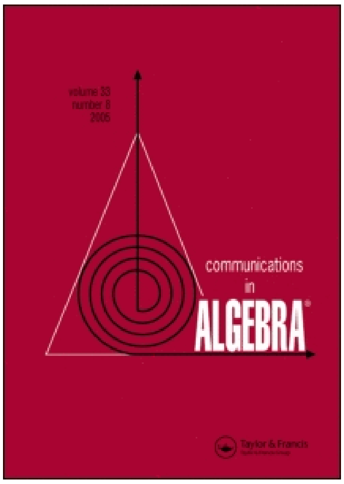

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A Note on the 4-Rank Densities of K_2O<i>F</i> for Quadratic Number Fields *F*

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A NOTE ON THE 4-RANK DENSITIES OF K_2O_F FOR QUADRATIC NUMBER FIELDS F

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In this article, we compute the 4-rank densities of the tame kernel of the quadratic number fields $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-2pl})$ with p, l odd rational prime, using a result related to a basic problem of characterizing the primes represented by $x^2 + 64y^2$.

Key Words: Positive definite quadratic forms; Tame kernel; 4-Rank densities.

2000 Mathematics Subject Classification: 11R70; 19F99.

1. INTRODUCTION

Let F be a quadratic number field and O_F its ring of integers. The structure of the 2-Sylow subgroup of K_2O_F has been intensively studied. As we know, the 2^j-rank of K_2O_F is the number of cyclic factors of K_2O_F of order divisible by 2^j. Tate (1976) gave the well-known formula of the 2-rank of the tame kernel for any number field. If F is a quadratic number field, Browkin and Schinzel (1982) simplified Tate's 2-rank formula. Qin (1995a,b) gives an ingenious approach to compute the 4-rank of K_2O_F. Conner and Hurrelbrink (2001) characterize the 4-rank of K_2O_F for certain quadratic number fields in terms of positive definite binary quadratic forms. Recently, Osburn and Murray considered the densities of 4-rank of the tame kernel of F. Osburn and Murray (2003) computed the 4-rank densities of the quadratic number fields of $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ with p, l odd rational prime. In this article, we focus on the 4-rank densities of the tame kernel of $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for odd primes p, l.

In Section 2, the definition and some known results are given. Put $\mathfrak{S} = \{p \text{ prime} \mid p = x^2 + 64y^2\}$ and we claim that $\mathfrak{S}$ has density $\frac{1}{8}$ as a subset of all primes and thus has density $\frac{1}{2}$ as a subset of the set of $p \equiv 1 \pmod 8$ with p prime. We will use this result to compute the 4-rank densities of the quadratic number fields $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for odd primes p, l.

From Qin's tables on the 4-rank of the tame kernel of quadratic number fields, we only need to deal with the 4-rank of the tame kernel of the quadratic number

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fields $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for odd primes $p \equiv l \equiv 1 \pmod 8$ and $p \equiv l \equiv 7 \pmod 8$. In fact, Osburn (2002) has solved the case when $p, l \equiv 1, 7 \pmod 8$.

Fix a prime $p \equiv 1 \pmod 8$ and consider the sets

$$A = \left\{ l \text{ rational primes} \mid l \equiv 1 \pmod 8 \text{ and } \left(\frac{l}{p}\right) = 1 \right\},$$

$$B = \left\{ l \text{ rational primes} \mid l \equiv 1 \pmod 8 \text{ and } \left(\frac{l}{p}\right) = -1 \right\}.$$

Similarly, fix a prime $p \equiv 7 \pmod 8$, we can define the following sets:

$$C = \left\{ l \text{ rational primes} \mid l \equiv 7 \pmod 8 \text{ and } \left(\frac{l}{p}\right) = 1 \right\},$$

$$D = \left\{ l \text{ rational primes} \mid l \equiv 7 \pmod 8 \text{ and } \left(\frac{l}{p}\right) = -1 \right\}.$$

In Section 3 we prove that (i) for real quadratic number field $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 appear with natural density $\frac{3}{4}$ and $\frac{1}{4}$ in A. For the imaginary number field $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 1 and 2 appear with natural density $\frac{3}{4}$ and $\frac{1}{4}$ in A; (ii) for real quadratic number field $\mathbb{Q}(\sqrt{2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in B. For the imaginary number field $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in B; (iii) for the imaginary number field $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in C; (iv) for the imaginary number field $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in D.

As a result, for square-free, even integers d, we can obtain the 4-rank densities in X and Y, respectively, where

$$X = \{d \mid d = 2pl\}$$

and

$$Y = \{d \mid d = -2pl\}$$

for distinct odd primes p and l.

2. 4-RANK OF K_2O_F IN POSITIVE DEFINITE FORMS

We start with the following definition on positive definite binary quadratic forms which will be often used throughout this article.

Definition 2.1. For primes $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = 1$, $\mathcal{K} = \mathbb{Q}(\sqrt{2p})$, and $h^+(\mathcal{K})$ the narrow class number of $\mathcal{K}$, we say:

l satisfies $\langle 1, 32 \rangle$ if and only if $l = x^2 + 32y^2$ for some $x, y \in \mathbb{Z}$;

l satisfies $\langle 1, 64 \rangle$ if and only if $l = x^2 + 64y^2$ for some $x, y \in \mathbb{Z}$;

l satisfies $\langle p, -2 \rangle$ if and only if $l^{h^+(\mathcal{K})} = pn^2 - 2m^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod l$.

l satisfies $\langle 1, -2p\rangle$ if and only if $l^{h^+(\mathcal{K})} = n^2 - 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{l}$.

From Cox (1989, Theorems 9.8 and 9.12), we can obtain some properties of the primes that satisfy $\langle 1, 64\rangle$.

Lemma 2.2. *If p is an odd prime, then p satisfies $\langle 1, 64\rangle \iff p \equiv 1 \pmod{4}$ and $x^4 \equiv 2 \pmod{p}$ has an integer solution, in other words, $\left(\frac{\sqrt{2}}{p}\right) = 1$.*

Lemma 2.3. *Let $ax^2 + bxy + cy^2$ be a primitive positive definite quadratic form of discriminant $D < 0$, and let $\mathfrak{S}$ be the set of primes represented by $ax^2 + bxy + cy^2$. Then the Dirichlet density $\delta(\mathfrak{S})$ exists and is given by the formula*

$$\delta(\mathfrak{S}) = \begin{cases} \dfrac{1}{2h(D)}, & \text{if } ax^2 + bxy + cy^2 \text{ is properly equivalent to the opposite,} \\ \dfrac{1}{h(D)}, & \text{otherwise,} \end{cases}$$

where $h(D)$ is the number of classes of primitive positive definite forms of discriminant D. In particular, $ax^2 + bxy + cy^2$ represents infinitely many prime numbers.

It is easy to see that $x^2 + 64y^2$ is a primitive positive definite quadratic form of discriminant $-256 < 0$. Let $\mathfrak{S}$ be the set of primes represented by $x^2 + 64y^2$. Since $h(-256) = 4$, the Dirichlet density $\delta(\mathfrak{S}) = \frac{1}{8}$. Now we have the following proposition.

Proposition 2.4. *$\mathfrak{S}$ has density $\frac{1}{8}$ as a subset of all primes, and hence $\mathfrak{S}$ has density $\frac{1}{2}$ as a subset of the set of $p \equiv 1 \pmod{8}$.*

Now let us recall some results from Conner and Hurrelbrink (2001) and Osburn and Murray (2003). Let us consider the quadratic number field $L = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. Let $\varepsilon = 1 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ denote the fundamental unit of $\mathbb{Q}(\sqrt{2})$ which has norm -1. Then the degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1})$. Set $N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1})$. Note that $\mathrm{Gal}(N_1/\mathbb{Q})$ is a dihedral group of order 8. We have the following lemma.

Lemma 2.5. *$\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{id\} \iff l$ splits completely in $N_1 \iff l$ satisfies $\langle 1, 32\rangle$.*

Consider a fixed prime $p \equiv 1 \pmod{8}$. Then p splits completely in $L = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ and so $pO_L = \mathfrak{P}\mathfrak{P}'$, where $\mathfrak{P}$ and are different prime ideals of the field L. We have

Lemma 2.6. *The prime $\mathfrak{P}$ which occurs in the decomposition of pO_L has a generator $\pi = a + b\sqrt{2} \in O_L$, unique up to a sign and to multiplication by the square of a unit in O_L^* for which $N_{L/\mathbb{Q}}(\pi) = a^2 - 2b^2 = p$.*

Set $N_2 = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p})$, which is the normal closure of $\mathbb{Q}(\sqrt{2}, \sqrt{\pi})$. Note that $\mathrm{Gal}(N_2/\mathbb{Q})$ is also a dihedral group of order 8. We have the following lemma.

Lemma 2.7. *Let* $l \equiv 1 \pmod 8$ *with* $\left(\frac{l}{p}\right) = 1$. *Then*:

$$\left(\frac{N_2/\mathbb{Q}}{l}\right) = \{id\} \Leftrightarrow l \text{ splits completely in } N_2 \Leftrightarrow l \text{ satisfies } \langle 1, -2p \rangle \Longleftrightarrow \left(\frac{\pi}{l}\right) = 1,$$

$$\left(\frac{N_2/\mathbb{Q}}{l}\right) \neq \{id\} \Leftrightarrow l \text{ does not split completely in}$$

$$N_2 \Leftrightarrow l \text{ satisfies } \langle p, -2 \rangle \Longleftrightarrow \left(\frac{\pi}{l}\right) = -1.$$

Additionally, the following lemma is known.

Lemma 2.8. $\left(\frac{\mathbb{Q}(\zeta_{16})/\mathbb{Q}}{l}\right) = \{id\} \Longleftrightarrow l$ *splits completely in* $\mathbb{Q}(\zeta_{16}) \Longleftrightarrow l \equiv 1 \pmod{16} \Longleftrightarrow \left(\frac{2+\sqrt{2}}{l}\right) = 1$.

Let M be a Galois extension of $\mathbb{Q}$, and $G = \mathrm{Gal}(M/\mathbb{Q})$. Let $Z(G)$ denote the center of G and $M^{Z(G)}$ denote the fixed field of $Z(G)$. Let p be a rational prime which is unramified in M. Let $\left(\frac{M/\mathbb{Q}}{p}\right)$ denote the Artin symbol of p and $\{g\}$ the conjugated class containing one element $g \in G$. From Osburn (2002), we have the following lemma.

Lemma 2.9. $\left(\frac{M/\mathbb{Q}}{p}\right) = \{g\}$ *for some* $g \in Z(G)$ *if and only if* p *splits completely in* $M^{Z(G)}$.

Set

$$H = N_1 N_2 \mathbb{Q}(\zeta_{16}),$$

then $[H : \mathbb{Q}] = 64$. Let $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. Since l splits completely in $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{p})$, it is easy to see that l splits completely in M. Similarly to Osburn (2002, Lemma 4.1), we have the following lemma.

Lemma 2.10. $Z(\mathrm{Gal}(H/\mathbb{Q})) = \mathrm{Gal}(H/M)$ *is elementary abelian of order* 8, *where* $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$.

Next we will recall some results from algebraic K-theory. Hurrelbrink and Kolster (1998) generalize Qin (1994, 1995a,b) approach and obtain 4-rank of the tame kernel of quadratic number fields by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. Specially, let $F = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$ and square-free. Let $p_1, p_2, \ldots, p_t$ denote the odd primes dividing d. Recall that 2 is a norm from F if and only if all p_i's are $\equiv \pm 1 \pmod 8$. If so, then $d = u^2 - 2w^2$ for $u, w \in \mathbb{Z}$. Set $v = u + w$ in this case. If 2 is not a norm from F, set $v = 2$. Now consider two matrices.

If $d < 0$,

$$M'_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \cdots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \cdots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \cdots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \\ (-d, -1)_2 & (-d, -1)_{p_1} & \cdots & (-d, -1)_{p_t} \end{pmatrix}$$

and if $d > 0$,

$$M_{F/\mathbb{Q}} = \begin{pmatrix} (-d, p_1)_2 & (-d, p_1)_{p_1} & \cdots & (-d, p_1)_{p_t} \\ (-d, p_2)_2 & (-d, p_2)_{p_1} & \cdots & (-d, p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \cdots & (-d, p_{t-1})_{p_t} \\ (-d, v)_2 & (-d, v)_{p_1} & \cdots & (-d, v)_{p_t} \\ (d, -1)_2 & (d, -1)_{p_1} & \cdots & (d, -1)_{p_t} \end{pmatrix}$$

Replacing the 1's by 0's and the -1's by 1's, we calculate the matrix rank over $\mathbb{F}_2$. Now from Hurrelbrink and Kolster (1998) or Osburn and Murray (2003), the 4-rank of the tame kernel of quadratic number fields can be characterized in this way.

Lemma 2.11. *$F = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$ and square-free. Then*:

(i) *If $d < 0$, then 4-rank $K_2O_F = t - rk(M'_{F/\mathbb{Q}})$;*
(ii) *If $d > 0$, then 4-rank $K_2O_F = t - rk(M_{F/\mathbb{Q}}) + a' - a$,*

where

$$a = \begin{cases} 0, & \text{if } 2 \text{ is a norm from } F \\ 1, & \text{otherwise} \end{cases}$$

and

$$a' = \begin{cases} 0, & \text{if both } -1 \text{ and } 2 \text{ are norms from } F \\ 1, & \text{if exactly one of } -1, 2 \text{ or } -2 \text{ is a norm from } F \\ 2, & \text{if none of } -1 \text{ or } 2 \text{ or } -2 \text{ are norms from } F. \end{cases}$$

3. 4-RANK DENSITIES OF THE TAME KERNEL

In this section, we focus on the 4-rank densities of the tame kernel of quadratic number fields $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-2pl})$ with p, l odd prime. Note that we discuss the following cases:

(i) $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-2pl})$ where $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = 1$;
(ii) $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-2pl})$ where $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = -1$;

(iii) $\mathbb{Q}(\sqrt{-2pl})$ where $p \equiv l \equiv 7 \pmod 8$ with $\left(\frac{l}{p}\right) = 1$;
(iv) $\mathbb{Q}(\sqrt{-2pl})$ where $p \equiv l \equiv 7 \pmod 8$ with $\left(\frac{l}{p}\right) = -1$.

It is easy to see that in both cases, 2 is a norm from $\mathbb{Q}(\sqrt{2pl})$ and $\mathbb{Q}(\sqrt{-2pl})$.

Recall from Hurrelbrink and Kolster (1998) the following lemma.

Lemma 3.1. (1) $(-2pl, v)_2 = 1 \Leftrightarrow$ *both* p, l *satisfy* $\langle 1, 64\rangle$ *or neither* p, l *satisfy* $\langle 1, 64\rangle$. (2) $(2pl, v)_2 = 1 \Leftrightarrow pl \equiv 1 \pmod{16}$.

Consider the following matrices.

For $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = 1$, we have

$$M_{\mathbb{Q}(\sqrt{2pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ (-2pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix},$$

$$M'_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ (2pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix}$$

For $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = -1$, we have

$$M_{\mathbb{Q}(\sqrt{2pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 1 & 1 \\ (-2pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix},$$

$$M'_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 1 & 1 \\ (2pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 0 & 0 \end{pmatrix}.$$

For $p \equiv l \equiv 7 \pmod 8$ with $\left(\frac{l}{p}\right) = 1$, we have

$$M'_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 1 & 1 \\ (2pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 1 & 1 \end{pmatrix}.$$

For $p \equiv l \equiv 7 \pmod 8$ with $\left(\frac{l}{p}\right) = -1$, we have

$$M'_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ (2pl, v)_2 & \left(\frac{v}{p}\right) & \left(\frac{v}{l}\right) \\ 0 & 1 & 1 \end{pmatrix}.$$

By Lemma 2.11, we can see the following remark.

Remark 3.2. For $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = 1$, we have:

(i) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 1 \iff$ rank $M_{\mathbb{Q}(\sqrt{2pl})/\mathbb{Q}} = 1 \iff (-2pl, v)_2 = 1, \left(\frac{v}{l}\right) = -1$ or $(-2pl, v)_2 = -1 \iff$ both p, l satisfy $\langle 1, 64\rangle$, $\left(\frac{v}{l}\right) = -1$ or neither p, l satisfy $\langle 1, 64\rangle$, $\left(\frac{v}{l}\right) = -1$, or exactly one of p, l satisfies $\langle 1, 64\rangle$;
(ii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 2 \iff$ rank $M_{\mathbb{Q}(\sqrt{2pl})/\mathbb{Q}} = 0 \iff (-2pl, v)_2 = 1, \left(\frac{v}{l}\right) = 1 \iff$ both p, l satisfy $\langle 1, 64\rangle$, $\left(\frac{v}{l}\right) = 1$ or neither p, l satisfy $\langle 1, 64\rangle$, $\left(\frac{v}{l}\right) = 1$;
(iii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 1 \iff$ rank $M_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = 1 \iff (-2pl, v)_2 = 1, \left(\frac{v}{l}\right) = -1$ or $(-2pl, v)_2 = -1 \iff p \equiv l \equiv 1 \pmod{16}, \left(\frac{v}{l}\right) = -1$ or $p \equiv l \equiv 9 \pmod{16}, \left(\frac{v}{l}\right) = -1$ or $p \equiv 1 \pmod{16}, l \equiv 9 \pmod 8$ or $p \equiv 9 \pmod{16}$, $l \equiv 1 \pmod{16}$;
(iv) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 2 \iff$ rank $M_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = 0 \iff (2pl, v)_2 = 1, \left(\frac{v}{l}\right) = 1 \iff p \equiv l \equiv 1 \pmod{16}, \left(\frac{v}{l}\right) = 1$ or $p \equiv l \equiv 9 \pmod{16}, \left(\frac{v}{l}\right) = 1$.

Remark 3.3. For $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = -1$, we have:

(i) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 1 \iff$ rank $M_{\mathbb{Q}(\sqrt{2pl})/\mathbb{Q}} = 1 \iff (-2pl, v)_2 = 1 \iff$ both p, l satisfy $\langle 1, 64\rangle$ or neither p, l satisfy $\langle 1, 64\rangle$;
(ii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 0 \iff$ rank $M_{\mathbb{Q}(\sqrt{2pl})/\mathbb{Q}} = 2 \iff (-2pl, v)_2 = -1 \iff$ one of p, l satisfies $\langle 1, 64\rangle$;
(iii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 1 \iff$ rank $M_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = 1 \iff (2pl, v)_2 = 1 \iff pl \equiv 1 \pmod{16} \iff p \equiv l \equiv 1 \pmod{16}$ or $p \equiv l \equiv 9 \pmod{16}$;
(vi) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 0 \iff$ rank $M_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = 2 \iff (2pl, v)_2 = -1 \iff pl \equiv 9 \pmod{16} \iff p \equiv 1 \pmod{16}, l \equiv 9 \pmod{16}$ or $p \equiv 9 \pmod{16}, p \equiv 1 \pmod{16}$.

Remark 3.4. For $p \equiv l \equiv 7 \pmod 8$, we have:

(i) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 1 \iff$ rank $M'_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = 1 \iff (2pl, 2) = 1 \iff pl \equiv 1 \pmod{16} \iff p \equiv l \equiv 7 \pmod{16}$ or $p \equiv l \equiv 15 \pmod{16}$;
(ii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 0 \iff$ rank $M'_{\mathbb{Q}(\sqrt{-2pl})/\mathbb{Q}} = 2 \iff (2pl, 2) = -1 \iff pl \equiv 9 \pmod{16} \iff p \equiv 7 \pmod{16}, l \equiv 15 \pmod{16}$ or $p \equiv 15 \pmod{16}$, $l \equiv 7 \pmod{16}$.

Proposition 3.5. *Let $d = \pm 2pl$ be as above, $d = u^2 - 2w^2$ with $u, w \in \mathbb{Z}$. Then*

$$\left(\frac{v}{l}\right) = \left(\frac{\pi}{l}\right)\left(\frac{2+\sqrt{2}}{l}\right)$$

if $d = 2pl$ and

$$\left(\frac{v}{l}\right) = \left(\frac{\pi}{l}\right)\left(\frac{\sqrt{2}}{l}\right)$$

if $d = -2pl$.

Proof. We use the identity

$$\left(\frac{v}{l}\right) = \left(\frac{\gamma + \delta\sqrt{2}}{l}\right)\left(\frac{1+\sqrt{2}}{l}\right),$$

where $\frac{d}{l} = N_{L/\mathbb{Q}}(\gamma + \delta\sqrt{2})$ for $\gamma, \delta \in \mathbb{Z}$. For $d = 2pl$, we can see that $\frac{d}{l} = 2p$ and thus $\gamma + \delta\sqrt{2} = \sqrt{2}\pi$, up to squares. For $d = -2pl$, we have $\frac{d}{l} = -2p = -2N_{L/\mathbb{Q}}(\pi)$ and so $\gamma + \delta\sqrt{2} = (2+\sqrt{2})\pi$, up to squares.

Combining Proposition 3.5 with Remarks 3.2–3.4, we have the following proposition.

Proposition 3.6. *For $p \equiv l \equiv 1 \pmod 8$ with $\left(\frac{l}{p}\right) = 1$:*

(i) *4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 1 \iff$ both p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle 1, 32\rangle$ and l satisfies $\langle p, -2\rangle$ or both p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle 1, -2p\rangle$ and l does not satisfy $\langle 1, 32\rangle$ or neither p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle 1, 32\rangle$ and l satisfies $\langle 1, -2p\rangle$ or neither p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle p, -2\rangle$ and l does not satisfy $\langle 1, 32\rangle$ or one of p, l satisfies $\langle 1, 64\rangle$;*

(ii) *4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 2 \iff$ both p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle 1, -2p\rangle$ and l satisfies $\langle 1, 32\rangle$ or both p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle p, -2\rangle$ and l does not satisfy $\langle 1, 32\rangle$ or neither p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle p, -2\rangle$ and l satisfies $\langle 1, 32\rangle$ or neither p, l satisfy $\langle 1, 64\rangle$, l satisfies $\langle 1, -2p\rangle$ and l does not satisfy $\langle 1, 32\rangle$.*

Let A, B, C, and D be defined as in Section 1. We can now prove the following theorem.

Theorem 3.7. *For real quadratic number field $\mathbb{Q}(\sqrt{2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in B. For the imaginary number field $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in B.*

Proof. First we consider the case of $\mathbb{Q}(\sqrt{2pl})$. Consider the sets

$$\mathscr{A}_1 = \{l \text{ prime} \mid l \equiv 1 \pmod 8 \text{ and } l \text{ satisfies } \langle 1, 64\rangle\},$$
$$\mathscr{A}_2 = \{l \text{ prime} \mid l \equiv 1 \pmod 8 \text{ and } l \text{ does not satisfy } \langle 1, 64\rangle\}.$$

By Proposition 2.4, $\mathscr{A}_1, \mathscr{A}_2$ each have density $\frac{1}{2}$ in the set of all primes $l \equiv 1 \pmod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, $\mathscr{A}_1$ and $\mathscr{A}_2$ each have density $\frac{1}{8}$ in the set of all primes l. It is easy to see that for primes $p \equiv 1 \pmod 8$, the sets

$$\mathscr{B}_1 = \left\{l \text{ prime} \mid l \equiv 1 \pmod 8, \left(\frac{l}{p}\right) = -1, \text{ and } l \text{ satisfies } \langle 1, 64\rangle\right\},$$

$$\mathscr{B}_2 = \{l \text{ prime} \mid l \equiv 1 \pmod 8, \left(\frac{l}{p}\right) = -1, \text{ and } l \text{ does not satisfy } \langle 1, 64\rangle\Big\}$$

each have density $\frac{1}{2}$ in $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. Thus $\mathscr{B}_1$ and $\mathscr{B}_2$ have densities $\frac{1}{16}$ in the set of all primes l. If p satisfies $\langle 1, 64\rangle$, then by Remark 3.3,

$$\mathscr{B}_1 = \left\{ l \text{ prime} \mid l \equiv 1 \pmod 8, \left(\frac{l}{p}\right) = -1, \text{ and 4-rank } K_2 O_{\mathbb{Q}(\sqrt{-2pl})} = 1 \right\},$$

$$\mathscr{B}_2 = \left\{ l \text{ prime} \mid l \equiv 1 \pmod 8, \left(\frac{l}{p}\right) = -1, \text{ and 4-rank } K_2 O_{\mathbb{Q}(\sqrt{-2pl})} = 0 \right\}.$$

Recall that

$$B = \left\{ l \text{ rational primes} \mid l \equiv 1 \pmod 8 \text{ and } \left(\frac{l}{p}\right) = -1 \right\}.$$

For each $\mathscr{B}_i$, $i = 1, 2$ we have:

$$\begin{aligned} &\{\text{Density of } \mathscr{B}_i \text{ in the set of all primes } l\} \\ &\quad = \{\text{Density of } \mathscr{B}_i \text{ in } B\} \times \{\text{Density of } B \text{ in the set of all primes } l\}, \end{aligned}$$

where B has density $\frac{1}{8}$ in the set of all primes l. Thus 4-rank 0 and 4-rank 1 each appear with natural density $\frac{1}{2}$ in B. A similar argument works if p does not satisfy $\langle 1, 64\rangle$.

Now we consider the case of $\mathbb{Q}(\sqrt{-2pl})$, $p \equiv l \equiv 1 \pmod 8$ with $(\frac{l}{p}) = -1$. If $p \equiv 1 \pmod{16}$, then we can see that the sets

$$\begin{aligned} \mathscr{C}_1 &= \left\{ l \text{ prime } \mid l \equiv 1 \pmod 8, \left(\frac{l}{p}\right) = -1 \text{ and 4-rank } K_2 O_{\mathbb{Q}(\sqrt{-2pl})} = 1 \right\} \\ &= \left\{ l \text{ prime } \mid l \equiv 1 \pmod{16}, \left(\frac{l}{p}\right) = -1 \right\}, \\ \mathscr{C}_2 &= \left\{ l \text{ prime } \mid l \equiv 1 \pmod 8, \left(\frac{l}{p}\right) = -1 \text{ and 4-rank } K_2 O_{\mathbb{Q}(\sqrt{-2pl})} = 0 \right\} \\ &= \left\{ l \text{ prime } \mid l \equiv 9 \pmod{16}, \left(\frac{l}{p}\right) = -1 \right\}, \end{aligned}$$

By the Dirichlet's Theorem on primes in arithmetic progressions, it is easy to see that $\mathscr{C}_1, \mathscr{C}_2$ each appear with natural density $\frac{1}{2}$ in B. The argument is similar if $p \equiv 9 \pmod{16}$. □

Similarly, we can prove the following two theorems.

Theorem 3.8. *For the imaginary number field* $\mathbb{Q}(\sqrt{-2pl})$, 4*-rank* 0 *and* 1 *each appear with natural density* $\frac{1}{2}$ *in* C.

Theorem 3.9. *For the imaginary number field* $\mathbb{Q}(\sqrt{-2pl})$, 4*-rank* 0 *and* 1 *each appear with natural density* $\frac{1}{2}$ *in* D.

Now we will prove the following theorem.

Theorem 3.10. *For real quadratic number field $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 appear with natural density $\frac{3}{4}$ and $\frac{1}{4}$ in A. For the imaginary number field $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 1 and 2 appear with natural density $\frac{3}{4}$ and $\frac{1}{4}$ in A.*

Recall that $N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1})$, $N_2 = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p})$, and $H = N_1N_2\mathbb{Q}(\zeta_{16})$. For $l \equiv 1 \pmod 8$ with $(\frac{l}{p}) = 1$, l splits completely in $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{p})$, and so splits completely in the composite field $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$ and $H^{Z(\mathrm{Gal}(H/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. By Lemma 2.9, we have

$$\left(\frac{H/\mathbb{Q}}{l}\right) = k \subset Z(\mathrm{Gal}(H/\mathbb{Q})) \qquad \text{for some } k \in \mathrm{Gal}(H/\mathbb{Q}).$$

As $Z(\mathrm{Gal}(H/\mathbb{Q}))$ has order 8, there are eight possible choices for $(\frac{H/\mathbb{Q}}{l})$. Using Lemmas 2.5, 2.7, and 2.8, we can make the following correspondences.

Remark 3.11.

(i) $(\frac{H/\mathbb{Q}}{l}) = \{id\} \Longleftrightarrow l$ splits completely in H

$$\Longleftrightarrow \left\{\begin{array}{c} l \text{ splits completely in} \\ N_1, N_2 \text{ and } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, -2p\rangle \\ l \equiv 1 \pmod{16} \end{array}\right\}$$

(ii) $(\frac{L/\mathbb{Q}}{l}) \neq \{id\} \Longleftrightarrow l$ does not split completely in H. There are seven cases:

$$(1)\ \left\{\begin{array}{c} l \text{ splits completely in } N_1, \\ \text{but does not in } N_2 \text{ or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle p, -2\rangle \\ l \equiv 9 \pmod{16} \end{array}\right\};$$

$$(2)\ \left\{\begin{array}{c} l \text{ splits completely in } N_1 \\ \text{and } N_2, \text{ but does not in } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, -2p\rangle \\ l \equiv 9 \pmod{16} \end{array}\right\};$$

$$(3)\ \left\{\begin{array}{c} l \text{ splits completely in } N_2, \\ \text{but does not in } N_1 \text{ or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, -2p\rangle \\ l \equiv 9 \pmod{16} \end{array}\right\};$$

$$(4)\ \left\{\begin{array}{c} l \text{ splits completely in } N_2 \\ \text{and } \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_1 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, -2p\rangle \\ l \equiv 1 \pmod{16} \end{array}\right\};$$

$$(5)\ \left\{\begin{array}{c} l \text{ splits completely in } N_1 \\ \text{and } \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle p, -2\rangle \\ l \equiv 1 \pmod{16} \end{array}\right\};$$

$$(6)\ \left\{\begin{array}{c} l \text{ splits completely in } \mathbb{Q}(\zeta_{16}), \\ \text{but does not in } N_1 \text{ or } N_2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle p, -2\rangle \\ l \equiv 1 \pmod{16} \end{array}\right\};$$

$$(7)\ \left\{\begin{array}{c} l \text{ does not split completely in} \\ N_1, N_2, \text{ or } \mathbb{Q}(\zeta_{16}), \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle p, -2\rangle \\ l \equiv 9 \pmod{16} \end{array}\right\}.$$

Proof of Theorem 3.10. Consider the set $W = \{l \text{ prime} \mid l$ is unramified in H and $(\frac{H/\mathbb{Q}}{l}) = \{k\} \subset Z(\mathrm{Gal}(H/\mathbb{Q}))\}$ for some $k \in \mathrm{Gal}(H/\mathbb{Q})$. By the Chebotarev Density Theorem, the set W has natural density $\frac{1}{64}$ in the set of all primes. Since the set A has natural density $\frac{1}{8}$ in the set of all primes. Thus W has natural density $\frac{1}{8}$ in A.

From Proposition 3.6, for real quadratic number field $\mathbb{Q}(\sqrt{2pl})$, we have:

(a) If p satisfies $\langle 1, 64\rangle$, then:

(i) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 1$ in cases (ii)(3), (5) or l does not satisfy $\langle 1, 64\rangle$;
(ii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 2$ in cases (i), (ii)(7).

(b) If p does not satisfy $\langle 1, 64\rangle$, then:

(i) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 1$ in cases (ii)(2), (6) or l satisfies $\langle 1, 64\rangle$;
(ii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{2pl})} = 2$ in cases (ii)(1), (4).

For imaginary number field $\mathbb{Q}(\sqrt{-2pl})$, we have:

(a) If $p \equiv 1 \pmod{16}$, then:

(i) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 1$ in cases (ii)(4), (5) or $l \equiv 9 \pmod{16}$;
(ii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 2$ in cases (i), (ii)(6).

(b) If $p \equiv 9 \pmod{16}$, then:

(i) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 1$ in cases (ii)(2), (7) or $l \equiv 1 \pmod{16}$;
(ii) 4-rank $K_2O_{\mathbb{Q}(\sqrt{-2pl})} = 2$ in cases (ii)(1), (3).

It is easy to see that for both the fields $\mathbb{Q}(\sqrt{2pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 1 appears with natural density $\frac{1}{8} \times 2 + \frac{1}{2} = \frac{3}{4}$, 4-rank 2 appears with natural density $\frac{1}{8} \times 2 = \frac{1}{4}$ in A. □

As a consequence of the former four theorems, we may compute the 4-rank densities of X and Y. Recall that for square-free, even integers d, the sets $X = \{d \mid d = 2pl\}$ and $Y = \{d \mid d = -2pl\}$ for distinct primes p and l. For our convenience we may divide X into four subsets, i.e., $X = X_1 \cup X_3 \cup X_5 \cup X_7$, where $X_i = \{d \mid d = 2pl, p \equiv i \pmod 8\}$. Similarly, $Y = Y_1 \cup Y_3 \cup Y_5 \cup Y_7$, where $Y_i = \{d \mid d = -2pl, p \equiv i \pmod 8\}$. Additionally, consider the sets

$$X_{ij} = \{d \mid d = 2pl, p \equiv i \pmod 8, l \equiv j \pmod 8\},$$
$$Y_{ij} = \{d \mid d = -2pl, p \equiv i \pmod 8, l \equiv j \pmod 8\}.$$

Now we can prove the following corollaries.

Corollary 3.12. *For the real quadratic fields* $\mathbb{Q}(\sqrt{2pl})$, 4-*rank* 0, 1, *and* 2 *appear with natural density* $\frac{13}{64}$, $\frac{97}{128}$, *and* $\frac{5}{128}$, *respectively, in* X.

Corollary 3.13. *For the imaginary quadratic fields* $\mathbb{Q}(\sqrt{-2pl})$, 4-*rank* 0, 1, 2 *appear with natural density* $\frac{49}{64}$, $\frac{29}{128}$, $\frac{1}{128}$ *respectively, in* Y.

In fact, by GP-PARI, we have computed the following. For $30 \le d \le 10^6$, there are 86157 d's in X. Among them, there are 16625 d's (19.30%) yielding 4-rank 0, 67463 d's (78.30%) yielding 4-rank 1 and 2069 d's (2.40%) yielding 4-rank 2. For $-10^6 \le d \le -30$, there are 86157 d's in Y. Among them, there are 70694 d's (82.05%) yielding 4-rank 0, 15180 d's (17.62%) yielding 4-rank 1 and 283 d's (0.33%) yielding 4-rank 2.

Proof of Corollary 3.12. Note that the following tables are from Qin (1995a,b).

By Theorem 3.10, $p \equiv l \equiv 1 \pmod 8$ with $(\frac{l}{p}) = 1$ yields 4-rank 1 and 2 with densities $\frac{3}{32}$ and $\frac{1}{32}$, respectively, in X_1. By Theorem 3.7, $p \equiv l \equiv 1 \pmod 8$ with $(\frac{l}{p}) = -1$ yields 4-rank 0 and 1 each with $\frac{1}{16}$ in X_1.

Regarding the set X_1 (Table 2):

(i) 4-rank 0, 1, and 2 appear with natural densities $\frac{1}{16}$, $\frac{3}{32} + \frac{1}{16} = \frac{5}{32}$, and $\frac{1}{32}$ in $X_{1,1}$;
(ii) 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,3}$;
(iii) 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,5}$;
(iv) 4-rank 1 and 2 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $X_{1,7}$.

Thus 4-rank 0, 1, and 2 appear with natural densities $\frac{5}{16}$, $\frac{19}{32}$, and $\frac{3}{32}$ in X_1. Similarly, from Tables 1 and 2, we can see that 4-rank 0 and 1 appear with natural densities $\frac{3}{8}$ and $\frac{5}{8}$ in X_3, 4-rank 0 and 1 appear with natural densities $\frac{1}{8}$ and $\frac{7}{8}$ in X_5 and 4-rank 1 and 2 appear with natural densities $\frac{15}{16}$ and $\frac{1}{16}$ in X_7. As each X_i has density $\frac{1}{4}$ in X, so we have:

(i) 4-rank 0 appears with natural density $\frac{5}{64} + \frac{3}{32} + \frac{1}{32} = \frac{13}{64}$ in X;
(ii) 4-rank 1 appears with natural density $\frac{19}{128} + \frac{5}{32} + \frac{7}{32} + \frac{15}{64} = \frac{97}{128}$ in X;
(iii) 4-rank 2 appears with natural density $\frac{3}{128} + \frac{1}{64} = \frac{5}{128}$ in X. □

Similarly, we can prove Corollary 3.13. Note that Osburn (2002) has proved that for fixed prime $p \equiv 7 \pmod 8$, set

$$\Omega = \left\{ l \text{ rational prime} \mid l \equiv 1 \pmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1 \right\},$$

then for imaginary quadratic number field, 4-rank 0 and 1 each appear with density $\frac{1}{2}$ in Ω.

Table 1

$\mathbb{Q}(\sqrt{2pl})$		
$p, l \pmod 8$	4-rank	Densities
3, 3	0	$\frac{1}{4}$ in X_3
5, 5	1	$\frac{1}{4}$ in X_5
7, 7	1	$\frac{1}{4}$ in X_7
3, 5	1	$\frac{1}{4}$ in X_3 and X_5
3, 7	1	$\frac{1}{4}$ in X_3 and X_7
5, 7	1	$\frac{1}{4}$ in X_5 and X_7

Table 2

$\mathbb{Q}(\sqrt{2pl})$			
$p, l \pmod 8$	Legendre symbols	4-rank	Densities
1, 3	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in X_1 and X_3
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in X_1 and X_3
1, 5	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in X_1 and X_5
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in X_1 and X_5
1, 7	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in X_1 and X_7
	$(\frac{l}{p}) = 1$	1	$\frac{1}{16}$ in X_1 and X_7
		2	$\frac{1}{16}$ in X_1 and X_7

Proof of Corollary 3.13. First we also give the tables on the 4-rank of the tame kernel.

By Theorem 3.10, $p \equiv l \equiv 1 \pmod 8$ with $(\frac{l}{p}) = 1$ yields 4-rank 1 and 2 with natural densities $\frac{3}{32}$ and $\frac{1}{32}$ in Y_1. By Theorem 3.7, $p \equiv l \equiv 1 \pmod 8$ with $(\frac{l}{p}) = -1$ yields 4-rank 0 and 1 appear with natural densities $\frac{1}{16}$ in Y_1. By Theorem 3.8, $p \equiv l \equiv 7 \pmod 8$ with $(\frac{l}{p}) = 1$ yields 4-rank 0 and 1 appear with natural densities $\frac{1}{16}$ in Y_7. And $p \equiv l \equiv 7 \pmod 8$ with $(\frac{l}{p}) = -1$ yields 4-rank 0 and 1 appear with natural densities $\frac{1}{16}$ in Y_7 by Theorem 3.9.

Regarding the set Y_1, we can see that (from Table 3):

(i) 4-rank 0, 1 and 2 appear with natural densities $\frac{1}{16}$, $\frac{3}{32} + \frac{1}{16} = \frac{5}{32}$ and $\frac{1}{32}$ in $Y_{1,1}$;
(ii) 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,3}$;
(iii) 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,5}$;
(iv) 4-rank 0 and 1 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $Y_{1,7}$.

Thus 4-rank 0, 1 and 2 appear with natural density $\frac{1}{2}$, $\frac{15}{32}$, and $\frac{1}{32}$ in Y_1.

For the set Y_3, we have (from Tables 3 and 4):

(i) 4-rank 0, 1 each appear with natural densities $\frac{1}{8}$ in $Y_{3,1}$;
(ii) 4-rank 0 appears with natural densities $\frac{1}{4}$ in $Y_{3,3}$;
(iii) 4-rank 0 appears with natural densities $\frac{1}{4}$ in $Y_{3,5}$;
(iv) 4-rank 0 appears with natural densities $\frac{1}{4}$ in $Y_{3,7}$.

Table 3

$\mathbb{Q}(\sqrt{-2pl})$		
$p, l \pmod 8$	4-rank	Densities
3, 3	0	$\frac{1}{4}$ in Y_3
5, 5	0	$\frac{1}{4}$ in Y_5
3, 5	0	$\frac{1}{4}$ in Y_3 and Y_5
3, 7	0	$\frac{1}{4}$ in Y_3 and Y_7
5, 7	0	$\frac{1}{4}$ in Y_5 and Y_7

Table 4

$\mathbb{Q}(\sqrt{-2pl})$			
$p, l \pmod 8$	Legendre symbols	4-rank	Densities
1, 3	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_3
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in Y_1 and Y_3
1, 5	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_5
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in Y_1 and Y_5
1, 7	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_7
	$(\frac{l}{p}) = 1$	0	$\frac{1}{16}$ in Y_1 and Y_7
		1	$\frac{1}{16}$ in Y_1 and Y_7

Thus 4-rank 0, 1 appear with natural densities $\frac{7}{8}, \frac{1}{8}$ in Y_3. Similarly, 4-rank 0 and 1 appear with natural densities in $\frac{7}{8}, \frac{1}{8}$ in Y_5 and 4-rank 0 and 1 appear with natural densities $\frac{13}{16}$ and $\frac{3}{16}$ in Y_7. As each Y_i has density $\frac{1}{4}$ in Y, we have:

(i) 4-rank 0 appears with natural density $\frac{1}{8} + \frac{7}{32} + \frac{7}{32} + \frac{13}{64} = \frac{49}{64}$ in Y;
(ii) 4-rank 1 appears with natural density $\frac{15}{128} + \frac{1}{32} + \frac{1}{32} + \frac{3}{64} = \frac{29}{128}$ in Y;
(iii) 4-rank 2 appears with natural density $\frac{1}{128}$ in Y. □

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