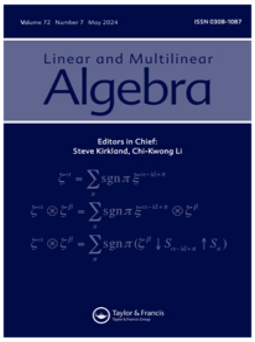

Linear and Multilinear Algebra

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/glma20

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To cite this article: Xuejun Guo, Xin Li, Zhengyu Tao & Tao Wei (2024) The eigenvectors-eigenvalues identity and Sun's conjectures on determinants and permanents, Linear and Multilinear Algebra, 72:7, 1071-1077, DOI: 10.1080/03081087.2023.2172380

To link to this article: https://doi.org/10.1080/03081087.2023.2172380

LINEAR AND MULTILINEAR ALGEBRA
2024, VOL. 72, NO. 7, 1071–1077
https://doi.org/10.1080/03081087.2023.2172380

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The eigenvectors-eigenvalues identity and Sun's conjectures on determinants and permanents

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ABSTRACT
In this paper, we prove several conjectures raised by Zhi-Wei Sun on determinants and permanents by the eigenvectors-eigenvalues identity recently highlighted by Denton, Parke, Tao and Zhang.

ARTICLE HISTORY
Received 15 July 2022
Accepted 12 September 2022

COMMUNICATED BY
B. Kuzma

KEYWORDS
Eigenvectors-eigenvalues identity; determinants; permanents

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
Primary 15A15; 11C20; Secondary 05A19; 11A07; 33B10

1. Introduction

In 2018, Zhi-Wei Sun began a systematical research on permanents and determinants of matrices in number theory. He found many identities with rich arithmetic meanings and also raised many open problems. His results are in [1–4]. One of Sun's conjectures is the following one which was firstly raised in 2018.

Conjecture 1.1: *[Zhi-Wei Sun, Conjecture 4.3 of [4]] Let $n > 1$ be an integer and $\zeta = e^{\frac{2\pi i}{n}}$.*

(1) *If n is an even number, then*

$$\sum_{\tau\in D(n)}\prod_{j=1}^{n}\frac{1}{1-\zeta^{j-\tau(j)}}=\frac{((n-1)!!)^2}{2^n}=\frac{n!}{4^n}\binom{n}{n/2}; \tag{1}$$

(2) *if n is an odd number, then*

$$\sum_{\tau\in D(n-1)}\prod_{j=1}^{n-1}\frac{1}{1-\zeta^{j-\tau(j)}}=\frac{1}{n}\left(\frac{n-1}{2}!\right)^2, \tag{2}$$

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and

$$\sum_{\tau\in D(n-1)} \operatorname{sign}(\tau)\prod_{j=1}^{n-1}\frac{1}{1-\zeta^{j-\tau(j)}} = \frac{(-1)^{\frac{n-1}{2}}}{n}\left(\frac{n-1}{2}!\right)^2, \tag{3}$$

where $D(m)$ is the set of all derangements τ of indices $j = 1,\dots,m$ such that $\tau(j)\neq j$ for all $j = 1,\dots,m$.

The left-hand side of (1) can be viewed as the permanent of the matrix

$$B = \left((1-\delta_{jk})\left\{\frac{1}{1-\zeta^{j-k}}\right\}\right)_{n\times n}, \tag{4}$$

where

$$\delta_{jk} = \begin{cases}1, & \text{if } j = k,\\ 0, & \text{if } j\neq k,\end{cases} \qquad \{x\} = \begin{cases}1, & \text{if } x=\infty,\\ x, & \text{otherwise.}\end{cases}$$

The matrix B is a circulant matrix. Hence $\det(B)$ can be easily found by computing the product of the eigenvalues of B. We know from [4] that

$$\det(B) = \begin{cases}(-1)^{\frac{n}{2}}\dfrac{((n-1)!!)2}{2^n}, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd}.\end{cases} \tag{5}$$

Also, we will see in Section 2 that the left-hand side of (3) and (2) are related respectively to the determinant and the permanent of an $(n-1)\times(n-1)$ sub-matrix A of B. Note that A is no longer a circulant determinant. Hence it is very difficult to find the eigenvalues of A.

In this note we prove Conjecture 1.1 based on the eigenvector-eigenvalue identity recently discussed by Denton, Parke, Tao and Zhang [5].

2. Eigenvalues from eigenvectors

In this section, we prove identity (3).

Theorem 2.1: [*F. Calogero and A.M. Perelomov, Theorem* 1 *of* [6]] *The off-diagonal hermitian matrix of order n whose elements are defined by the formula*

$$a_{jk} = \left(1-\delta_{jk}\right)\left\{1+\mathrm{i}\cot\frac{(j-k)\pi}{n}\right\}$$

has the integer eigenvalues

$$\lambda_i = 2i-n-1, \quad i = 1,2,\dots,n$$

and the corresponding eigenvectors v_i with norm 1 *have components*

$$v_{i,j} = \exp\left(-\frac{2\pi\mathrm{i}ij}{n}\right)/\sqrt{n}, \quad j = 1,2,\dots,n.$$

If A is an $n \times n$ Hermitian matrix, then we denote its n real eigenvalues by $\lambda_1(A), \dots, \lambda_n(A)$. We can find an orthonormal basis $v_1, v_2, \dots, v_n \in \mathbb{C}^n$ such that v_i is associated with $\lambda_i(A), 1 \le i \le n$. For any $i, j = 1, \dots, n$, let $v_{i,j}$ denote the j^{th} component of v_i. For $1 \le j \le n$, let M_j denote the $(n-1) \times (n-1)$ minor formed from A by deleting the j^{th} row and column from A. The matrix M_j is also a Hermitian matrix, and thus has $n-1$ real eigenvalues $\lambda_1(M_j), \dots, \lambda_{n-1}(M_j)$.

Theorem 2.2: [*Eigenvector-eigenvalue identity, Theorem* 1 *of* [5]] *Let the notations be as above. We have*

$$\left|v_{i,j}\right|^2 \prod_{k=1;k\neq i}^{n} (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n-1} \left(\lambda_i(A) - \lambda_k\left(M_j\right)\right). \tag{6}$$

Proof of Equation (3).: Recall that in Conjecture 1.1 we set $\zeta = e^{\frac{2\pi \mathrm{i}}{n}}$. Note that

$$\begin{aligned} \frac{1}{1-\zeta^{j-k}} &= \frac{1}{2}\left(1 + \frac{1+\zeta^{j-k}}{1-\zeta^{j-k}}\right) \\ &= \frac{1}{2}\left(1 + \mathrm{i}\cot\frac{\pi(j-k)}{n}\right). \end{aligned}$$

Let $A = (A_{jk})$ be the $n \times n$ Hermitian matrix defined in Theorem 2.1. By Theorem 2.1, the eigenvalues of A are

$$\lambda_1 = n-1, \lambda_2 = n-3, \dots, \lambda_i = n+1-2i, \dots, \lambda_n = 1-n. \tag{7}$$

Let $i = \frac{n+1}{2}$ and $j = n$. Then $\lambda_i = 0$. By Theorem 2.2,

$$\left|v_{i,n}\right|^2 \prod_{k=1;k\neq i}^{n} (-\lambda_k(A)) = \prod_{k=1}^{n-1} \left(-\lambda_k\left(M_n\right)\right).$$

By Theorem 2.1, $|v_{s,n}|^2 = \frac{1}{n}$. Hence

$$|M_n| = \prod_{k=1}^{n-1} \lambda_k\left(M_n\right) = \frac{(-1)^{\frac{n-1}{2}}((n-1)!!)^2}{n},$$

which implies that

$$\begin{aligned} \sum_{\tau \in D(n-1)} \operatorname{sign}(\tau) \prod_{j=1}^{n-1} \frac{1}{1-\zeta^{j-\tau(j)}} &= 2^{1-n}|M_n| = \frac{2^{1-n}(-1)^{\frac{n-1}{2}}((n-1)!!)^2}{n} \\ &= \frac{(-1)^{\frac{n-1}{2}}}{n}\left(\frac{n-1}{2}!\right)^2, \end{aligned}$$

and Equation (3) is proved. ∎

Han Wang and Zhi-Wei Sun proved [7] another conjecture involving derangements and roots of unity by similar arguments. Keqin Liu proves (3) by a different strategy in [8]. Although the eigenvalues of A are rational integers as in Theorem 2.1, some numerical computations showed that there is no simple pattern of the eigenvalues of M_n in the proof of Equation (3). However, the eigenvalues of M_nB are very simple for certain diagonal matrices B. Let B_s the diagonal matrix whose diagonal entries are given as $1-\zeta^{is}$, $1 \le i \le n-1$, where $s \in \{-\frac{n-1}{2}, -\frac{n-3}{2}, \dots, -1, 1, 2, \dots, \frac{n-1}{2}\}$. Let $M_{n,s} = M_nB_s$. Then (3) is equivalent to

$$\det\left(M_{n,1}\right) = (-1)^{\frac{n-1}{2}}\left(\frac{n-1}{2}!\right)^2. \tag{8}$$

Liu [8] proved that the eigenvalues of $M_{n,1}$ are

$$-\frac{n-1}{2}, \dots, -1, 1, \dots, \frac{n-1}{2},$$

which can imply (3).

3. The permanents of matrices

In this section, we prove Equations (1) and (2).

Lemma 3.1: *Let $n > 2$ be an integer, and $x_1, \dots, x_n$ pairwise distinct complex numbers. Let $L(n)$ be the set of all elements $\tau \in S(n)$ such that τ is an n-cycle. Then*

$$\sum_{\tau \in L(n)} \prod_{j=1}^{n} \frac{1}{x_{\tau(j)} - x_j} = 0.$$

Proof: For $\tau_1 = (1a_2a_3\cdots a_n), \tau_2 = (1b_2b_3\cdots b_n) \in L(n)$, define $\tau_1 \sim \tau_2$ if and only if $(a_2a_3\cdots a_n) = (b_2b_3\cdots b_n) \in L(n-1)$. One can easily see that $\sim$ is an equivalence relation on $L(n)$. Obviously, there are $n-1$ elements in each equivalence class. Take an arbitrary equivalence class

$$\begin{aligned}\sigma_1 &= (1a_2a_3\cdots a_{n-1}a_n),\\ \sigma_2 &= (1a_3a_4\cdots a_na_2),\\ &\vdots\\ \sigma_{n-1} &= (1a_na_2\cdots a_{n-2}a_{n-1}).\end{aligned}$$

We just need to show that

$$\sum_{i=1}^{n-1}\prod_{j=1}^{n} \frac{1}{x_{\sigma_i(j)} - x_j} = 0.$$

Let $\alpha_i = \prod_{j=1}^{n} \frac{1}{x_{\sigma_i(j)} - x_j}$, obviously, $\alpha_i \neq 0$ and

$$\frac{\alpha_i}{\alpha_1} = \begin{cases} \dfrac{(x_1 - x_{a_2})(x_1 - x_{a_n})}{x_{a_2} - x_{a_n}}\left(\dfrac{1}{x_1 - x_{a_{i+1}}} - \dfrac{1}{x_1 - x_{a_i}}\right), & i \neq 1,\\ 1, & i = 1.\end{cases}$$

Thus

$$\sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_1} = 1 + \frac{(x_1 - x_{a_2})(x_1 - x_{a_n})}{x_{a_2} - x_{a_n}} \left(\frac{1}{x_1 - x_{a_n}} - \frac{1}{x_1 - x_{a_2}} \right) = 0,$$

which implies $\sum_{i=1}^{n-1} \alpha_i = 0$. This completes the proof of Lemma 3.1. ■

Theorem 3.2: *Let n be a positive integer, and $x_1, \ldots, x_n$ pairwise distinct complex numbers. Assume*

$$A = (a_{ij})_{n\times n} = \left((1-\delta_{ij}) \left\{ \frac{1}{x_j - x_i} \right\} \right)_{n\times n}.$$

We have

(1) *If n is odd, then*

$$\sum_{\substack{\tau\in D(n) \\ \text{sign}(\tau)=1}} \prod_{j=1}^{n} a_{j\tau(j)} = \sum_{\substack{\tau\in D(n) \\ \text{sign}(\tau)=-1}} \prod_{j=1}^{n} a_{j\tau(j)} = 0,$$

i.e. $\det(A) = \text{per}(A) = 0$;

(2) *If n is even, then*

$$\sum_{\substack{\tau\in D(n) \\ \text{sign}(\tau)=(-1)^{\frac{n}{2}+1}}} \prod_{j=1}^{n} a_{j\tau(j)} = 0,$$

i.e. $\text{per}(A) = (-1)^{\frac{n}{2}} \det(A)$.

Proof: First, we will check this theorem for $n = 1,2,3$. If $n = 1$, then $A = 0$ and (1) holds; if $n = 2$, then

$$A = \begin{pmatrix} 0 & \dfrac{1}{x_2 - x_1} \\ \dfrac{1}{x_1 - x_2} & 0 \end{pmatrix}.$$

Since there is only one odd permutation in $D(2)$, thus (2) holds; if $n = 3$, then

$$A = \begin{pmatrix} 0 & \dfrac{1}{x_2 - x_1} & \dfrac{1}{x_3 - x_1} \\ \dfrac{1}{x_1 - x_2} & 0 & \dfrac{1}{x_3 - x_2} \\ \dfrac{1}{x_1 - x_3} & \dfrac{1}{x_2 - x_3} & 0 \end{pmatrix}.$$

Since there are only two even permutations (1 2 3), (1 3 2) in $D(3)$ and

$$\frac{1}{x_2 - x_1} \cdot \frac{1}{x_3 - x_2} \cdot \frac{1}{x_1 - x_3} + \frac{1}{x_3 - x_1} \cdot \frac{1}{x_1 - x_2} \cdot \frac{1}{x_2 - x_3} = 0.$$

This implies that (1) holds.

If $n \geqslant 4$ is odd, then

$$\sum_{\substack{\tau\in D(n)\\ \mathrm{sign}(\tau)=1}} \prod_{j=1}^{n} a_{j\tau(j)} = \sum_{\substack{\tau\in D(n)\\ \mathrm{sign}(\tau)=1}} \prod_{j=1}^{n} \frac{1}{x_{\tau(j)} - x_j}.$$

For $Y = \{y_1, \dots, y_r\} \subset X = \{x_1, \dots, x_n\}$, let

$$f(Y) = \sum_{\tau\in L(r)} \prod_{j=1}^{r} \frac{1}{y_{\tau(j)} - y_j}.$$

By Lemma 3.1 we know that $f(Y) = 0$ if $|Y| > 2$. Note that

$$\sum_{\substack{\tau\in D(n)\\ \mathrm{sign}(\tau)=1}} \prod_{j=1}^{n} \frac{1}{x_{\tau(j)} - x_j} = \sum_{\substack{X=\bigsqcup_{i=1}^{s} X_i\\ |X_i|\geqslant 2\\ s \text{ odd}}} \left(\prod_{i=1}^{s} f(X_i)\right), \tag{9}$$

where the disjoint unions $X = \bigsqcup_{i=1}^{s} X_i$ in the right-hand side of (9) come form cycle decompositions of each $\tau \in D(n)$. Since n is odd, every disjoint union $X = \bigsqcup_{i=1}^{s} X_i$ has an X_i with $|X_i| > 2$, thus

$$\sum_{\substack{\tau\in D(n)\\ \mathrm{sign}(\tau)=1}} \prod_{j=1}^{n} \frac{1}{x_{\tau(j)} - x_j} = 0.$$

Similarly, it can be shown that $\sum_{\substack{\tau\in D(n)\\ \mathrm{sign}(\tau)=-1}} \prod_{j=1}^{n} \frac{1}{x_{\tau(j)}-x_j} = 0$.

If n is even, one can prove (2) in the same way by using Lemma 3.1. This complete the proof of this theorem. ■

Proof of Equations (1) and (2).: If n is even, then by Theorem 3.2(2) and (5), we have

$$\begin{aligned}\sum_{\tau\in D(n)} \prod_{j=1}^{n} \frac{1}{1-\zeta^{j-\tau(j)}} &= \mathrm{per}\left((1-\delta_{ij})\left\{\frac{1}{1-\zeta^{i-j}}\right\}\right)_{n\times n}\\ &= (-1)^{\frac{n}{2}} \det\left((1-\delta_{ij})\left\{\frac{1}{1-\zeta^{i-j}}\right\}\right)_{n\times n}\\ &= \frac{((n-1)!!)^2}{2^n}.\end{aligned}$$

If n is odd, then by Theorem 3.2(2) and (3), we have

$$\begin{aligned}\sum_{\tau\in D(n-1)} \prod_{j=1}^{n-1} \frac{1}{1-\zeta^{j-\tau(j)}} &= (-1)^{\frac{n-1}{2}}\left(\sum_{\tau\in D(n-1)} \mathrm{sign}(\tau) \prod_{j=1}^{n-1} \frac{1}{1-\zeta^{j-\tau(j)}}\right)\\ &= \frac{1}{n}\left(\frac{n-1}{2}!\right)^2.\end{aligned}$$

Thus we complete the proof of (1) and (2). ■

Acknowledgments

The authors are deeply grateful to Zhi-Wei Sun and Keqin Liu for very helpful discussions. They also would like to thank the referee for many valuable comments and suggestions.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The authors are supported by the National Nature Science Foundation of China [grant numbers 11971226, 11631009].

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References

[1] Sun Z-W. On some determinants with Legendre symbol entries. Finite Fields Appl. 2019;56:285–307.
[2] Sun Z-W. On some determinants involving the tangent function. Available from: arXiv:1901.04837.
[3] Sun Z-W. Permuations of $\{1, \ldots, n\}$ and related topics. J Algebraic Combin. in press. Available from: arXiv:1811.10503.
[4] Sun Z-W. Arithmetic properties of some permanents. Available from: arxiv:2108.07723.
[5] Denton PB, Parke SJ, Tao T, et al. Eigenvectors from eigenvalues: a survey of a basic identity in linear algebra. Bull Am Math Soc. 59(1):31–58.
[6] Calogero F, Perelomov AM. Some diophantine relations involving circular functions of rational angles. Lin Alg Appl. 1979;25:91–94.
[7] Wang H, Sun Z-W. Proof of a conjecture involving derangements and roots of unity. Available from: arxiv:2206.02589v3.
[8] Liu K. Generalization and another proof of two conjectural identities posed by Sun. Available from: arXiv:2206.05021.