MATHEMATICS OF COMPUTATION
https://doi.org/10.1090/mcom/3961
Article electronically published on March 12, 2024

CM POINTS, CLASS NUMBERS, AND THE MAHLER MEASURES OF $x^3 + y^3 + 1 - kxy$

ZHENGYU TAO AND XUEJUN GUO

ABSTRACT. We study the Mahler measures of the polynomial family $Q_k(x, y) = x^3 + y^3 + 1 - kxy$ using the method previously developed by the authors. An algorithm is implemented to search for complex multiplication points with class numbers $\leqslant 3$, we employ these points to derive interesting formulas that link the Mahler measures of $Q_k(x, y)$ to L-values of modular forms. As by-products, some conjectural identities of Samart are confirmed, one of them involves the modified Mahler measure $\tilde{n}(k)$ introduced by Samart recently. For $k = \sqrt[3]{729 \pm 405\sqrt{3}}$, we also prove an equality that expresses a 2×2 determinant with entries the Mahler measures of $Q_k(x, y)$ as some multiple of the L-value of two isogenous elliptic curves over $\mathbb{Q}(\sqrt{3})$.

1. Introduction

For any non-zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$, the (logarithmic) Mahler measure of P is defined by

$$m(P) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \log |P(e^{i\theta_1}, \cdots, e^{i\theta_n})| d\theta_1 \cdots d\theta_n.$$

Initiated by the insights of Deninger and Boyd, the relation between multivariate Mahler measures and special values of L-functions has attracted a significant amount of research. In [2], Deninger conjectured that

$$m\Big(x + \frac{1}{x} + y + \frac{1}{y} + 1\Big) \stackrel{?}{=} L'(E, 0), \tag{1.1}$$

where E is the conductor 15 elliptic curve defined by the projective closure of $x + \frac{1}{x} + y + \frac{1}{y} + 1 = 0$. Later, based on numerical experiments, Boyd [4] made similar conjectures of the form

$$m(P_k) \stackrel{?}{=} r_k L'(E_k, 0) \tag{1.2}$$

for many $k \in \mathbb{Z} - \{0, \pm 4\}$, where $P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k, r_k \in \mathbb{Q}$ and E_k is the elliptic curve associated to $P_k(x, y) = 0$. He also formulated analogous conjectures for many other families, among which is the family

$$Q_k(x, y) = x^3 + y^3 + 1 - kxy. \tag{1.3}$$

Received by the editor November 14, 2023, and, in revised form, February 5, 2024, and February 15, 2024.
2020 *Mathematics Subject Classification.* Primary 11R06, 11F67; Secondary 11Y40, 19F27.
Key words and phrases. Mahler measure, CM point, L-function, elliptic curve.
The authors were supported by NSFC 11971226 and NSFC 12231009.
The second author is the corresponding author.

Note that $Q_k(x,y)=0$ is the Hesse pencil of elliptic curves with Weierstrass model

$$C_k : Y^2 = X^3 - 27k^6X^2 + 216k^9(k^3-27)X - 432k^{12}(k^3-27)^2. \tag{1.4}$$

The rational transformation that converts C_k to the curve $Q_k(x,y)=0$ is

$$X = \frac{12k^4(k^3-27)x}{kx+3y+3}, \quad Y = \frac{108k^6(k^3-27)(y-1)}{kx+3y+3}. \tag{1.5}$$

During the period when Boyd's work appeared, motivated by mirror symmetry in physics, Rodriguez Villegas [10] represented $m(P_k)$ and $m(Q_k)$ as Kronecker-Eisenstein series. This led to the proof of (1.2) in some cases when the elliptic curves E_k and C_k *have* complex multiplication (usually abbreviated as CM). Specifically, Rodriguez Villegas proved (1.2) for the polynomials $P_{4\sqrt{2}}, P_{2\sqrt{2}}$ and Q_{-6}.

Since the curve E in (1.1) has *no* CM, Rodriguez Villegas' method doesn't work in this case. The question mark in equation (1.1) was finally removed by Rogers and Zudilin [17] nearly twenty years after Deninger made his conjecture. Their approach, now known as the Rogers-Zudilin method [9, Chapter 9], can be successfully applied to prove a number of non-CM cases of (1.2). However, the use of Rogers-Zudilin method relies heavily on the modular unit parametrization of elliptic curves and Brunault [8] proved that there are only finitely many elliptic curves over $\mathbb{Q}$ that can be parametrized by modular units. Interested readers can refer to the tables in [5,6], where the proven cases of (1.2) related to $m(P_k)$ and $m(Q_k)$ are listed (whether CM or non-CM).

Although much of the current literature focuses on the study of non-CM cases, we believe that there are still some veins to be mined in the CM cases. In our previous work [24], we proved that when τ is a CM point (i.e., imaginary quadratic numbers in the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}$), the degree of $k = k(\tau)$ as an algebraic number in Rodriguez Villegas' formula that expresses $m(P_k)$ as Kronecker-Eisenstein series can be bounded by the class number of the CM point τ. This fact together with a systematic search for CM points with class numbers $\leqslant 2$ enabled us to prove over twenty identities of the form

$$m(P_k) = \frac{r_k s_k}{\pi^2} L(f_k, 2),$$

where $r_k \in \mathbb{Q}, s_k \in \{1, \sqrt{2}, \sqrt{3}, \sqrt{7}\}$ and f_k are weight two cusp forms of levels $28, 48, 56, 64, 112,\ 128, 192, 256$ and 448. Guided by Beilinson's conjecture, we also proved 5 identities connecting L-values of CM elliptic curves over real quadratic fields to 2×2 determinants with $m(P_k)$ as entries. These identities extend the recent work [22] of Guo, Ji, Liu and Qin, in which they dealt with the case when $k = 4 \pm 4\sqrt{2}$. As an example, we provide here one of our results:

$$\left| \det \begin{pmatrix} m(P_{12+8\sqrt{2}}) & m(P_{12-8\sqrt{2}}) \\ m(P_{12-8\sqrt{2}}) & m(P_{12+8\sqrt{2}}) \end{pmatrix} \right| = \frac{1024}{\pi^4} L(E_{12\pm 8\sqrt{2}}, 2). \tag{1.6}$$

The present paper is devoted to treating the polynomial family $Q_k(x,y)$. Since a change of variables shows that $m(Q_k)$ only depends on k^3 [10, §14], in the rest of this paper, we will use the following notation introduced by Samart [7]:

$$m_3(t) := 3m(Q_{\sqrt[3]{t}}) = 3m(x^3 + y^3 + 1 - \sqrt[3]{t}xy).$$

The main difference between $P_k(x,y)$ and $Q_k(x,y)$ is that the family $P_k(x,y)$ is reciprocal while $Q_k(x,y)$ is non-reciprocal, where recall that a multivariable Laurent polynomial $P(x_1,\cdots,x_n)$ is *reciprocal* if

$$\frac{P(x_1,\cdots,x_n)}{P(1/x_1,\cdots,1/x_n)}=x_1^{b_1}\cdots x_n^{b_n}$$

for some $b_1,\cdots,b_n\in\mathbb{Z}$ and *non-reciprocal* otherwise. Let $\mathcal{K}_Q$ (resp. $\mathcal{K}_P$) be the set of $k\in\mathbb{C}$ such that $Q_k(x,y)$ (resp. $P_k(x,y)$) vanishes on $\mathbb{T}^2=\{(x,y)\in\mathbb{C}^2 \mid |x|=|y|=1\}$. As explained in [4,6], $\mathcal{K}_P\subset\mathbb{R}$ since P_k is reciprocal, in fact $\mathcal{K}_P=[-4,4]$. However, for the non-reciprocal $Q_k(x,y)$, the set $\mathcal{K}_Q$ has non-empty interior: it is the compact region consisting of a hypocycloid with vertices at $3,3e^{\frac{2\pi i}{3}},3e^{\frac{4\pi i}{3}}$ (see [6] for the picture). We can now state Rodriguez Villegas' formula for $m_3(t)$:

Theorem 1.1 (Rodriguez Villegas [10, §14]). *Let $\mathcal{F}'\subset\mathcal{H}$ be the fundamental domain of the congruence subgroup $\Gamma_0(3)$ formed by the geodesic triangle of vertices $i\infty,0,(1+i/\sqrt{3})/2$ and its reflection along the imaginary axis. For any $\tau\in\mathcal{F}'$, if $\sqrt[3]{t(\tau)}\in\mathbb{C}-\mathcal{K}_Q^\circ$, where $t(\tau)=27+\big(\eta(\tau)/\eta(3\tau)\big)^{12}$, then we have*

$$m_3(t(\tau))=\frac{81\sqrt{3}\,\mathrm{Im}(\tau)}{4\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-3}(n)(3m\,\mathrm{Re}(\tau)+n)}{|3m\tau+n|^4}, \tag{1.7}$$

where $\chi_{-3}(\cdot)=\left(\frac{-3}{\cdot}\right)$ and $\sum'_{m,n\in\mathbb{Z}}$ means that $(m,n)=(0,0)$ is excluded from the summation.

It is known that $t(\tau)$ is a Hauptmodul for $\Gamma_0(3)$, i.e., a generator of the function field of the modular curve $X_0(3)$. Let $\tau\in\mathcal{H}$ be a CM point. As mentioned earlier, this means that τ is an imaginary quadratic number in $\mathcal{H}$. Thus, there must exist three uniquely determined integers a,b,c with $a>0,\gcd(a,b,c)=1$ such that $a\tau^2+b\tau+c=0$. In this paper, we will simply write τ as $[a,b,c]$. Recall that for any negative integer D with $D\equiv 0$ or $1 \mod 4$, the class number

$$h(D)=\#\{\text{primitive binary quadratic forms with discriminant } D\}/\sim,$$

where "$\sim$" is the equivalence relation that identifies equivalent quadratic forms as the same. As a slight abuse of notation, we define the *class number* $h(\tau)$ of τ to be the class number of b^2-4ac, the discriminant of τ. According to the theory of complex multiplication, $t(\tau)$ are algebraic numbers if τ takes CM points. Moreover, the algebraic degree of $t(\tau)$ can be bounded by $h(\tau)h(3\tau)$ (see Theorem 2.2).

In [7], Samart proved a number of formulas that express the Mahler measures of certain polynomials in two or three variables in terms of linear combinations of L-values of multiple modular forms. For the family $Q_k(x,y)$, he proved that

$$m_3\big(6-6\sqrt[3]{2}+18\sqrt[3]{4}\big)=\frac{3}{2}\big(L'(f_{108},0)+L'(f_{36},0)-3L'(f_{27},0)\big), \tag{1.8}$$

where $f_{27}(\tau)=\eta(3\tau)^2\eta(9\tau)^2\in\mathcal{S}_2(\Gamma_0(27)),f_{36}(\tau)=\eta(6\tau)^4\in\mathcal{S}_2(\Gamma_0(36))$ and

$$\begin{aligned} f_{108}(\tau)&=\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv\pm1,\pm2,n\equiv5\\ \pmod 6}}(4m+3n)q^{4m^2+6mn+3n^2}\\ &=q+5q^7-7q^{13}-q^{19}-5q^{25}-4q^{31}-q^{37}+8q^{43}+\cdots \end{aligned}$$

is the unique normalized newform in $\mathcal{S}_2(\Gamma_0(108))$. Since the elliptic curve C_k has CM when $k=\sqrt[3]{6-6\sqrt[3]{2}+18\sqrt[3]{4}}$ (this can be easily checked by using the SageMath command `has_cm()`), Theorem 1.1 should be able to resolve (1.8). Indeed, Samart proved (1.8) by taking $\tau = \frac{i\sqrt{3}}{9}$ in (1.7). Since $\frac{i\sqrt{3}}{9}$ satisfies $27\tau^2+1$, we have $h\big(\frac{i\sqrt{3}}{9}\big) = h(-108) = 3$. Based on some numerical values of the hypergeometric representation of $m_3(t)$, Samart also discovered the following conjectural identities:

$$m_3\big(17766 + 14094\sqrt[3]{2} + 11178\sqrt[3]{4}\big) \stackrel{?}{=} \frac{3}{2}\big(L'(f_{108},0) + 3L'(f_{36},0) + 3L'(f_{27},0)\big), \tag{1.9}$$

$$m_3(\alpha \pm i\beta) \stackrel{?}{=} \frac{3}{2}\big(L'(f_{108},0) + 3L'(f_{36},0) - 6L'(f_{27},0)\big), \tag{1.10}$$

$$m_3\left(\frac{(7+\sqrt{5})^3}{4}\right) \stackrel{?}{=} \frac{1}{8}\big(9L'(f_{100},0) + 38L'(f_{20},0)\big), \tag{1.11}$$

$$m_3\left(\frac{(7-\sqrt{5})^3}{4}\right) \stackrel{?}{=} \frac{1}{4}\big(9L'(f_{100},0) - 38L'(f_{20},0)\big), \tag{1.12}$$

where

$$\alpha = 17766 - 7047\sqrt[3]{2} - 5589\sqrt[3]{4},$$
$$\beta = 243\sqrt{3}(29\sqrt[3]{2} - 23\sqrt[3]{4}),$$
$$f_{20}(\tau) = \eta(2\tau)^2\eta(10\tau)^2$$

and $f_{100}(\tau)$ is a cusp form of level 100. One can check that the elliptic curves C_k related to (1.9) and (1.10) have CM, while those related to (1.11) and (1.12) have no CM. It is also worth noting that $17766 + 14094\sqrt[3]{2} + 11178\sqrt[3]{4}$ and $\alpha \pm i\beta$ are the three roots of

$$T^3 - 53298T^2 + 1635876T - 19683000 = 0.$$

In this paper, we apply the method developed in [24] to the family $Q_k(x,y)$ and obtain the following results for $k \in \mathbb{C} - \mathcal{K}_Q^\circ$.

Theorem 1.2. *The following identities are true:*

$$m_3\big(-4320 - 1944\sqrt{5}\big) = \frac{405}{4\pi^2}L(F_{225},2), \tag{1.13}$$

$$m_3\big(-163296 - 35640\sqrt{21}\big) = \frac{567}{4\pi^2}L(F_{441},2), \tag{1.14}$$

$$m_3\big(729 + 405\sqrt{3}\big) = \frac{81}{\pi^2}L(F_{144},2), \quad m_3\big(729 - 405\sqrt{3}\big) = \frac{324}{\pi^2}L(\widetilde{F}_{144},2), \tag{1.15}$$

$$\begin{aligned} m_3\big(17766 + 14094\sqrt[3]{2} + 11178\sqrt[3]{4}\big) &= \frac{243}{2\pi^2}L(F_{108},2),\\ m_3(\alpha \pm i\beta) &= \frac{486}{\pi^2}L(\widetilde{F}_{108},2), \end{aligned} \tag{1.16}$$

$$\begin{aligned} &m_3\big(-216(18964 + 13149\sqrt[3]{3} + 9117\sqrt[3]{9})\big) = \frac{729}{4\pi^2}L(F_{243},2),\\ &m_3\big(-108(37928 - 13149\sqrt[3]{3}(1 \pm i\sqrt{3}) - 9117\sqrt[3]{9}(1 \mp i\sqrt{3}))\big)\\ &\quad = \frac{5103}{4\pi^2}L(\widetilde{F}_{243},2), \end{aligned} \tag{1.17}$$

$$m_3\big(6 + 3\sqrt[3]{2} - 9\sqrt[3]{4} \pm 3i\sqrt{3}(\sqrt[3]{2} + 3\sqrt[3]{4})\big) = \frac{81}{2\pi^2}L(G_{108},2), \tag{1.18}$$

$$m_3\big(96 - 28\sqrt[3]{3} + 36\sqrt[3]{9} \pm 4i\sqrt{3}(7\sqrt[3]{3} + 9\sqrt[3]{9})\big) = \frac{243}{4\pi^2}L(G_{243},2), \tag{1.19}$$

where α, β are the same as those appearing in (1.10) *and $F_N, \widetilde{F}_N, G_N$ are normalized (i.e., with leading coefficient* 1*) weight* 2 *cusp forms of the form*

$$r \sum_{m,n\in\mathbb{Z}} \chi_{-3}(n)(lm+sn)q^{am^2+bmn+cn^2}$$

in $\mathcal{S}_2(\Gamma_0(N))$. We make them clear in Table 1 *by listing the corresponding r, l, s, a, b, c.*

TABLE 1. Cusp forms in Theorem 1.2

Cusp forms	r	l	s	a	b	c	Cusp forms	r	l	s	a	b	c
F_{108}	1/2	0	1	27	0	1	F_{243}	1/4	3	2	63	3	1
$\widetilde{F}_{108}$	1/8	3	4	9	6	4	$\widetilde{F}_{243}$	1/28	3	14	9	3	7
F_{144}	1/2	0	1	12	0	1	F_{441}	1/4	3	2	39	3	1
$\widetilde{F}_{144}$	1/2	0	1	3	0	4	G_{108}	1/2	1	1	4	2	1
F_{225}	1/4	3	2	21	3	1	G_{243}	1/4	1	2	7	1	1

Since $6-6\sqrt[3]{2}+18\sqrt[3]{4}$ and $6+3\sqrt[3]{2}-9\sqrt[3]{4}\pm 3i\sqrt{3}(\sqrt[3]{2}+3\sqrt[3]{4})$ are the three roots of

$$T^3-18T^2+756T-27000=0,$$

our result (1.18) can be seen as a supplement to Samart's identity (1.8). Moreover, (1.16) and some linear combinations of modular forms yield:

Corollary 1.3. *The identities* (1.9) *and* (1.10) *are true.*

When $k=\sqrt[3]{729\pm 405\sqrt{3}}$, guided by Beilinson's conjecture for curves over number fields, we can also prove the following result similar to (1.6).

Theorem 1.4. *Consider $C_{\sqrt[3]{729\pm405\sqrt{3}}}$ as elliptic curves defined over $\mathbb{Q}(\sqrt{3})$ (see* (1.4)*). Then we have*

$$\left|\det\begin{pmatrix} m_3(729+405\sqrt{3}) & m_3(729-405\sqrt{3}) \\ m_3(729-405\sqrt{3}) & 4m_3(729+405\sqrt{3})\end{pmatrix}\right| = \frac{19683}{\pi^4}L(C_{\sqrt[3]{729\pm405\sqrt{3}}},2)$$
$$= \frac{243}{8}L''(C_{\sqrt[3]{729\pm405\sqrt{3}}},0).$$

For $k\in\mathcal{K}_Q^\circ$, the zero locus of $Q_k(x,y)$ will intersect with $\mathbb{T}^2$. We cannot expect $m_3(k^3)$ to be related to L-values of modular forms or elliptic curves in this case, since we cannot write $m_3(k^3)$ as a regulator integral over some *closed* path and thus Beilinson's conjecture does not work. In [6], Samart turned to the tempered polynomial family

$$\widetilde{Q}_k(x,y)=y^2+(x^2-kx)y+x.$$

Note that $m(Q_k)=m(\widetilde{Q}_k)$ because $(x^2y)^3Q_k(y/x^2,1/xy)=\widetilde{Q}_k(x^3,y^3)$. When $k\in\mathcal{K}_Q^\circ\cap\mathbb{R}=(-1,3)$, he proved that the zero locus of $\widetilde{Q}_k(x,y)$ intersects with $\mathbb{T}^2$ at

$$\left\{(e^{i\theta},\tilde{y}^{\pm}(e^{i\theta}))\mid \theta=0,\pm\cos^{-1}\left(\frac{k-1}{2}\right)\right\},$$

where $\tilde{y}^{\pm}(x) = -(x^2 - kx)\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{x(x-k)^2}}\right)$ with the square root chosen to be the principal branch. Clearly, one can factorize $\widetilde{Q}_k(x, y)$ as $(y - \tilde{y}^+(x))(y - \tilde{y}^-(x))$. Samart then introduced the modified Mahler measure

$$\tilde{n}(k) := m(\widetilde{Q}_k) - \frac{3}{\pi} \int_{\cos^{-1}\left(\frac{k-1}{2}\right)}^{\pi} \log |\tilde{y}^+(e^{i\theta})| d\theta. \tag{1.20}$$

This modification allowed him to interpret $\tilde{n}(k)$ as the regulator integral over a carefully chosen closed path on the Riemann surface associated to $\{(x, y) \in \mathbb{C}^2 \mid \widetilde{Q}_k(x, y) = 0\}$ for $k \in (-1, 3)$. By using the modular unit parametrization for $\widetilde{Q}_2(x, y) = 0$, he proved that

$$\tilde{n}(2) = -3L'(C_2, 0).$$

Based on numerical evidences, he also made other conjectures for $\tilde{n}(k)$ with $k^3 = 1, 2, \cdots, 26$. Observe that when $k = \sqrt[3]{24}$, the elliptic curve

$$C_{\sqrt[3]{24}} : Y^2 = X^3 - 15552X^2 - 8957952X - 1289945088$$

has CM. The last result of this paper is the following identity that was conjectured by Samart in [6, Table 2].

Theorem 1.5. *Let $\tilde{n}(k)$ be Samart's modified Mahler measure* (1.20). *Then we have*

$$\tilde{n}(\sqrt[3]{24}) = -3L'(C_{\sqrt[3]{24}}, 0).$$

This paper is organized as follows. In Section 2, we briefly introduce the theory of complex multiplication. An algorithm is designed to search for all CM points in $\mathcal{F}'$ such that $h(\tau) \leqslant 3, h(\tau)h(3\tau) \leqslant 4$. In Section 3, we will prove Theorem 1.2 and Corollary 1.3. A transformation formula for general theta functions is used to verify that the modular forms in Table 1 are indeed cusp forms for $\Gamma_0(N)$. In Section 4, we construct the Beilinson regulator that relates $m_3(729 \pm 405\sqrt{3})$ to $L(C_{\sqrt[3]{729 \pm 405\sqrt{3}}}, 2)$. This can help us to prove Theorem 1.4. Finally, according to Samart's hypergeometric formula for $\tilde{n}(k)$, we will prove Theorem 1.5.

2. CM points and the algorithm

Let E be an elliptic curve over $\mathbb{C}$. Then $E = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathcal{H}$ that is unique up to an action by $\mathrm{SL}(2, \mathbb{Z})$. Recall that the j-invariant of E is defined by

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots,$$

where $q = e^{2\pi i \tau}$. It is well known that E has CM if and only if τ is a CM point. Furthermore, $j(\tau)$ are algebraic integers of degree $h(\tau)$ if τ takes CM points, these algebraic integers are called *singular moduli*. When $\tau = [a, b, c]$ is a CM point, we call the singular modulus $j(\tau)$ is of *discriminant* $b^2 - 4ac$. This is a convenience borrowed from [23]. For every negative integer D with $D \equiv 0$ or $1 \mod 4$, there are exactly $h(D)$ different singular moduli of discriminant D which form a full Galois orbit over $\mathbb{Q}$. Let $\mathcal{F} = \{\tau \in \mathcal{H} \mid -1/2 \leqslant \mathrm{Re}(\tau) \leqslant 1/2, |\tau| \geqslant 1\}$ be the fundamental domain of $\mathrm{SL}(2, \mathbb{Z})$. We can find $h(D)$ CM points $\tau_1, \cdots, \tau_{h(D)} \in \mathcal{F}$

with discriminant D that are in different $\mathrm{SL}(2,\mathbb{Z})$-orbits. Then $j(\tau_1),\cdots,j(\tau_{h(D)})$ are all the different singular moduli of discriminant D. Moreover,

$$\prod_{i=1}^{h(D)}\bigl(X-j(\tau_i)\bigr)$$

is the monic polynomial in $\mathbb{Z}[X]$ that makes $j(\tau_1),\cdots,j(\tau_{h(D)})$ algebraic integers (see, for instance, [3, 14]).

For a general congruence subgroup $\Gamma_0(N), N \geqslant 1$, its modular functions also take algebraic values at CM points under suitable conditions. Let $\{\gamma_1,\cdots,\gamma_r\}$ be a set of right coset representatives of $\Gamma_0(N)$ in $\mathrm{SL}(2,\mathbb{Z})$. It is known that

$$r=\bigl[\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(N)\bigr]=N\prod_{\substack{p\mid N\\ p \text{ prime}}}\Bigl(1+\frac{1}{p}\Bigr),$$

and there exists a polynomial $\Phi_N(X,Y)\in\mathbb{Z}[X,Y]$ (the so-called *modular equation* for $\Gamma_0(N)$) such that

$$\Phi_N(X,j(\tau))=\prod_{i=1}^{r}\bigl(X-j(N\gamma_i\tau)\bigr). \tag{2.1}$$

Proposition 2.1 ([3, Proposition 12.7])**.** *Let $f(\tau)$ be a modular function for $\Gamma_0(N)$ whose q-expansion has rational coefficients. Then:*

(1) $f(\tau)\in\mathbb{Q}(j(\tau),j(N\tau))$.
(2) *Assume in addition that $f(\tau)$ is holomorphic on $\mathcal{H}$, and let $\tau_0\in\mathcal{H}$. If*

$$\frac{\partial\Phi_N}{\partial X}\bigl(j(N\tau_0),j(\tau_0)\bigr)\neq 0,$$

then $f(\tau_0)\in\mathbb{Q}(j(\tau_0),j(N\tau_0))$.

In the case when $N=3$, we have $r=3\cdot(1+\frac{1}{3})=4$. Let $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}, T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ be the generators of $\mathrm{SL}(2,\mathbb{Z})$, then

$$\gamma_1=I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \gamma_2=S,\quad \gamma_3=ST,\quad \gamma_4=ST^{-1} \tag{2.2}$$

form a set of right coset representatives of $\Gamma_0(3)$ in $\mathrm{SL}(2,\mathbb{Z})$. These matrices can also transform the fundamental domain $\mathcal{F}$ to cover $\mathcal{F}'$:

$$\mathcal{F}'=\bigcup_{i=1}^{4}\gamma_i\mathcal{F}. \tag{2.3}$$

See Figure 1 for the picture.

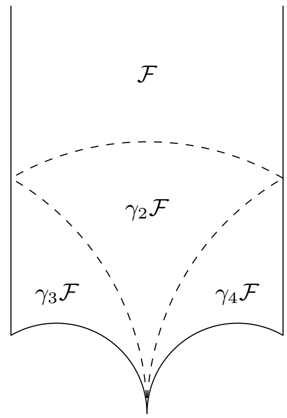

FIGURE 1. $\mathcal{F}'$ and its covering (2.3)

Since $\eta(\tau) = e^{\frac{2\pi i\tau}{24}} \prod_{n=1}^{\infty}(1-q^n)$, the modular function $t(\tau) = 27 + \big(\eta(\tau)/\eta(3\tau)\big)^{12}$ for $\Gamma_0(3)$ in Theorem 1.1 has q-expansion

$$t(\tau) = \frac{1}{q} + 15 + 54q - 76q^2 - 243q^3 + 1188q^4 + \cdots,$$

the coefficients are rational integers. It is also easily seen that $t(\tau)$ is holomorphic on $\mathcal{H}$ since $\eta(\tau)$ has no zeros on $\mathcal{H}$. Thus, we can apply Proposition 2.1 to $t(\tau)$ and prove the following result that is similar to [24, Theorem 2.2].

Theorem 2.2. *Let $\tau_0 \in \mathcal{H}$ be a CM point. If $j(3\tau_0) \neq j(3\gamma_i\tau_0)$ for $i = 2, 3, 4$, where γ_i are as in (2.2), then $t(\tau_0)$ is an algebraic number with degree no more than $h(\tau_0)h(3\tau_0)$.*

Theorem 2.2 implies that, in order to obtain some interesting CM points that keep the degrees of $t(\tau)$ not too high, we can search for CM points in $\mathcal{F}'$ with class numbers relatively small. In this work, we focus our attention on CM points with class numbers $\leqslant 3$ and will use them to search for $t(\tau)$ that have degrees $\leqslant 4$ as algebraic numbers. To achieve this, our first task is to determine discriminants with some small class numbers. It is well known that for each positive integer n, only finitely many negative discriminants $D \equiv 0$ or $1 \mod 4$ have $h(D) = n$. We list all negative discriminants that have class numbers $1, 2$ and 3 as follows.

$$\begin{aligned} h(D) = 1 \iff D = &-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67,\\ &-163;\\ h(D) = 2 \iff D = &-15, -20, -24, -32, -35, -36, -40, -48, -51, -52, -60,\\ &-64, -72, -75, -88, -91, -99, -100, -112, -115, -123,\\ &-147, -148, -187, -232, -235, -267, -403, -427;\\ h(D) = 3 \iff D = &-23, -31, -44, -59, -76, -83, -92, -107, -108, -124,\\ &-139, -172, -211, -243, -268, -283, -307, -331, -379,\\ &-499, -547, -643, -652, -883, -907. \end{aligned}$$

Just like what Huber, Schultz and Ye did in [20, Algorithm 4.1], we can apply Algorithm 2.3 to determine all CM points $\tau \in \mathcal{F}'$ such that $h(\tau) \leqslant 3, h(\tau)h(3\tau) \leqslant 4$.

Algorithm 2.3. *For each discriminant D listed above, perform the following operations.*

(1) *Determine all CM points $\tau = [a, b, c]$ in $\mathcal{F}$ with discriminant D by solving the system*

$$\begin{cases} a, b, c \in \mathbb{Z}, a > 0, \\ \gcd(a, b, c) = 1, \\ b^2 - 4ac = D, \\ |b| \leqslant a \leqslant c. \end{cases}$$

(2) *Use the right coset representatives* (2.2) *to translate these points obtained in* (1) *to get a full list of CM points in $\mathcal{F}'$ with discriminant D.*

(3) *For each point τ obtained in* (2)*, verify whether $h(\tau)h(3\tau) \leqslant 4$. If it passes this test, then output τ.*

We implement Algorithm 2.3 in a Mathematica notebook. Thanks to the built-in function `DedekindEta[]`, $t(\tau)$ can be calculated to any given precision. This can help us to filter out the points that make $\sqrt[3]{t(\tau)} \in \mathbb{C} - \mathcal{K}_Q^\circ$ from the outputs of Algorithm 2.3. We list these lucky ones and the corresponding numerical approximations (accurate to five decimal places) of $t(\tau)$ in Table 2. According to Theorem 2.2, all values in Table 2 are in fact approximations of algebraic numbers with degrees $\leqslant 4$.

TABLE 2. CM points and the numerical approximations of $t(\tau)$

τ	$t(\tau)$	τ	$t(\tau)$	τ	$t(\tau)$
[1, −1, 1]	*−216.00000*	[2, 0, 1]	100.64395	[9, 2, 1]	25.99494 + 1.63455*i*
[1, −1, 2]	−4056.94536	[3, 2, 1]	4.00000 + 14.14213*i*	[9, −2, 1]	25.99494 − 1.63455*i*
[2, 1, 1]	15.01873 + 62.96451*i*	[3, −2, 1]	4.00000 − 14.14213*i*	[3, 2, 3]	−171.99494 + 323.63455*i*
[2, −1, 1]	15.01873 − 62.96451*i*	*[1, 0, 3]*	*53267.29623*	[3, −2, 3]	−171.99494 − 323.63455*i*
[1, −1, 3]	−33491.14467	*[4, 2, 1]*	*15.35188 + 11.56864i*	[9, 0, 1]	28.39230
[3, 1, 1]	32.00000 + 26.53299*i*	*[4, −2, 1]*	*15.35188 − 11.56864i*	*[12, 0, 1]*	*27.51942*
[3, −1, 1]	32.00000 − 26.53299*i*	*[3, 0, 1]*	*54.00000*	*[3, 0, 4]*	*1430.48057*
[1, −1, 5]	−885464.77774	[1, 0, 4]	286766.31332	[15, 0, 1]	27.21952
[5, 1, 1]	30.24531 + 7.39216*i*	[4, 0, 1]	40.31662	[3, 0, 5]	3347.78047
[5, −1, 1]	30.24531 − 7.39216*i*	[5, 2, 1]	20.68502 + 7.84745*i*	[18, 0, 1]	27.10102
[1, −1, 7]	*−12288728.98398*	[5, −2, 1]	20.68502 − 7.84745*i*	[9, 0, 2]	36.89897
[7, 1, 1]	*28.49199 + 2.87797i*	[1, 0, 7]	16580645.98812	[3, −3, 2]	−43.68691
[7, −1, 1]	*28.49199 − 2.87797i*	[7, 0, 1]	29.99744	[3, −3, 5]	−1754.59081
[1, −1, 11]	−884736728.99977	[8, 2, 1]	25.50721 + 2.35941*i*	*[3, −3, 7]*	*−8666.91614*
[11, 1, 1]	27.37486 + 0.66585*i*	[8, −2, 1]	25.50721 − 2.35941*i*	[3, −3, 11]	−110618.99341
[11, −1, 1]	27.37486 − 0.66585*i*	[9, 1, 1]	27.72446 + 1.31876*i*	*[3, −3, 13]*	*−326618.99776*
[1, −1, 17]	−147197952728.99999	[9, −1, 1]	27.72446 − 1.31876*i*	[3, −3, 23]	−27000026.99997
[17, 1, 1]	27.06887 + 0.11984*i*	[3, 1, 3]	260.27553 + 424.63897*i*	[9, −9, 5]	−18.97825
[17, −1, 1]	27.06887 − 0.11984*i*	[3, −1, 3]	260.27553 − 424.63897*i*	*[27, 0, 1]*	*27.01369*
[1, −1, 41]	−262537412640768728.99999	[6, 2, 1]	23.30495 + 5.19349*i*	*[9, 6, 4]*	*−4.50684 + 31.29187i*
[41, 1, 1]	27.00056 + 0.00098*i*	[6, −2, 1]	23.30495 − 5.19349*i*	*[9, −6, 4]*	*−4.50684 − 31.29187i*
[41, −1, 1]	27.00056 − 0.00098*i*	[3, 2, 2]	−39.30495 + 93.19349*i*	*[9, 3, 7]*	*130.50002 + 199.64657i*
[1, 0, 1]	550.59223	[3, −2, 2]	−39.30495 − 93.19349*i*	*[9, −3, 7]*	*130.50002 − 199.64657i*
[2, 2, 1]	−10.59223	[6, 0, 1]	31.63246		
[1, 0, 2]	7243.35604	[3, 0, 2]	184.36753		

In general, each CM point τ_0 in Table 2 will produce an identity of the form

$$m_3(t(\tau_0)) = \frac{c_{\tau_0}}{\pi^2} L(f_{\tau_0}, 2),$$

with f_{τ_0} a normalized cusp form and c_{τ_0} a real quadratic number such that $c_{\tau_0}^2 \in \mathbb{Q}$. To limit the length of this paper, we only deal with those points that make

$c_{\tau_0} \in \mathbb{Q}$, and they are italicized in Table 2. Note that $[1,-1,1] = \frac{1+i\sqrt{3}}{2}$, $[3,0,1] = \frac{i\sqrt{3}}{3}$ and $[27,0,1] = \frac{i\sqrt{3}}{9}$ correspond to the already proven results [7, 10, 16] for $m_3(-216), m_3(54)$ and $m_3\big(6 - 6\sqrt[3]{2} + 18\sqrt[3]{4}\big)$, respectively.

3. The relations with cusp forms

In order to recognize the lattice sums appearing in (1.7) as L-values of modular forms, we need some facts about theta functions associated to lattices. Recall that an *even* lattice L of rank n is the submodule $\mathbb{Z}^n$ of $\mathbb{R}^n$ equipped with a non-degenerate quadratic form

$$\mathcal{Q}(X) = \frac{1}{2}X^t AX, \quad X \in \mathbb{Z}^n,$$

where A is an *even* matrix of rank n, that is, A is an $n \times n$ symmetric matrix with integer entries and even integer diagonals. Obviously, this definition makes $\mathcal{Q}(X) \in \mathbb{Z}$ for every $X \in \mathbb{Z}^n$. The *level* of L is defined to be the smallest positive integer N such that NA^{-1} is an even matrix. We also define $L^* = \{Y \in \mathbb{R}^n \mid X^t AY \in \mathbb{Z}, \forall X \in \mathbb{Z}^n\}$ to be the *dual lattice* of L.

Proposition 3.1 ([12, Corollary 14.3.16]). *Let L be a positive definite even lattice of* even *rank n, level N, and quadratic form $\mathcal{Q}(X) = \frac{1}{2}X^t AX$. Assume that $A^{-1} = (b_{i,j})_{1\leqslant i,j\leqslant n}$ and let $P(x_1, \cdots, x_n)$ be a homogeneous polynomial of degree $(k-n)/2$ such that $\Delta_{\mathcal{Q}}(P) = 0$, where*

$$\Delta_{\mathcal{Q}} = \sum_{1\leqslant i,j\leqslant n} b_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then for all $Y \in L^*$, the theta function

$$\Theta(P, L, Y; \tau) := \sum_{X\in\mathbb{Z}^n} P(X+Y) q^{\mathcal{Q}(X+Y)}$$

is in $\mathcal{M}_{k/2}(\Gamma(N))$. In addition, if $k > n$, then Θ is also a cusp form.

It is clear from the definition that $\Theta(P, L, Y; \tau)$ only depends on the class of Y in $L^*/\mathbb{Z}^n$. Let k be an integer. Recall that the weight k slash operator $\big|_k$ is given by

$$(f\big|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau), \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \text{ and } f : \mathcal{H} \to \mathbb{C}.$$

For elements in $\Gamma_0(N)$, we also have the following transformation formula.

Proposition 3.2 ([1, Chapter IX, §4, Theorem 5]). *Let the assumptions of Proposition* 3.1 *hold. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have*

$$\big(\Theta(P, L, Y; \cdot)\big|_{k/2}\gamma\big)(\tau) = v(d) e^{2\pi i ab\mathcal{Q}(Y)} \Theta(P, L, aY; \tau),$$

where $v(d) = \left(\frac{(-1)^{n/2}\det A}{d}\right)$ if $d > 0$, and $v(d) = (-1)^{n/2}\left(\frac{(-1)^{n/2}\det A}{-d}\right)$ if $d < 0$.

With these tools in hand, we can now prove the identities in Theorem 1.2. We first deal with the cases when the class numbers are equal to 1.

3.1. **Proofs of the cases when** $h(\tau)=1$.

Proof of (1.16). Since $17766+14094\sqrt[3]{2}+11178\sqrt[3]{4}=53267.29623\cdots$, by examining Table 2, we observe that $\tau_0=[1,0,3]=i\sqrt{3}$ seems to be the candidate such that

$$t(\tau_0)=17766+14094\sqrt[3]{2}+11178\sqrt[3]{4}. \tag{3.1}$$

This can be rigorously proved by using the identity [7, §2]

$$j(\tau)=\frac{t(\tau)(t(\tau)+216)^3}{(t(\tau)-27)^3} \tag{3.2}$$

together with the fact that $j(i\sqrt{3})$ is a rational integer because $h(i\sqrt{3})=h(-12)=1$. A numerical calculation shows that $j(i\sqrt{3})=54000$, and thus (3.1) is confirmed. It turns out that $\sqrt[3]{t(i\sqrt{3})}\notin\mathcal{K}_Q^\circ$ since $\mathcal{K}_Q\cap\mathbb{R}=[-1,3]$. Taking $\tau=\tau_0$ in Theorem 1.1 then yields

$$\begin{aligned} m_3\big(17766+14094\sqrt[3]{2}+11178\sqrt[3]{4}\big)&=\frac{243}{4\pi^2}\sum_{m,n\in\mathbb{Z}}{}^{'}\frac{\chi_{-3}(n)n}{(27m^2+n^2)^2}\\ &=\frac{243}{2\pi^2}L(F_{108},2), \end{aligned}$$

where

$$\begin{aligned} F_{108}(\tau)&=\frac{1}{2}\sum_{m,n\in\mathbb{Z}}\chi_{-3}(n)nq^{27m^2+n^2}\\ &=\frac{1}{2}\left(\sum_{m,n\in\mathbb{Z}}(3n+1)q^{27m^2+(3n+1)^2}-\sum_{m,n\in\mathbb{Z}}(3n+2)q^{27m^2+(3n+2)^2}\right)\\ &=\frac{3}{2}\left(\sum_{m,n\in\mathbb{Z}}(n+\tfrac{1}{3})q^{27m^3+9(n+\frac{1}{3})^2}-\sum_{m,n\in\mathbb{Z}}(n+\tfrac{2}{3})q^{27m^3+9(n+\frac{2}{3})^2}\right). \end{aligned}$$

Let L be the rank 2 lattice with quadratic form $\mathcal{Q}(x_1,x_2)=27x_1^2+9x_2^2$. The Gram matrix is $A=\begin{pmatrix}54&0\\0&18\end{pmatrix}$ and thus the level $N=108$. We have

$$F_{108}(\tau)=\frac{3}{2}\big(\Theta(P,L,Y_1;\tau)-\Theta(P,L,Y_2;\tau)\big),$$

where $P(x_1,x_2)=x_2$ and $Y_1=(0,1/3)^t,Y_2=(0,2/3)^t\in L^*$. Proposition 3.1 immediately implies that $F_{108}(\tau)\in\mathcal{S}_2(\Gamma(108))$. In fact, we can prove that $F_{108}(\tau)\in\mathcal{S}_2(\Gamma_0(108))$ by verifying

$$(F_{108}\big|_2\gamma)(\tau)=F_{108}(\tau),\quad\forall\gamma\in\Gamma_0(108). \tag{3.3}$$

Take $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(108)$. We have $(a,d)\equiv(\pm1,\pm1)\pmod 6$ because $ad\equiv 1\pmod{108}$. Also note that $\det A=972$ and

$$\left(\frac{-972}{d}\right)=\begin{cases}1, & \text{if } d\equiv 1\pmod 6,\\ -1, & \text{if } d\equiv -1\pmod 6,\\ 0, & \text{otherwise.}\end{cases}$$

If $(a,d) \equiv (1,1) \pmod 6$ and $d>0$, then, according to Proposition 3.2, we have

$$\begin{aligned}(F_{108}\big|_2\gamma)(\tau) &= \frac{3}{2}\Big(\big(\Theta(P,L,Y_1;\cdot)\big|_2\gamma\big)(\tau) - \big(\Theta(P,L,Y_2;\cdot)\big|_2\gamma\big)(\tau)\Big)\\ &= \frac{3}{2}\left(\frac{-972}{d}\right)\Big(e^{2\pi iab}\Theta(P,L,aY_1;\tau) - e^{8\pi iab}\Theta(P,L,aY_2;\tau)\Big)\\ &= \frac{3}{2}\big(\Theta(P,L,aY_1;\tau) - \Theta(P,L,aY_2;\tau)\big)\\ &= \frac{3}{2}\big(\Theta(P,L,Y_1;\tau) - \Theta(P,L,Y_2;\tau)\big)\\ &= F_{108}(\tau).\end{aligned}$$

The fourth equality holds because $Y_1 - aY_1, Y_2 - aY_2 \in \mathbb{Z}^2$. If $(a,d) \equiv (-1,-1) \pmod 6$ and $d>0$, we have $Y_2 - aY_1, Y_1 - aY_2 \in \mathbb{Z}^2$, thus

$$\begin{aligned}(F_{108}\big|_2\gamma)(\tau) &= -\frac{3}{2}\big(\Theta(P,L,aY_1;\tau) - \Theta(P,L,aY_2;\tau)\big)\\ &= -\frac{3}{2}\big(\Theta(P,L,Y_2;\tau) - \Theta(P,L,Y_1;\tau)\big)\\ &= F_{108}(\tau).\end{aligned}$$

In the cases when $(a,d) \equiv (\pm1,\pm1) \pmod 6$ and $d<0$, (3.3) also holds because

$$-\left(\frac{-972}{-d}\right) = \begin{cases}1, & \text{if } d\equiv 1 \pmod 6,\\ -1, & \text{if } d \equiv -1 \pmod 6.\end{cases}$$

This completes our verification of (3.3).

To prove the identities $m_3(\alpha \pm i\beta) = \frac{486}{\pi^2}L(\widetilde{F}_{108},2)$, we calculate that

$$\alpha = 15.35188\cdots, \quad \beta = 11.56864\cdots.$$

This time, according to Table 2, we need to prove that

$$t([4,\pm2,1]) = t\left(\frac{\mp1 + i\sqrt{3}}{4}\right) = \alpha \pm i\beta.$$

Indeed, this can be confirmed by the facts that $h([4,\pm2,1]) = h(-12) = 1$ and $j([4,\pm2,1]) = 54000$. It follows from Theorem 1.1 that

$$\begin{aligned}m_3(\alpha\pm i\beta) &= \frac{243}{4\pi^2}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{\chi_{-3}(n)(3m+4n)}{(9m^2+6mn+4n^2)^2}\\ &= \frac{486}{\pi^2}L(\widetilde{F}_{108},2),\end{aligned}$$

where

$$\begin{aligned}\widetilde{F}_{108}(\tau) &= \frac{1}{8}\sum_{m,n\in\mathbb{Z}}\chi_{-3}(n)(3m+4n)q^{9m^2+6mn+4n^2}\\ &= \frac{3}{8}\Bigg(\sum_{m,n\in\mathbb{Z}}(m+4(n+\tfrac{1}{3}))q^{9m^3+18m(n+\frac{1}{3})+36(n+\frac{1}{3})^2}\\ &\qquad - \sum_{m,n\in\mathbb{Z}}(m+4(n+\tfrac{2}{3}))q^{9m^3+18m(n+\frac{2}{3})+36(n+\frac{2}{3})^2}\Bigg).\end{aligned}$$

Similarly, we can write $\widetilde{F}_{108}(\tau)$ as $\frac{3}{8}\big(\Theta(P_1, L_1, Y_1; \tau) - \Theta(P_1, L_1, Y_2; \tau)\big)$ with L_1 the level 108 lattice associated to $\mathcal{Q}_1(x_1, x_2) = 9x_1^2 + 18x_1x_2 + 36x_2^2$ and $P_1(x_1, x_2) = x_1 + 4x_2$. A completely parallel argument can be made to show that $\widetilde{F}_{108}(\tau) \in \mathcal{S}_2(\Gamma_0(108))$. □

Now, we can provide a proof for Samart's conjectural identities (1.9) and (1.10).

Proof of Corollary 1.3. According to the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f, s) = \pm\left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(f, 2-s)$$

for L-functions of newforms of weight 2 and level N, we have

$$L'(f_{108}, 0) = \frac{27}{\pi^2}L(f_{108}, 2), \quad L'(f_{36}, 0) = \frac{9}{\pi^2}L(f_{36}, 2), \quad L'(f_{27}, 0) = \frac{27}{4\pi^2}L(f_{27}, 2).$$

Thus, one can rewrite the right-hand sides of (1.9) and (1.10) as

$$\begin{aligned}
&\frac{3}{2}(L'(f_{108}, 0) + 3L'(f_{36}, 0) + 3L'(f_{27}, 0)) \\
&\qquad = \frac{81}{2\pi^2}L(f_{108}, 2) + \frac{81}{2\pi^2}L(f_{36}, 2) + \frac{243}{8\pi^2}L(f_{27}, 2) \\
&\qquad = \frac{243}{2\pi^2}\left(\frac{1}{3}L(f_{108}, 2) + \frac{1}{3}L(f_{36}, 2) + \frac{1}{4}L(f_{27}, 2)\right), \\
&\frac{3}{2}(L'(f_{108}, 0) + 3L'(f_{36}, 0) - 6L'(f_{27}, 0)) \\
&\qquad = \frac{81}{2\pi^2}L(f_{108}, 2) + \frac{81}{2\pi^2}L(f_{36}, 2) - \frac{243}{4\pi^2}L(f_{27}, 2) \\
&\qquad = \frac{486}{\pi^2}\left(\frac{1}{12}L(f_{108}, 2) + \frac{1}{12}L(f_{36}, 2) - \frac{1}{8}L(f_{27}, 2)\right).
\end{aligned}$$

By (1.16), it is hence enough to prove that

$$\begin{aligned}
\frac{1}{3}L(f_{108}, 2) + \frac{1}{3}L(f_{36}, 2) + \frac{1}{4}L(f_{27}, 2) &= L(F_{108}, 2), \\
\frac{1}{12}L(f_{108}, 2) + \frac{1}{12}L(f_{36}, 2) - \frac{1}{8}L(f_{27}, 2) &= L(\widetilde{F}_{108}, 2).
\end{aligned}$$

For this, we can first write $F_{108}(\tau)$ and $\widetilde{F}_{108}(\tau)$ as

$$\begin{aligned}
F_{108}(\tau) &= \frac{1}{3}f_{108}(\tau) + \frac{1}{3}f_{36}(\tau) + \frac{1}{3}f_{27}(\tau) - \frac{4}{3}f_{27}(4\tau), \\
\widetilde{F}_{108}(\tau) &= \frac{1}{12}f_{108}(\tau) + \frac{1}{12}f_{36}(\tau) - \frac{1}{6}f_{27}(\tau) + \frac{2}{3}f_{27}(4\tau)
\end{aligned}$$

by using the fact that the Sturm bound for $\mathcal{M}_2(\Gamma_0(108))$ is 36. Then

$$\begin{aligned}
L(F_{108}, 2) &= \frac{1}{3}L(f_{108}, 2) + \frac{1}{3}L(f_{36}, 2) + \frac{1}{3}L(f_{27}, 2) - \frac{4}{3} \cdot \frac{1}{4^2}L(f_{27}, 2) \\
&= \frac{1}{3}L(f_{108}, 2) + \frac{1}{3}L(f_{36}, 2) + \frac{1}{4}L(f_{27}, 2), \\
L(\widetilde{F}_{108}, 2) &= \frac{1}{12}L(f_{108}, 2) + \frac{1}{12}L(f_{36}, 2) - \frac{1}{6}L(f_{27}, 2) + \frac{2}{3} \cdot \frac{1}{4^2}L(f_{27}, 2) \\
&= \frac{1}{12}L(f_{108}, 2) + \frac{1}{12}L(f_{36}, 2) - \frac{1}{8}L(f_{27}, 2),
\end{aligned}$$

as desired. □

In the proofs of the remaining identities of Theorem 1.2, we will give the CM points directly and omit the constructions of theta functions. They are all similar and not hard to do.

Proof of (1.17). We just need to show that

$$t([1,-1,7]) = t\left(\frac{1+3i\sqrt{3}}{2}\right) = -216(18964 + 13149\sqrt[3]{3} + 9117\sqrt[3]{9}),$$
$$t([7,\pm 1,1]) = t\left(\frac{\mp 1+3i\sqrt{3}}{14}\right) = -108\big(37928 - 13149\sqrt[3]{3}(1\pm i\sqrt{3}) - 9117\sqrt[3]{9}(1\mp i\sqrt{3})\big).$$

These equalities are all the consequences of $h([1,-1,7]) = h([7,\pm 1,1]) = h(-27) = 1$,

$$j([1,-1,7]) = j([7,\pm 1,1]) = -12288000,$$

and (3.2). □

For CM points with class numbers greater than 1, the values of $j(\tau)$ are no longer rational integers. However, according to the theory of complex multiplication mentioned in Section 2, these values are always determinable.

3.2. **Proofs of the cases when $h(\tau) = 2, 3$.**

Proofs of (1.13), (1.14) *and* (1.15). We can prove these identities by establishing the following equalities:

$$t([3,-3,7]) = t\left(\frac{3+5i\sqrt{3}}{6}\right) = -4320 - 1944\sqrt{5}, \tag{3.4}$$
$$t([3,-3,13]) = t\left(\frac{3+7i\sqrt{3}}{6}\right) = -163296 - 35640\sqrt{21},$$
$$t([3,0,4]) = t\left(\frac{2i\sqrt{3}}{3}\right) = 729 + 405\sqrt{3}, \quad t([12,0,1]) = t\left(\frac{i\sqrt{3}}{6}\right) = 729 - 405\sqrt{3}.$$

Since $h([3,-3,7]) = h([3,-3,13]) = h([3,0,4]) = h([12,0,1]) = 2$, the values of $j(\tau)$ at these points are all algebraic integers of degree 2. In fact, we have

$$j([3,-3,7]) = -327201914880 + 146329141248\sqrt{5}, \tag{3.5}$$
$$j([3,-3,13]) = -17424252776448000 + 3802283679744000\sqrt{21},$$
$$j([3,0,4]) = 1417905000 - 818626500\sqrt{3},$$
$$j([12,0,1]) = 1417905000 + 818626500\sqrt{3}.$$

In the following, we will derive (3.4) by proving (3.5) and then using (3.2). The others can be done in the same manner.

Note that the discriminant of $\tau_1 = [3,-3,7]$ is -75, one can choose τ_2 to be the CM point with discriminant -75 that is not in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit with τ_1, for instance, we choose $\tau_2 = [1,-1,19] = \frac{1+5i\sqrt{3}}{2}$. Then $j(\tau_1)$ and $j(\tau_2)$ are all the 2 different singular moduli of discriminant -75, so the polynomial $(X - j(\tau_1))(X - j(\tau_2))$ must be monic and has integer coefficients. By a numerical calculation, we find that this polynomial should be

$$X^2 + 654403829760X + 5209253090426880.$$

Immediately, we can confirm (3.5) since $j(\tau_1)$ is a root of this polynomial. $\square$

Finally, let us handle the cases when $h(\tau) = 3$.

Proofs of (1.18) *and* (1.19). This time, the CM points $[9, \pm 6, 4]$ and $[9, \pm 3, 7]$ with class numbers $h(-108) = h(-243) = 3$ should be employed. We have

$$t([9, \pm 6, 4]) = t\left(\frac{\mp 1 + i\sqrt{3}}{3}\right) = 6 + 3\sqrt[3]{2} - 9\sqrt[3]{4} \pm 3i\sqrt{3}(\sqrt[3]{2} + 3\sqrt[3]{4}),$$

because $j([9, \pm 6, 4]) = \alpha_1 \pm i\beta_1$ are the complex roots of

$$X^3 - 151013228706000X^2 + 224179462188000000X - 2^{12} \cdot 3^3 \cdot 5^9 \cdot 11^6 \cdot 17^3,$$

where

$$\alpha_1 = 6000(8389623817 - 3329424418\sqrt[3]{2} - 2642565912\sqrt[3]{4}),$$
$$\beta_1 = -12000\sqrt{3}(1664712209\sqrt[3]{2} - 1321282956\sqrt[3]{4}).$$

And we have

$$t([9, \pm 3, 7]) = t\left(\frac{\mp 1 + 3i\sqrt{3}}{6}\right) = 96 - 28\sqrt[3]{3} + 36\sqrt[3]{9} \pm 4i\sqrt{3}(7\sqrt[3]{3} + 9\sqrt[3]{9}),$$

because $j([9, \pm 3, 7]) = \alpha_2 \pm i\beta_2$ are the complex roots of

$$X^3 + 1855762905734664192000X^2 - 2^{30} \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 29 \cdot 1097 \cdot 37181X + 2^{45} \cdot 3 \cdot 5^9 \cdot 11^3 \cdot 23^3,$$

where

$$\alpha_2 = -4096000\big(151022371885959 - 52356532113152\sqrt[3]{3} - 36301991826555\sqrt[3]{9}\big),$$
$$\beta_2 = 69632000\sqrt[6]{3}\big(6406233851745 - 3079796006656\sqrt[3]{9}\big).$$

The above equalities can be verified using the same fact as used in the proofs of (1.18) and (1.19). The difference is that algebraic numbers involved here are cubic. $\square$

4. Linking to the L-values of elliptic curves

The goal of this section is to prove Theorem 1.4 and Theorem 1.5. Our process for the former is directed by Beilinson's conjecture. Readers interested in a detailed formulation of Beilinson's conjecture for curves over number fields can consult [19].

First, for simplicity, we denote the elliptic curve

$$C_{\sqrt[3]{729+405\sqrt{3}}}: Y^2 = X^3 - 1062882(26+15\sqrt{3})X^2 + 18596183472(23859+13775\sqrt{3})X - 488038239039168(3650401 + 2107560\sqrt{3})$$

as C directly. It is defined over $K = \mathbb{Q}(\sqrt{3})$, and the non-trivial element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ converts C to the curve $C^\sigma := C_{\sqrt[3]{729-405\sqrt{3}}}$. Since C has j-invariant $1417905000 - 818626500\sqrt{3}$, conductor norm 36 and torsion structure $\mathbb{Z}/2\mathbb{Z}$. One can search in LMFDB [21] and find that C is isomorphic over K to the elliptic curve with LMFDB label `2.2.12.1-36.1-a3`. Its L-function $L(C, s)$ with label `4-72e2-1.1-c1e2-0-2` can be written as

$$L(C, s) = L(f_{36}, s)L(f_{144}, s),$$

where

$$\begin{aligned} f_{36}(\tau) &= \eta(6\tau)^4 = q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} - 4q^{31} - 10q^{37} + 8q^{43} + 9q^{49} \\ &\quad + 14q^{61} + \cdots, \\ f_{144}(\tau) &= \frac{\eta(12\tau)^{12}}{\eta(6\tau)^4\eta(24\tau)^4} = q + 4q^7 + 2q^{13} - 8q^{19} - 5q^{25} + 4q^{31} - 10q^{37} - 8q^{43} \\ &\quad + 9q^{49} + \cdots. \end{aligned}$$

Moreover, C is a $\mathbb{Q}$-curve, i.e., it is isogenous over $\overline{K}$ to C^σ, the only Galois conjugate of C. In fact, there is an isogeny $\phi : C \to C^\sigma$ defined over K with kernel

$$\Big\{O, \big(39366(362 + 209\sqrt{3}), 0\big), \\ \Big(78732(265 + 153\sqrt{3}), \pm 38263752\sqrt{3388314 + 1956244\sqrt{3}}\Big)\Big\}.$$

Thus, we have $L(C^\sigma, s) = L(C, s)$ since two isogenous elliptic curves over a number field have the same L-function [11]. By using Vélu's formula [15, Theorem 12.16], one can write out this isogeny explicitly:

$$\phi : (X, Y) \mapsto \left(\frac{X\phi_3(X)}{4\phi_1(X)\phi_2(X)^2}, \frac{-Y\phi_4(X)}{8\phi_1(X)^2\phi_2(X)^3}\right),$$

where

$$\begin{aligned} \phi_1(X) &= X - 39366\big(362 + 209\sqrt{3}\big), \quad \phi_2(X) = X - 78732\big(265 + 153\sqrt{3}\big), \\ \phi_3(X) &= \big(1351 - 780\sqrt{3}\big)X^3 + 629856\big(45 - 26\sqrt{3}\big)X^2 + 86782189536\big(3 + \sqrt{3}\big)X \\ &\quad - 3904305912313344\big(362 + 209\sqrt{3}\big), \\ \phi_4(X) &= \big(70226 - 40545\sqrt{3}\big)X^5 + 78732\big(698 - 403\sqrt{3}\big)X^4 \\ &\quad - 24794911296\big(82 - 49\sqrt{3}\big)X^3 - 1952152956156672\big(425 + 246\sqrt{3}\big)X^2 \\ &\quad + 153696906544127099904\big(56089 + 32383\sqrt{3}\big)X \\ &\quad - 12100864846032214829641728\big(2672279 + 1542841\sqrt{3}\big). \end{aligned}$$

We can also obtain an isogeny $\phi^\sigma : C^\sigma \to C$ by applying σ to the coefficients of ϕ. Some (complicated) algebraic calculations imply that

$$\phi \circ \phi^\sigma = [4], \quad \phi^*\omega_{C^\sigma} = -(52 + 30\sqrt{3})\omega_C,$$

where ω_C and ω_{C^σ} are the invariant differentials of C and C^σ defined by $\frac{dX}{2Y}$, respectively.

Recall that for a pair of meromorphic functions f, g on the Riemann surface $C(\mathbb{C})$ or $C^\sigma(\mathbb{C})$, there is a classical differential form

$$\eta(f, g) = \log|f| d\arg g - \log|g| d\arg f.$$

To construct regulator integrals that relate to $m_3(729 \pm 405\sqrt{3})$ and match them with Beilinson's conjecture, we need to find a pair of rational functions f, g on C that are defined over K (this is for some K-theoretical considerations). Since the inverse of (1.5) is

$$x = \frac{-18k^2 X}{3k^3 X + Y - 36k^9 + 972k^6}, \quad y = 1 - \frac{2Y}{3k^3 X + Y - 36k^9 + 972k^6},$$

the functions $f = x^3, g = y$ on C are defined over K. From now on, we fix this pair of f, g.

Next, we construct the integral paths of $\eta(f,g)$ and $\eta(f^\sigma, g^\sigma)$. Let

$$u^{\pm}(x) = \sqrt[3]{-1 - x^3 + \sqrt{x^6 - (106 \pm 60\sqrt{3})x^3 + 1}},$$

where $\sqrt{\cdot}$ and $\sqrt[3]{\cdot}$ are chosen to be the principal branches, i.e.,

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{\frac{i\theta}{2}}, \quad \sqrt[3]{re^{i\theta}} = \sqrt[3]{r}e^{\frac{i\theta}{3}}, \quad \text{for } r \in \mathbb{R}^{\geqslant 0}, \theta \in (-\pi, \pi].$$

We can factorize $x^3+y^3+1-\sqrt[3]{729 \pm 405\sqrt{3}}xy$ as $(y-y_1^{\pm}(x))(y-y_2^{\pm}(x))(y-y_3^{\pm}(x))$, where

$$\begin{aligned} y_1^{\pm}(x) &= \frac{(3 \pm \sqrt{3})x}{u^{\pm}(x)} + \frac{u^{\pm}(x)}{\sqrt[3]{2}}, \\ y_2^{\pm}(x) &= e^{\frac{4\pi i}{3}} \frac{(3 \pm \sqrt{3})x}{u^{\pm}(x)} + e^{\frac{2\pi i}{3}} \frac{u^{\pm}(x)}{\sqrt[3]{2}}, \\ y_3^{\pm}(x) &= e^{\frac{2\pi i}{3}} \frac{(3 \pm \sqrt{3})x}{u^{\pm}(x)} + e^{\frac{4\pi i}{3}} \frac{u^{\pm}(x)}{\sqrt[3]{2}}. \end{aligned}$$

One can check that

$$\begin{aligned} \{\theta \in [-\pi, \pi] \mid |y_1^{\pm}(e^{i\theta})| \geqslant 1\} &= (-2\pi/3, 2\pi/3], \\ \{\theta \in [0, 2\pi] \mid |y_2^{\pm}(e^{i\theta})| \geqslant 1\} &= (2\pi/3, 2\pi], \\ \{\theta \in [0, 2\pi] \mid |y_3^{\pm}(e^{i\theta})| \geqslant 1\} &= (0, 4\pi/3]. \end{aligned}$$

Let

$$\begin{aligned} \gamma_1^{\pm} &= \{(e^{i\theta}, y_1^{\pm}(e^{i\theta})) \mid -2\pi/3 < \theta \leqslant 2\pi/3\}, \\ \gamma_2^{\pm} &= \{(e^{i\theta}, y_2^{\pm}(e^{i\theta})) \mid 2\pi/3 < \theta \leqslant 2\pi\}, \\ \gamma_3^{\pm} &= \{(e^{i\theta}, y_3^{\pm}(e^{i\theta})) \mid 0 < \theta \leqslant 4\pi/3\}. \end{aligned}$$

Some calculations show that the boundary points of $\gamma_1^{\pm}, \gamma_2^{\pm}$ and $\gamma_3^{\pm}$ are

$$(e^{-\frac{2\pi i}{3}}, r^{\pm}e^{-\frac{\pi i}{3}}), \quad (e^{\frac{2\pi i}{3}}, r^{\pm}e^{\frac{\pi i}{3}}), \quad (1, -r^{\pm}),$$

where $r^+ = \sqrt[3]{20 + 12\sqrt{3}}, r^- = \sqrt[3]{6\sqrt{3} + \frac{9}{2}(\sqrt{6} - \sqrt{2}) - 7}$. Thus $\gamma^{\pm} = \gamma_1^{\pm} \cup \gamma_2^{\pm} \cup \gamma_3^{\pm}$ are continuous closed paths on the Riemann surfaces

$$\left\{(x, y) \in \mathbb{C}^2 \;\middle|\; x^3 + y^3 + 1 - \sqrt[3]{729 \pm 405\sqrt{3}}xy = 0\right\}.$$

The paths of $y_1^{\pm}(e^{i\theta}), y_2^{\pm}(e^{i\theta})$ and $y_3^{\pm}(e^{i\theta})$ are shown in Figure 2 to illustrate this. We define the positive orientations of $\gamma^{\pm}$ in terms of $e^{i\theta}$ running counterclockwise and regard them also as closed paths on $C(\mathbb{C})$ and $C^\sigma(\mathbb{C})$ via (1.5). Since complex conjugation reverses the orientations of $\gamma^{\pm}$, we have $\gamma^+ \in H_1(C(\mathbb{C}), \mathbb{Z})^-$ and $\gamma^- \in H_1(C^\sigma(\mathbb{C}), \mathbb{Z})^-$.

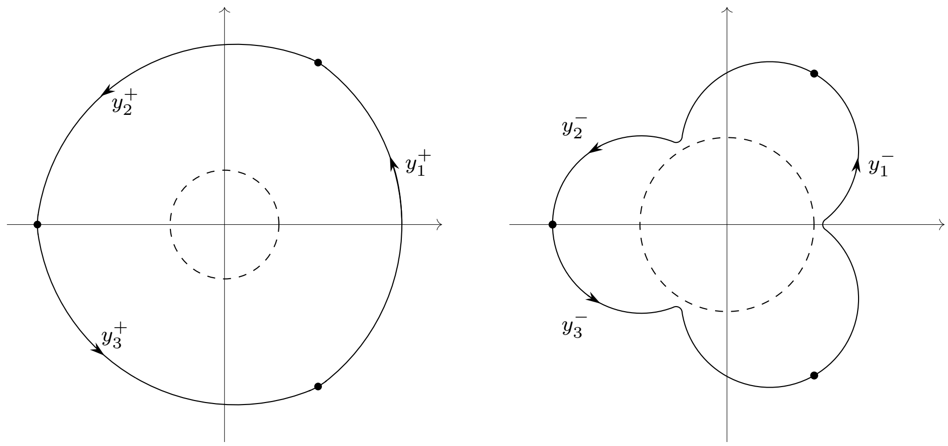

FIGURE 2. The paths of $y_1^{\pm}(e^{i\theta}), y_2^{\pm}(e^{i\theta})$ and $y_3^{\pm}(e^{i\theta})$ on $\mathbb{C}$

After completing all these preparations, Beilinson's conjecture then predicts that the regulator

$$\left|\det\begin{pmatrix} \frac{1}{2\pi}\int_{\gamma^+}\eta(f,g) & \frac{1}{2\pi}\int_{\gamma^-}\eta(f^\sigma,g^\sigma) \\ \frac{1}{2\pi}\int_{\gamma^+}\eta(\phi^*f^\sigma,\phi^*g^\sigma) & \frac{1}{2\pi}\int_{\gamma^-}\eta((\phi^*f^\sigma)^\sigma,(\phi^*g^\sigma)^\sigma)\end{pmatrix}\right| \tag{4.1}$$

should be some rational multiple of $\frac{1}{\pi^4}L(C,2)$. By Jensen's formula, we can calculate that

$$\begin{aligned}\frac{1}{2\pi}\int_{\gamma^+}\eta(f,g) &= \frac{1}{2\pi}\int_{\gamma^+}\eta(x^3,y)\\ &= \frac{1}{2\pi}\int_{\gamma^+}\log|x^3|\operatorname{Im}\left(\frac{dy}{y}\right)-\log|y|\operatorname{Im}\left(\frac{dx^3}{x^3}\right)\\ &= -\frac{3}{2\pi}\int_{\gamma^+}\log|y|\operatorname{Im}\left(\frac{dx}{x}\right)\\ &= -\frac{3}{2\pi}\left(\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}\log|y_1^+(e^{i\theta})|d\theta+\int_{\frac{2\pi}{3}}^{2\pi}\log|y_2^+(e^{i\theta})|d\theta\right.\\ &\qquad\left.+\int_0^{\frac{4\pi}{3}}\log|y_3^+(e^{i\theta})|d\theta\right)\\ &= -m_3(729+405\sqrt{3}).\end{aligned}$$

Similarly, we have $\frac{1}{2\pi}\int_{\gamma^-}\eta(f^\sigma,g^\sigma)=-m_3(729-405\sqrt{3})$ and

$$\begin{aligned}\frac{1}{2\pi}\int_{\gamma^+}\eta(\phi^*f^\sigma,\phi^*g^\sigma) &= \frac{1}{2\pi}\int_{\phi_*\gamma^+}\eta(f^\sigma,g^\sigma),\\ \frac{1}{2\pi}\int_{\gamma^-}\eta((\phi^*f^\sigma)^\sigma,(\phi^*g^\sigma)^\sigma) &= \frac{1}{2\pi}\int_{\gamma^-}\eta((\phi^\sigma)^*f,(\phi^\sigma)^*g)\\ &= \frac{1}{2\pi}\int_{(\phi^\sigma)_*\gamma^-}\eta(f,g).\end{aligned}$$

Thus, in order to work out the second row of (4.1), we should determine the push-forward of the paths γ^+ and γ^- by ϕ and ϕ^σ.

Lemma 4.1. *We have* $\phi_*\gamma^+ = -\gamma^- \in H_1(C^\sigma(\mathbb{C}),\mathbb{Z})^-$ *and* $(\phi^\sigma)_*\gamma^- = -4\gamma^+ \in H_1(C(\mathbb{C}),\mathbb{Z})^-$.

Proof. Again, we can prove this by "numerical method". By (1.5), the invariant differential of C_k can be written as

$$\frac{dX}{2Y} = \frac{1+y-\frac{x(3x^2-ky)}{kx-3y^2}}{6k^2(y-1)(kx+3y+3)}dx.$$

This enables us to calculate numerically (by using Mathematica) the integrations of ω_C and ω_{C^σ} along γ^+ and γ^-:

$$\int_{\gamma^+}\omega_C \approx 0.000735163130i, \quad \int_{\gamma^-}\omega_{C^\sigma} \approx 0.076428679590i. \tag{4.2}$$

Comparing these values with the real and complex periods of the lattices corresponding to C and C^σ calculated by SageMath, we find that γ^+ and γ^- are in fact generators of $H_1(C(\mathbb{C}),\mathbb{Z})^-$ and $H_1(C^\sigma(\mathbb{C}),\mathbb{Z})^-$, respectively. Thus, there must exist integers $a,b\in\mathbb{Z}$ such that $\phi_*\gamma^+ = a\gamma^-, (\phi^\sigma)_*\gamma^- = b\gamma^+$. Since $\phi\circ\phi^\sigma = [4]$, we also have $ab=4$. Moreover, one can calculate that

$$\int_{\phi_*\gamma^+}\omega_{C^\sigma} = \int_{\gamma^+}\phi^*\omega_{C^\sigma} = -(52+30\sqrt{3})\int_{\gamma^+}\omega_C \approx -0.076428679590i. \tag{4.3}$$

Comparing (4.2) and (4.3), we immediately observe that $a=-1$ and thus $b=-4$. ☐

According to Lemma 4.1, the regulator (4.1) equals

$$\left|\det\begin{pmatrix} m_3(729+405\sqrt{3}) & m_3(729-405\sqrt{3}) \\ m_3(729-405\sqrt{3}) & 4m_3(729+405\sqrt{3})\end{pmatrix}\right|.$$

Proof of Theorem 1.4. Since the Sturm bound for $\mathcal{M}_2(\Gamma_0(144))$ is 48, we can prove that

$$F_{144}(\tau) = \frac{1}{2}f_{36}(\tau) - 2f_{36}(4\tau) + \frac{1}{2}f_{144}(\tau), \quad \widetilde{F}_{144}(\tau) = -\frac{1}{4}f_{36}(\tau) + f_{36}(4\tau) + \frac{1}{4}f_{144}(\tau).$$

And thus

$$L(F_{144},2) = \frac{3}{8}L(f_{36},2) + \frac{1}{2}L(f_{144},2), \quad L(\widetilde{F}_{144},2) = -\frac{3}{16}L(f_{36},2) + \frac{1}{4}L(f_{144},2).$$

By (1.15), we have

$$\begin{aligned}
&4m_3(729+405\sqrt{3})^2 - m_3(729-405\sqrt{3})^2 \\
&= 4\left(\frac{81}{\pi^2}\left(\frac{3}{8}L(f_{36},2)+\frac{1}{2}L(f_{144},2)\right)\right)^2 - \left(\frac{324}{\pi^2}\left(-\frac{3}{16}L(f_{36},2)+\frac{1}{4}L(f_{144},2)\right)\right)^2 \\
&= \frac{19683}{\pi^4}L(f_{36},2)L(f_{144},2) \\
&= \frac{19683}{\pi^4}L(C,2) \\
&= \frac{243}{8}L''(C,0),
\end{aligned}$$

where the last equality follows by the functional equation of $L(C,s)$. ☐

Finally, let us prove Theorem 1.5. To achieve this, we need the following hypergeometric formula for $\tilde{n}(k)$ proved by Samart.

Theorem 4.2 ([6, Theorem 1]). *Let $\tilde{n}(k)$ be the modified Mahler measure* (1.20). *Then for $k \in (-1,3)-\{0\}$, the following identity is true:*

$$\tilde{n}(k) = \frac{4}{1-3\operatorname{sgn}(k)} \operatorname{Re}\left(\log k - \frac{2}{k^3}\, {}_4F_3\left(\begin{matrix} \frac{4}{3},\ \frac{5}{3},\ 1,\ 1 \\ 2,\ 2,\ 2 \end{matrix} \,\middle|\, \frac{27}{k^3}\right)\right).$$

For $k \in \mathbb{C} - \mathcal{K}_Q^\circ$, we also have [16, Theorem 3.1]

$$m(Q_k) = \operatorname{Re}\left(\log k - \frac{2}{k^3}\, {}_4F_3\left(\begin{matrix} \frac{4}{3},\ \frac{5}{3},\ 1,\ 1 \\ 2,\ 2,\ 2 \end{matrix} \,\middle|\, \frac{27}{k^3}\right)\right). \tag{4.4}$$

Proof of Theorem 1.5. Since $t(\tau)$ is a Hauptmodul for $\Gamma_0(3)$, the map $\tau \mapsto t(\tau)$ is a biholomorphic mapping form the genus zero Riemann surface $X_0(3)$ to $\mathbb{P}^1(\mathbb{C})$. Also, it is known from [10, §14] that $t\big(\pm\frac{1}{2} + \frac{i}{2\sqrt{3}}\big) = 0$. Thus, we have

$$t(\tau) \neq 0, \quad \forall \tau \in \mathcal{F}' - \left\{\pm\frac{1}{2} + \frac{i}{2\sqrt{3}}\right\}.$$

By Rodriguez Villegas' formula (1.7) and the above hypergeometric formula (4.4), the equation

$$\begin{aligned}\operatorname{Re}&\left(\frac{\log t(\tau)}{3} - \frac{2}{t(\tau)}\, {}_4F_3\left(\begin{matrix} \frac{4}{3},\ \frac{5}{3},\ 1,\ 1 \\ 2,\ 2,\ 2 \end{matrix} \,\middle|\, \frac{27}{t(\tau)}\right)\right) \\ &= \frac{27\sqrt{3}\operatorname{Im}(\tau)}{4\pi^2} \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-3}(n)(3m\operatorname{Re}(\tau)+n)}{|3m\tau+n|^4}\end{aligned}$$

holds on some open set of $\mathcal{F}'$. Since both sides of the above equation are harmonic on $\mathcal{F}'^\circ$, they must coincide for every $\tau \in \mathcal{F}' - \big\{\pm\frac{1}{2} + \frac{i}{2\sqrt{3}}\big\}$. Hence, according to Theorem 4.2, we just proved that for $\tau \in \mathcal{F}'$ with $\sqrt[3]{t(\tau)} \in (-1,3)-\{0\}$, the following formula for $\tilde{n}(k)$ holds:

$$\tilde{n}(\sqrt[3]{t(\tau)}) = \frac{27\sqrt{3}\operatorname{Im}(\tau)}{\big(1-3\operatorname{sgn}(\sqrt[3]{t(\tau)})\big)\pi^2} \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-3}(n)(3m\operatorname{Re}(\tau)+n)}{|3m\tau+n|^4}. \tag{4.5}$$

Now, we are back to our familiar track. From the outputs of Algorithm 2.3, we find that $t([9,3,1]) = t\big(-\frac{1}{6} + \frac{i}{2\sqrt{3}}\big) = 24$. Note that this point is not listed in Table 2, because we only listed in Table 2 those points that make $\sqrt[3]{t(\tau)} \in \mathbb{C} - \mathcal{K}_Q^\circ$. By (4.5), we immediately obtain that

$$\tilde{n}(\sqrt[3]{24}) = -\frac{81}{4\pi^2} L(f_{27}, 2) = -3L'(f_{27}, 0),$$

where $f_{27}(\tau) = \frac{1}{6} \sum_{m,n\in\mathbb{Z}} \chi_{-3}(n)(m+2n)q^{m^2+mn+n^2} = \eta(3\tau)^2\eta(9\tau)^2$ is the unique normalized cusp form in $\mathcal{M}_2(\Gamma_0(27))$. This is exactly the newform that corresponds to $C_{\sqrt[3]{24}}$ which is isomorphic over $\mathbb{Q}$ to the elliptic curve with LMFDB label `27.a1`. □

5. Some remarks

Our approach was originally inspired by the work [20] of Huber, Schultz and Ye on $1/\pi$-series. In fact, Q. He and Ye have already proved in [18] all formulas conjectured by Samart in [7] that involve the Mahler measure of the trivariate Laurent polynomial

$$\left(x+\frac{1}{x}\right)^2\left(y+\frac{1}{y}\right)^2\frac{(1+z)^3}{z^2}-s.$$

They also suggested that Samart's conjectural identities associated to

$$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)+\sqrt{s} \quad \text{and} \quad x^4+y^4+z^4+1-\sqrt[4]{s}xyz$$

might be proved using their method. Moreover, in [13], Fei expressed the Mahler measures of 23 families of Laurent polynomials in terms of Kronecker-Eisenstein series. There seems to be a huge number of identities that can be proved. Finally, it will be interesting if one could prove an identity that relates a 3×3 or higher order determinant with Mahler measures as entries to the L-value of an elliptic curve.

Acknowledgments

We would like to thank Detchat Samart for the valuable discussions about his modified Mahler measure $\tilde{n}(k)$. We also express our thanks to Dr. Yu Xu for his helpful comments. Finally, we are grateful to the referees for their comments and suggestions to improve this paper.

References

[1] B. Schoeneberg, *Elliptic Modular Functions: An Introduction*, Die Grundlehren der mathematischen Wissenschaften, Band 203, Springer-Verlag, New York-Heidelberg, 1974. Translated from the German by J. R. Smart and E. A. Schwandt. MR412107

[2] C. Deninger, *Deligne periods of mixed motives, K-theory and the entropy of certain* $\mathbf{Z}^n$*-actions*, J. Amer. Math. Soc. **10** (1997), no. 2, 259–281, DOI 10.1090/S0894-0347-97-00228-2. MR1415320

[3] D. A. Cox, *Primes of the Form* x^2+ny^2, 2nd ed., Pure and Applied Mathematics (Hoboken), John Wiley & Sons, Inc., Hoboken, NJ, 2013. Fermat, class field theory, and complex multiplication, DOI 10.1002/9781118400722. MR3236783

[4] D. W. Boyd, *Mahler's measure and special values of L-functions*, Experiment. Math. **7** (1998), no. 1, 37–82. MR1618282

[5] D. Samart, *A functional identity for Mahler measures of non-tempered polynomials*, Integral Transforms Spec. Funct. **32** (2021), no. 1, 78–89, DOI 10.1080/10652469.2020.1799357. MR4238025

[6] D. Samart, *Mahler measure of a non-reciprocal family of elliptic curves*, Q. J. Math. **74** (2023), no. 3, 1187–1208, DOI 10.1093/qmath/haad016. MR4642252

[7] D. Samart, *Mahler measures as linear combinations of L-values of multiple modular forms*, Canad. J. Math. **67** (2015), no. 2, 424–449, DOI 10.4153/CJM-2014-012-8. MR3314841

[8] F. Brunault, *Parametrizing elliptic curves by modular units*, J. Aust. Math. Soc. **100** (2016), no. 1, 33–41, DOI 10.1017/S1446788715000233. MR3436769

[9] F. Brunault and W. Zudilin, *Many Variations of Mahler Measures—A Lasting Symphony*, Australian Mathematical Society Lecture Series, vol. 28, Cambridge University Press, Cambridge, 2020. MR4382435

[10] F. R. Villegas, *Modular Mahler measures. I*, Topics in Number Theory (University Park, PA, 1997), Math. Appl., vol. 467, Kluwer Acad. Publ., Dordrecht, 1999, pp. 17–48. MR1691309

[11] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern* (German), Invent. Math. **73** (1983), no. 3, 349–366, DOI 10.1007/BF01388432. MR718935

[12] H. Cohen and F. Strömberg, *Modular Forms*, Graduate Studies in Mathematics, vol. 179, American Mathematical Society, Providence, RI, 2017. A classical approach, DOI 10.1090/gsm/179. MR3675870

[13] J. Fei, *Mahler measure of 3D Landau-Ginzburg potentials*, Forum Math. **33** (2021), no. 5, 1369–1401, DOI 10.1515/forum-2020-0339. MR4308625

[14] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994, DOI 10.1007/978-1-4612-0851-8. MR1312368

[15] L. C. Washington, *Elliptic Curves*, 2nd ed., Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2008. Number theory and cryptography, DOI 10.1201/9781420071474. MR2404461

[16] M. Rogers, *Hypergeometric formulas for lattice sums and Mahler measures*, Int. Math. Res. Not. IMRN **17** (2011), 4027–4058, DOI 10.1093/imrn/rnq240. MR2836402

[17] M. Rogers and W. Zudilin, *On the Mahler measure of* $1 + X + 1/X + Y + 1/Y$, Int. Math. Res. Not. IMRN **9** (2014), 2305–2326, DOI 10.1093/imrn/rns285. MR3207368

[18] Q. He and D. Ye, *On conjectures of Samart*, Manuscripta Math. **167** (2022), no. 3-4, 545–588, DOI 10.1007/s00229-021-01279-6. MR4385383

[19] T. Dokchitser, R. de Jeu, and D. Zagier, *Numerical verification of Beilinson's conjecture for* K_2 *of hyperelliptic curves*, Compos. Math. **142** (2006), no. 2, 339–373, DOI 10.1112/S0010437X05001892. MR2218899

[20] T. Huber, D. Schultz, and D. Ye, *Series for* $1/\pi$ *of level 20*, J. Number Theory **188** (2018), 121–136, DOI 10.1016/j.jnt.2017.12.010. MR3778626

[21] The LMFDB Collaboration, *The L-functions and modular forms database*, 2023, `http://www.lmfdb.org`.

[22] X. Guo, Q. Ji, H. Liu, and H. Qin, *The Mahler measure of* $x + 1/x + y + 1/y + 4 \pm 4\sqrt{2}$ *and Beilinson's conjecture*, Int. J. Number Theory **20** (2024), no. 1, 185–197, DOI 10.1142/S1793042124500088. MR4688732

[23] Y. Bilu, P. Habegger, and L. Kühne, *No singular modulus is a unit*, Int. Math. Res. Not. IMRN **24** (2020), 10005–10041, DOI 10.1093/imrn/rny274. MR4190395

[24] Z. Tao, X. Guo, and T. Wei, *Mahler measures and L-values of elliptic curves over real quadratic fields*, Preprint, `arXiv:2209.14717`, 2022.

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

Email address: `taozhy@smail.nju.edu.cn`

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

Email address: `guoxj@nju.edu.cn`