s \geqslant n,$$

where $\lambda_{0,0} = n$ by the definition of s_0. For $1 \leqslant s \leqslant n-1$, it is easy to see that

$$sa_s = \begin{cases} -s\binom{n}{s}, & \text{if } s \text{ is even}, \\ 0, & \text{if } s \text{ is odd}. \end{cases}$$

Since $s\binom{n}{s} = n\binom{n-1}{s-1}$, thus $sa_s \in n\mathbb{Z}$ for $1 \leqslant s \leqslant n-1$. By using the two equations above and induction on s, we have $\lambda_{s,0} \in n\mathbb{Z}$. ∎

Recalling that the Chebyshev polynomials are two sequences of polynomials given by $T_k(\cos\theta) = \cos k\theta$, $W_k(\cos\theta)\sin\theta = \sin(k+1)\theta$, $k = 0, 1, 2, \ldots$, one can write down the recurrence relations for T_{k+1} and W_{k+1}:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \tag{3}$$
$$W_0(x) = 1, \quad W_1(x) = 2x, \quad W_{k+1}(x) = 2xW_k(x) - W_{k-1}(x). \tag{4}$$

Let $x_k = \cos\frac{k\pi}{n}$. Then

$$\begin{aligned}\lambda_{s,j} &= \sum_{k=0}^{n-1} \zeta^{jk}\left(\tan\frac{k\pi}{n}\right)^s \\ &= \sum_{k=0}^{n-1}\left(\cos\frac{2kj\pi}{n} + \mathrm{i}\sin\frac{2kj\pi}{n}\right)\left(\tan\frac{k\pi}{n}\right)^s \\ &= \begin{cases} i\sum_{k=0}^{n-1} W_{2j-1}(x_k)\sin\frac{k\pi}{n}\left(\tan\frac{k\pi}{n}\right)^s, & \text{if } s \text{ is odd}, \\ \sum_{k=0}^{n-1} T_{2j}(x_k)\left(\tan\frac{k\pi}{n}\right)^s, & \text{if } s \text{ is even}. \end{cases}\end{aligned} \tag{5}$$

If $s \geqslant 2$ is an even number, then by the recurrence relations (3), we have

$$\begin{aligned}\lambda_{s,j} &= \sum_{k=0}^{n-1} T_{2j}(x_k)\left(\tan\frac{k\pi}{n}\right)^s \\ &= \sum_{k=0}^{n-1}\left(2x_kT_{2j-1}(x_k) - T_{2j-2}(x_k)\right)\left(\tan\frac{k\pi}{n}\right)^s \\ &= \sum_{k=0}^{n-1}\left((2x_k)^{2j-1}T_1(x_k) - \sum_{l=0}^{2j-2}(2x_k)^{2j-2-l}T_l(x_k)\right)\left(\tan\frac{k\pi}{n}\right)^s.\end{aligned} \tag{6}$$

Lemma 3.3: *Let* $G_s(j,h) = \sum_{k=0}^{n-1}(2x_k)^j T_h(x_k)(\tan\frac{k\pi}{n})^s$. *Then* $G_s(2j,0) = 2^{2j}\sum_{k=0}^{n-1}\cos^{2j-s}\frac{k\pi}{n}\sin^s\frac{k\pi}{n} \in 2n\mathbb{Z}$ *for* $j = 1,\ldots,n-1$, *where* $s \geqslant 0$ *is an even number.*

Proof: We prove this lemma by induction on s, the case $s = 0$ can be calculated directly. Let $z_k = e^{\mathrm{i}\frac{k\pi}{n}}$. Then

$$\begin{aligned}
G_0(2j,0) &= 2^{2j}\sum_{k=0}^{n-1}\cos^{2j}\frac{k\pi}{n} = 2^{2j}\cdot\left(\frac{1}{2}\right)^{2j}\sum_{k=0}^{n-1}\left(z_k+\frac{1}{z_k}\right)^{2j}\\
&= \sum_{k=0}^{n-1}\Big(z_k^{2j}+\tbinom{2j}{1}z_k^{2j-2}+\tbinom{2j}{2}z_k^{2j-4}\cdots+\tbinom{2j}{j-1}z_k^{2}\\
&\quad+\tbinom{2j}{j}+\tbinom{2j}{j+1}z_k^{-2}+\cdots+\tbinom{2j}{2j-1}z_k^{2-2j}+z_k^{-2j}\Big)\\
&= 0+\cdots+0+\tbinom{2j}{j}n+0+\cdots+0\\
&= 2\tbinom{2j-1}{j-1}n.
\end{aligned}$$

Let $s \geqslant 2$. If $G_k(2j,0) \in 2n\mathbb{Z}$ for $k = 0,\ldots,s-2$, then we have to show that $G_s(2j,0) \in 2n\mathbb{Z}$ for $j = 1,\ldots,n-1$. When $j = 1$,

$$\begin{aligned}
G_s(2,0) &= 4\sum_{k=0}^{n-1}\cos^{2-s}\frac{k\pi}{n}\sin^s\frac{k\pi}{n}\\
&= 4\sum_{k=0}^{n-1}\cos^{2-s}\frac{k\pi}{n}\left(1-\cos^2\frac{k\pi}{n}\right)\sin^{s-2}\frac{k\pi}{n}\\
&= 4\left(\sum_{k=0}^{n-1}\tan^{s-2}\frac{k\pi}{n}-\sum_{k=0}^{n-1}\cos^{2-(s-2)}\frac{k\pi}{n}\sin^{s-2}\frac{k\pi}{n}\right)\\
&= 4\lambda_{s-2,0}-G_{s-2}(2,0).
\end{aligned}$$

Thus, $G_s(2,0) \in 2n\mathbb{Z}$ by Lemma 3.2 and induction hypothesis. On the other hand, when $j \geqslant 1$, similar to the above calculation, $G_s(2j,0) = 4G_{s-2}(2j-2,0) - G_{s-2}(2j,0) \in 2n\mathbb{Z}$. ∎

Lemma 3.4: *For* $j = 0,\ldots,n-1$,

$$\lambda_{s,j} \in \begin{cases} \mathrm{i}n\mathbb{Z}, & \text{if } s \text{ is odd,}\\ n\mathbb{Z}, & \text{if } s \text{ is even.}\end{cases}$$

Proof: We will only prove this lemma when s is even, and the case when s is odd is similar (need to use recurrence relations 4 of $W_{k+1}(x)$ instead). Now assume $s \geqslant 2$ is even. Since we have proved that $\lambda_{s,0} \in n\mathbb{Z}$ in Lemma 3.2, it is sufficient to show $\lambda_{s,j} = G_s(0,2j) \in n\mathbb{Z}$ for $j = 1,\ldots,n-1$. In fact, we will prove that $G_s(2j-l,l) \in n\mathbb{Z}$ for all $j = 1,\ldots,n-1$ and $l = 0,1,\ldots,2j$.

If $j = 1$, according to (6) and Lemma 3.3, we find that

$$\lambda_{s,1} = G_s(0,2) = G_s(1,1) - G_s(0,0) = \frac{1}{2}G_s(2,0) - \lambda_{s,0} \in n\mathbb{Z}.$$

Thus we have $G_s(0,2), G_s(1,1), G_s(2,0) \in n\mathbb{Z}$. Assume that we have proved that $G_s(2j-l,l) \in n\mathbb{Z}, l = 0,1,\ldots,2j$ for all $1 \leqslant j < r \leqslant n-1$, let us prove the case when $j = r$. By using the recurrence relations (3) repeatedly, we have

$$\begin{aligned} G_s(0,2r) &= G_s(1,2r-1) - G_s(0,2j-2) \\ &= G_s(2,2j-2) - G_s(1,2j-3) - G_s(0,2j-2) \\ &\ \ \vdots \\ &= G_s(2r-1,1) - \sum_{l=0}^{2r-2} G_s(2r-2-l,l) \\ &= \frac{1}{2}G_s(2r,0) - \sum_{l=0}^{2r-2} G_s(2r-2-l,l), \end{aligned}$$

thus $G_s(0,2r), G_s(1,2r-1),\ldots,G_s(2r,0) \in n\mathbb{Z}$ by Lemma 3.3 and inductive hypothesis ■

Let $c_1,\ldots,c_n$ be n distinct numbers and consider the Vandermonde matrix

$$J = \begin{pmatrix} 1 & c_1 & \cdots & c_1^{n-1} \\ 1 & c_2 & \cdots & c_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & \cdots & c_n^{n-1} \end{pmatrix}.$$

Since each leading principal minors of J is not equal to zero, we have the LU decomposition $J = LU$, where L is a lower triangular matrix and U is an upper triangular matrix. Generally this decomposition is not unique. For the inverse of L and U, we have

Lemma 3.5 (cf. [5]): *There is an LU decomposition of J with*

$$U^{-1} = (u_{i,j}) = \begin{pmatrix} 1 & -c_1 & c_1c_2 & -c_1c_2c_3 & \cdots \\ 0 & 1 & -(c_1+c_2) & c_1c_2+c_2c_3+c_3c_1 & \cdots \\ 0 & 0 & 1 & -(c_1+c_2+c_3) & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$L^{-1} = (l_{i,j}) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{c_1-c_2} & \frac{1}{c_2-c_1} & 0 & \cdots \\ \frac{1}{(c_1-c_2)(c_1-c_3)} & \frac{1}{(c_2-c_1)(c_2-c_3)} & \frac{1}{(c_3-c_1)(c_3-c_2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

more precisely

$$u_{i,i}=1,\quad u_{i,1}=0,\quad u_{i,j}=u_{i-1,j-1}-u_{i,j-1}c_{j-1}\quad \text{otherwise, where } u_{0,j}:=0;$$

$$l_{i,j}=0\quad \text{if } i<j,\ l_{1,1}=1,\ l_{i,j}=\prod_{k=1,k\neq j}^{i}\frac{1}{c_j-c_k}\ \text{otherwise.}$$

Recall from Section 2 that $A = PBP^{-1}$, where

$$P=\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & \zeta & \cdots & \zeta^{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \zeta^{n-1} & \cdots & \zeta^{(n-1)^2}\end{pmatrix}.$$

Now, applying Lemma 3.5, we can write P^{-1} as $P^{-1}=U^{-1}L^{-1}$, where

$$U^{-1}=\begin{pmatrix}1 & -1 & \zeta & \cdots\\ 0 & 1 & -(1+\zeta) & \cdots\\ 0 & 0 & 1 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix},$$

$$L^{-1}=\begin{pmatrix}1 & 0 & 0 & \cdots\\ \dfrac{1}{1-\zeta} & \frac{1}{\zeta-1} & 0 & \cdots\\ \dfrac{1}{(1-\zeta)(1-\zeta^2)} & \frac{1}{(\zeta-1)(\zeta-\zeta^2)} & \frac{1}{(\zeta^2-1)(\zeta^2-\zeta)} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

If we denote $PBU^{-1}=M$, then $A=ML^{-1}$, then by the Cauchy–Binet formula

$$\begin{aligned}A\begin{pmatrix}2 & \cdots & n\\ 2 & \cdots & n\end{pmatrix} &= (ML^{-1})\begin{pmatrix}2 & \cdots & n\\ 2 & \cdots & n\end{pmatrix}\\ &= \sum_{1\leqslant k_2<\cdots<k_n\leqslant n} M\begin{pmatrix}2 & \cdots & n\\ k_2 & \cdots & k_n\end{pmatrix}L^{-1}\begin{pmatrix}k_2 & \cdots & k_n\\ 2 & \cdots & n\end{pmatrix}\\ &= M\begin{pmatrix}2 & \cdots & n\\ 2 & \cdots & n\end{pmatrix}L^{-1}\begin{pmatrix}2 & \cdots & n\\ 2 & \cdots & n\end{pmatrix}\\ &= \frac{1}{\det P}M\begin{pmatrix}2 & \cdots & n\\ 2 & \cdots & n\end{pmatrix}.\end{aligned}\tag{7}$$

Notice that $M[{}^{2\ \cdots\ n}_{2\ \cdots\ n}]$ is the product of

$$S=\begin{pmatrix}\lambda_{s,0} & \lambda_{s,1}\zeta & \cdots & \lambda_{s,n-1}\zeta^{n-1}\\ \lambda_{s,0} & \lambda_{s,1}\zeta^2 & \cdots & \lambda_{s,n-1}\zeta^{2(n-1)}\\ \vdots & \vdots & \vdots & \ddots\\ \lambda_{s,0} & \lambda_{s,1}\zeta^{n-1} & \cdots & \lambda_{s,n-1}\zeta^{(n-1)^2}\end{pmatrix}_{(n-1)\times n}$$

and

$$T=\begin{pmatrix} -1 & \zeta & -\zeta^3 & \cdots \\ 1 & -(1+\zeta) & \zeta+\zeta^2+\zeta^3 & \cdots \\ 0 & 1 & -(1+\zeta+\zeta^2) & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{n\times(n-1)}.$$

Using the Cauchy–Binet formula again to $M\binom{2\ \cdots\ n}{2\ \cdots\ n}$, we have

$$\begin{aligned} M\begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} &= (ST)\begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \\ &= \sum_{1\leqslant j_1<\cdots<j_{n-1}\leqslant n} S\begin{pmatrix} 1 & \cdots & n-1 \\ j_1 & \cdots & j_{n-1} \end{pmatrix} T\begin{pmatrix} j_1 & \cdots & j_{n-1} \\ 1 & \cdots & n-1 \end{pmatrix}, \end{aligned} \tag{8}$$

It is obvious that $T\binom{j_1\ \cdots\ j_{n-1}}{1\ \cdots\ n-1} \in \{-1,1\}$ for any $1 \leqslant j_1 < \cdots < j_{n-1} \leqslant n$. In the following, we calculate $S\binom{1\ \cdots\ n-1}{j_1\ \cdots\ j_{n-1}}$. Let

$$N = N(c_1,\dots,c_m) = \begin{pmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{m-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{m-1} \end{pmatrix}$$

be the Vandermonde matrix with respect to $c_1,\dots,c_m$. Let

$$R_k := R_k(c_1,\dots,c_m) = \sum_{1\leqslant j_1<\cdots<j_k\leqslant m} c_{j_1}c_{j_2}\cdots c_{j_k}, \quad \text{for} \quad 1\leqslant k\leqslant m,$$

$$R_0 := R_0(c_1,\dots,c_m) = 1.$$

Lemma 3.6 ([6], Lemma 2.1): *For $j \in \{0,1,\dots,m\}$, the determinant of*

$$N_j(c_1,c_2,\dots,c_m) = \begin{pmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{j-1} & c_1^{j+1} & \cdots & c_1^m \\ 1 & c_2 & c_2^2 & \cdots & c_2^{j-1} & c_2^{j+1} & \cdots & c_2^m \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{j-1} & c_m^{j+1} & \cdots & c_m^m \end{pmatrix}$$

is given by $\det(N_j(c_1,c_2,\dots,c_m)) = R_{m-j}\det(N) = R_{m-j}\prod_{1\leqslant k<i\leqslant m}(c_i-c_k)$.

Proof: According to Lemma 3.6, we can calculate

$$\begin{aligned} S\begin{pmatrix} 1 & \cdots & n-1 \\ j_1 & \cdots & j_{n-1} \end{pmatrix} &= \lambda_{s,j_1-1}\lambda_{s,j_2-1}\cdots\lambda_{s,j_{n-1}-1}\det(N_{j'-1}(\zeta,\dots,\zeta^{n-1})) \\ &= \pm\lambda_{s,j_1-1}\lambda_{s,j_2-1}\cdots\lambda_{s,j_{n-1}-1}\det(N(\zeta,\dots,\zeta^{n-1})) \\ &= \pm\lambda_{s,j_1-1}\lambda_{s,j_2-1}\cdots\lambda_{s,j_{n-1}-1}\frac{\det P}{n}, \end{aligned}$$

where j' is the only element in $\{1, \ldots, n\} \setminus \{j_1, \ldots, j_{n-1}\}$. Thus, by Lemma 3.4,

$$S\begin{pmatrix} 1 & \cdots & n-1 \\ j_1 & \cdots & j_{n-1} \end{pmatrix} \in n^{n-2}(\det P)\mathbb{Z}. \tag{9}$$

Combining (7), (8), (9) and the fact that $T\left(\begin{smallmatrix} j_1 & \cdots & j_{n-1} \\ 1 & \cdots & n-1 \end{smallmatrix}\right) \in \{-1, 1\}$, we get

$$A\begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \in n^{n-2}\mathbb{Z}.$$

■

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