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Tame Kernels of Pure Cubic Fields

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Abstract In this paper, we study the p-rank of the tame kernels of pure cubic fields. In particular, we prove that for a fixed positive integer m, there exist infinitely many pure cubic fields whose 3-rank of the tame kernel equal to m. As an application, we determine the 3-rank of their tame kernels for some special pure cubic fields.

Keywords Tame kernel, pure cubic fields, class group, 3-rank

MR(2000) Subject Classification 11R70, 19C99

1 Introduction

Let F be a number field with $\mathcal{O}_F$ the ring of integers of F. The structure of $K_2\mathcal{O}_F$ has been intensively studied. The 2-part of the tame kernel of quadratic number fields is especially investigated by many authors. Recently, Browkin, Qin and Zhou discussed the p-primary part of the tame kernels of cubic cyclic fields and real number fields in [1] and [2].

We focus on the tame kernels of the pure cubic fields in the present paper. The 3-rank of the tame kernel is especially investigated. In Section 3, we give the independent generators of ${}_2K_2\mathcal{O}_F$ when F is a pure cubic field. In Section 4, we study the q-rank of the tame kernels of pure cubic fields. The q-rank of the tame kernels is expressed by the q-rank of certain subgroups of the ideal class groups. In the last section, we prove that for a fixed positive integer m, there exist infinitely many pure cubic fields whose 3-rank of the tame kernel is equal to m. As an application, we determine the 3-rank of the tame kernel of some special pure cubic fields.

2 Basic Information on Pure Cubic Fields

A number field F is called pure cubic if $F = \mathbb{Q}(\sqrt[3]{m})$ for some cube-free natural number m such that $\sqrt[3]{m} \notin \mathbb{Z}$. Note that no pure cubic field is Galois over $\mathbb{Q}$. Since m is cube-free, we can assume that $m = ab^2$, where a, b are square-free co-prime positive integers. Let

$$\alpha = \sqrt[3]{ab^2} = \sqrt[3]{m}, \quad \beta = \frac{\alpha^2}{b} = \sqrt[3]{a^2b}.$$

Then $F = \mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$.

The following basic properties of pure cubic fields can be found in [3].

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Proposition 2.1 *Let $F=\mathbb{Q}(\sqrt[3]{m}), a, b, \alpha, \beta$ as above.*

(1) *If $a^2 \not\equiv b^2 \pmod 9$, then $\mathcal{O}_F=\mathbb{Z}\oplus\mathbb{Z}\alpha\oplus\mathbb{Z}\beta$; if $a^2\equiv b^2 \pmod 9$, then $\mathcal{O}_F=\mathbb{Z}\alpha\oplus\mathbb{Z}\beta\oplus\mathbb{Z}\gamma$, where $\gamma=\frac{1+a\alpha+b\beta}{3}$.*

(2) *Let $d(F)$ be the discriminant of F. Then*

$$d(F)=\begin{cases} -27a^2b^2, & \text{if } a^2\not\equiv b^2 \pmod 9,\\ -3a^2b^2, & \text{if } a^2\equiv b^2 \pmod 9.\end{cases}$$

(3) *If $p|ab$, then p is totally ramified in F. In particular, if $p=3$, then*

$$3\mathcal{O}_F=\begin{cases} \mathfrak{P}^3, & \text{if } a^2\not\equiv b^2 \pmod 9,\\ \mathfrak{P}_1^2\mathfrak{P}_2, & \text{if } a^2\equiv b^2 \pmod 9,\end{cases}$$

where $\mathfrak{P}, \mathfrak{P}_1, \mathfrak{P}_2$ are the prime ideals of F and $\mathfrak{P}_1\neq\mathfrak{P}_2$.

(4) *In particular, if $p=3$, then*

$$3\mathcal{O}_F=\begin{cases} \mathfrak{P}^3, & \text{if } a^2\not\equiv b^2 \pmod 9,\\ \mathfrak{P}_1^2\mathfrak{P}_2, & \text{if } a^2\equiv b^2 \pmod 9,\end{cases}$$

where $\mathfrak{P}, \mathfrak{P}_1, \mathfrak{P}_2$ are the prime ideals of F and $\mathfrak{P}_1\neq\mathfrak{P}_2$.

3 The 2-primary Part of $K_2\mathcal{O}_F$

The general formula for the 2-rank of the tame kernel of a number field F is given by Tate in [4]:

$$\text{2-rank}\, K_2\mathcal{O}_F = r_1+g_2-1+\text{2-rank}\, Cl(\mathcal{O}_{F,2}),$$

where r_1 is the number of real places of F and g_2 is the number of dyadic prime ideals of F.

By Proposition 2.1, 2 is either unramified or totally ramified in a pure cubic field F. If 2 is unramified, then $2\nmid ab$ which implies both a and b are odd. By Proposition 2.1, $[\mathcal{O}_F:\mathbb{Z}[\sqrt[3]{m}]]=b$ if $a^2\not\equiv b^2 \pmod 9$ and $[\mathcal{O}_F:\mathbb{Z}[\sqrt[3]{m}]]=3b$ if $a^2\equiv b^2 \pmod 9$. Hence $2\nmid[\mathcal{O}_F:\mathbb{Z}[\sqrt[3]{m}]]$. It is easy to see that

$$x^3-m\equiv x^3-1=(x-1)(x^2+x+1) \pmod 2.$$

Therefore $2\mathcal{O}_F=\mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1\neq\mathfrak{p}_2$ if 2 is not ramified in F. Now we have

$$g_2=\begin{cases} 1, & \text{if 2 is totally ramified in } F,\\ 2, & \text{if 2 is unramified in } F.\end{cases}$$

Then the 2-rank formula takes the form

$$\text{2-rank}\, K_2\mathcal{O}_F=\text{2-rank}\, Cl(\mathcal{O}_{F,2})+\begin{cases} 1, & \text{if 2 is totally ramified in } F,\\ 2, & \text{if 2 is unramified in } F.\end{cases}$$

Lemma 3.1 *We have*

$$\text{2-rank}\, K_2\mathcal{O}_F=\text{2-rank}\, Cl(\mathcal{O}_{F,2})+\begin{cases} 1, & \text{if } 2\,|\,ab,\\ 2, & \text{if } 2\nmid ab.\end{cases}$$

Let W_2F denote the wild kernel of the field F. The description of $K_2\mathcal{O}_F/W_2F$ was given by Browkin in [5]. We have the following

Lemma 3.2 *Let W_2F be the wild kernel of the field F. Then the 2-part of $K_2\mathcal{O}_F/W_2F$ is equal to $(\mathbb{Z}/2\mathbb{Z})^r$, where*

$$r = \begin{cases} 1, & \textit{if 2 is totally ramified in } F, \\ 2, & \textit{if 2 is not ramified in } F. \end{cases}$$

Proof By Theorem 2 of [5], $K_2\mathcal{O}_F/W_2F$ is isomorphic to the abelian group defined by the generators g_v, where v runs through all real places of F, and such finite places that $\zeta_p \in F_v$ for $v|p$ and some relations.

If 2 is totally ramified, then the generators are g_∞, g_2 with relations $g_\infty^2 = 1, g_2^2 = 1$ and $g_\infty \cdot g_2 = 1$, hence the 2-Sylow subgroup of $K_2\mathcal{O}_F/W_2F$ is $\mathbb{Z}/2\mathbb{Z}$.

If 2 is unramified, then the generators are g_∞, g_{2_1} and g_{2_2} with relations $g_\infty^2 = 1, g_{2_1}^2 = 1$, $g_{2_2}^2 = 1$ and $g_\infty \cdot g_{2_1} \cdot g_{2_2}^3 = 1$, so the 2-part of $K_2\mathcal{O}_F/W_2F$ is $(\mathbb{Z}/2\mathbb{Z})^2$ in this case. □

Corollary 3.3 *For every pure cubic field F, we have*

$$\text{2-rank}\, K_2\mathcal{O}_F = \text{2-rank}\, Cl(\mathcal{O}_{F,2}) + \text{2-rank}\,(K_2\mathcal{O}_F/W_2F).$$

Lemma 3.4 *Let G be a finite abelian group. For any $x \in G$,*

$$rk_2(G/\langle x\rangle) = \begin{cases} rk_2(G), & \textit{if } x \in G^2, \\ rk_2(G) - 1, & \textit{otherwise.} \end{cases}$$

Proposition 3.5 *If F is a pure cubic field, then*

$$\text{2-rank}\, Cl(\mathcal{O}_{F,2}) = \text{2-rank}\, Cl(\mathcal{O}_F) \text{ or } \text{2-rank}\, Cl(\mathcal{O}_F) - 1.$$

Proof It is known that $Cl(\mathcal{O}_{F,2})$ is the quotient group of $Cl(\mathcal{O}_F)$ by the classes of prime ideals above (2).

If 2 is totally ramified in F, then $2\mathcal{O}_F = \mathfrak{p}^3$. So we have $Cl(\mathcal{O}_{F,2}) = Cl(\mathcal{O}_F)/\langle[\mathfrak{p}]\rangle$.

If 2 is unramified in F, then $2\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$. So we have $Cl(\mathcal{O}_{F,2}) = Cl(\mathcal{O}_F)/\langle[\mathfrak{p}_1],[\mathfrak{p}_2]\rangle = Cl(\mathcal{O}_F)/\langle[\mathfrak{p}_1]\rangle$ since $[\mathfrak{p}_1][\mathfrak{p}_2] = 1$. By Lemma 3.4, the result follows. □

Next we will describe the elements of order 2 in $K_2\mathcal{O}_F$. It is well known that if $x^2 = 1, x \in K_2F$, then $x = \{-1, a\}, a \in F^*$. Let $\Delta_F = \{a \in F^* \mid \{-1, a\} = 1\}$, then 2-rank$(\Delta_F/F^{*2}) = 1 + r_2 = 2$ by Theorem 6.3 of [4]. Hence $\Delta_F = F^{*2} \cup 2F^{*2} \cup \delta F^{*2} \cup 2\delta F^{*2}$ with $\delta \in F^*$.

Let ε be the fundamental unit of F. We may assume $N\varepsilon = 1$. If not, then change the sign. It is easy to see that $\{-1,-1\}, \{-1,\varepsilon\} \in K_2\mathcal{O}_F$.

In the following, we will determine other elements of order 2. Take independent generators of the group ${}_2Cl(\mathcal{O}_{F,2})$ of the form $Cl(\mathfrak{p}_i), i = 1, \dots, t$, where $t =$ 2-rank $Cl(\mathcal{O}_{F,2})$ and $\mathfrak{p}_i$ are prime ideals satisfying $\mathfrak{p}_i \nmid 2$. The ideals $\mathfrak{p}_i^2$ are obviously principal, $\mathfrak{p}_i^2 = (\gamma_i)$ for $i = 1, \dots, t$. Assume $N\gamma_j > 0$, too. Then $N\gamma_i = N\mathfrak{p}_i^2 = (N\mathfrak{p}_i)^2 \in F^{*2}$. It follows that $\{-1, \gamma_i\} \in K_2\mathcal{O}_F$ for $i = 1, \dots, t$.

If 2 is unramified in F, $(2) = \mathfrak{p}_1\mathfrak{p}_2$ and $[\mathfrak{p}_1]$ has order r, then the ideal $\mathfrak{p}_1^r$ is principal, $\mathfrak{p}_1^r = (\gamma)$ and $N\gamma = N(\mathfrak{p}_1^r) = 2^r$(or 2^{2r}). As a consequence, $\{-1, \gamma\} \in K_2\mathcal{O}_F$. Hence $-1, \varepsilon, \gamma_1, \gamma_2, \dots, \gamma_t$ are multiplicatively independent modulo Δ_F if 2 is totally ramified in F, and $-1, \varepsilon, \gamma, \gamma_1, \gamma_2, \dots, \gamma_t$ are multiplicatively independent modulo Δ_F if 2 is unramified in F.

So we have the following theorem.

Theorem 3.6 *If 2 is totally ramified in F, then the elements*

$$\{-1,-1\},\{-1,\varepsilon\},\{-1,\gamma_1\},\{-1,\gamma_2\},\ldots,\{-1,\gamma_t\},$$

are independent generators of the finite group ${}_2K_2\mathcal{O}_F$. If 2 is unramified in F, the same elements and $\{-1,\gamma\}$ are independent generators of ${}_2K_2\mathcal{O}_F$.

4 The q-rank of $K_2\mathcal{O}_F$ for q Odd

Let q be an odd prime number, ζ_q a primitive q-th root of unity. Let $G=\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$, then it is well known that

$$G=\{\sigma_a \mid 1\le a\le q-1\},$$

where $\sigma_a(\zeta_q)=\zeta_q^a$. There is a classical isomorphism: $(\mathbb{Z}/q\mathbb{Z})^*\longrightarrow G, a\mapsto\sigma_a$. So G is a cyclic group with a generator $\sigma:=\sigma_h$, where h is a fixed primitive root modulo q.

For an odd prime q, let ω be the q-adic Teichmüller character of the group $(\mathbb{Z}/q\mathbb{Z})^*$. Then for $1\le a\le q-1, \omega(a)$ satisfies

$$\omega(a)^{q-1}=1 \quad \text{and} \quad \omega(a)\equiv a \pmod q.$$

Let $\widehat{G}$ be the group of characters of G. Then $\widehat{G}=\{\omega^j \mid 1\le j\le q-1\}$. The corresponding primitive idempotents of the group ring $\mathbb{Z}_q[G]$ are

$$\varepsilon_j=\frac{1}{q-1}\sum_{a=1}^{q-1}\omega(a)^j\sigma_a^{-1},\quad 0\le j\le q-2.$$

Since $\sigma_a\varepsilon_j=\omega(a)^j\varepsilon_j=\varepsilon_j\sigma_a$ in $\mathbb{Z}_q[G]$, for a $Z_q[G]$-module M, we have

$$\varepsilon_jM=\{m\in M \mid \sigma_a(m)=\omega(a)^jm\},$$

and we can get a decomposition of M into a direct sum of $\mathbb{Z}_q[G]$-submodules:

$$M=\bigoplus_{j=0}^{q-2}\varepsilon_jM.$$

The group μ_q of q-th roots of unity has the natural structure as a $\mathbb{Z}_q[G]$-module. Now we define the action of G on $\mu_q\otimes M$ by

$$(\zeta\otimes m)^\tau=\zeta^\tau\otimes m^\tau,\quad \text{where } \zeta\in\mu_q, m\in M,\tau\in G.$$

It was proved by Browkin in [1] that

$$(\mu_q\otimes M)^G=\varepsilon_0(\mu_q\otimes M)=\mu_q\otimes\varepsilon_{q-2}M.$$

Let $E=F(\zeta_q)$, where F is the pure cubic field defined above. Let $\lambda: Cl(\mathcal{O}_E)\to Cl(\mathcal{O}_E[1/q])$ be the homomorphism of the class groups induced by the imbedding $\mathcal{O}_E\to\mathcal{O}_E[1/q]$, and $A=A_E$ the Sylow q-subgroup of $Cl(\mathcal{O}_E)$. Then $\lambda(A)$ is the Sylow q-subgroup of $Cl(\mathcal{O}_E[1/q])$ by the surjectivity of λ.

Since A is a q-group on which $G=\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})=\mathrm{Gal}(E/F)$ acts, we have

$$A=\bigoplus_{j=0}^{q-2}\varepsilon_jA.$$

Proposition 4.1 *If $a^2\not\equiv b^2 \pmod 9$, then 3 is totally ramified in the extension $E/\mathbb{Q}$.*

Proof Let $\mathfrak{P} \in E$ be a prime ideal above 3. And let $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F, \mathfrak{p}' = \mathfrak{P} \cap O_{\mathbb{Q}(\zeta_3)}$. Then we have

$$e(\mathfrak{P}/3) = e(\mathfrak{P}/\mathfrak{p})e(\mathfrak{p}/3) = e(\mathfrak{P}/\mathfrak{p}')e(\mathfrak{p}'/3).$$

We know that in this case, 3 is totally ramified in F, so $e(\mathfrak{p}/3) = 3$. It is easy to see that $e(\mathfrak{p}'/3) = 2$ since 3 is totally ramified in $\mathbb{Q}(\zeta_3)$. So both 2 and 3 divide $e(\mathfrak{P}/3)$. As a result, $6 \mid e(\mathfrak{P}/3)$. Since $[E : \mathbb{Q}] = 6$, we have $e(\mathfrak{P}/3) = 6$, and 3 is totally ramified in E. □

By the above lemma, it is easy to see that

Corollary 4.2 *Suppose that* $a^2 \not\equiv b^2 \pmod 9$. *Let* $\mathfrak{p} \in F$ *be a prime ideal above* 3. *Then* $\mathfrak{p}$ *is totally ramified in the extension* E/F.

Lemma 4.3 *If* $q \neq 3$ *and* $q \nmid ab$, *or* $q = 3$ *with* $a^2 \not\equiv b^2 \pmod 9$, *then for* $j \neq 0$ *the mapping* $\lambda : \varepsilon_j A \longrightarrow \varepsilon_j \lambda(A)$ *is an isomorphism. If* $q = 3$ *with* $a^2 \equiv b^2 \pmod 9$, *then* λ *is surjective.*

Proof Since λ commutes with the action of G, we have $\lambda(\varepsilon_j A) = \varepsilon_j \lambda(A) = \varepsilon_j A$. So λ is surjective.

By the definition of λ, the group $\ker\lambda$ is generated by classes containing primes ideals $\mathfrak{Q}$ of E that divide q. If $q \nmid ab$, then q does not ramify in F, it follows that the prime divisors $\mathfrak{q}$ of q totally ramify in E; as a result, $\sigma(\mathfrak{Q}) = \mathfrak{Q}$ for every $\sigma \in G$. If $3 \mid ab$ and $a^2 \not\equiv b^2 \pmod 9$, by Corollary 4.2, the 3-adic prime of F totally ramifies in E, so we have $\sigma\mathfrak{P} = \mathfrak{P}$, where $\mathfrak{P}$ is the prime ideal dividing 3.

Consequently, $\ker\lambda \cap A \subset A^G = \varepsilon_0 A$. Therefore $\ker\lambda \cap \varepsilon_j A = 0$ for $j \neq 0$, the result follows. □

Let S' be the set of prime ideals of F dividing q that split completely in E. For pure cubic fields, we have

Lemma 4.4 *If* $q \neq 3$ *or* $q = 3$ *with* $a^2 \not\equiv b^2 \pmod 9$, *then the set* S' *is empty.*

Proof By Proposition 2.1 (2), (3), (4), for a prime $q \neq 3$, q is unramified or totally ramified in F. For number fields K, L, where $\mathbb{Q} \subset K \subset L \subset E$, denote by $e_q(L/K)$ the ramification index in the extension L/K of a prime ideal of K that divides q. Then

$$q - 1 = e_q(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \mid e_q(E/\mathbb{Q}) = e_q(F/\mathbb{Q}) \cdot e_q(E/F).$$

Since $e_q(F/\mathbb{Q}) = 1$ or 3, $2|q-1$, it follows that $2|e_q(E/F)$. Hence every prime ideal of F dividing q ramifies in E, so it cannot split completely in E.

If $q = 3$ and $a^2 \not\equiv b^2 \pmod 9$, the result comes from Corollary 4.2. □

Lemma 4.5 *If* $q = 3$ *and* $a^2 \equiv b^2 \pmod 9$, *then* $|S'| = 1$.

Proof Let $\mathfrak{P}$ denote one of the 3-adic primes of E. And let $\mathfrak{p} = \mathfrak{P} \cap O_{\mathbb{Q}(\zeta_3)}$, where $O_{\mathbb{Q}(\zeta_3)}$ denote the ring of integers of $\mathbb{Q}(\zeta_3)$. It is obvious that 3 ramifies totally in $\mathbb{Q}(\zeta_3)$, so we have $3O_{\mathbb{Q}(\zeta_3)} = \mathfrak{p}^2$ and $e(\mathfrak{p}/3) = 2$. We can see that $E/\mathbb{Q}(\zeta_3)$ is a Galois extension and $[E : \mathbb{Q}(\zeta_3)] = 3$. Hence $e(\mathfrak{P}/\mathfrak{p}) = 1$ or 3. If $e(\mathfrak{P}/\mathfrak{p}) = 3$, then $e(\mathfrak{P}/3) = 6$, i.e., 3 ramifies totally in E, which is impossible since 3 splits in F. So $e(\mathfrak{P}/\mathfrak{p}) = 1$ and $\mathfrak{p}$ splits completely in E. As a result, $e(\mathfrak{P}/3) = 2$. Since $E/\mathbb{Q}$ is a Galois extension, the ramification index of the extension is 2. By Proposition 2.1 (3), $3\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_2$ in F. It is easy to see that $e(\mathfrak{p}_1/3) = 2$ and $e(\mathfrak{p}_2/3) = 1$. So $\mathfrak{p}_1$ splits completely and $\mathfrak{p}_2$ is totally ramified in E. Therefore $S' = \{\mathfrak{p}_1\}$. □

Theorem 4.6 *Let* $E = F(\zeta_q)$, *where* F *is a pure cubic field defined above.*

(1) *If* $q \neq 3$ *and* $q \nmid ab$, *then*

$$q\text{-rank } K_2\mathcal{O}_F = q\text{-rank } \varepsilon_{q-2}A_E.$$

(2) *If* $q = 3$ *and* $a^2 \not\equiv b^2 \pmod 9$, *then*

$$3\text{-rank } K_2\mathcal{O}_F = 3\text{-rank } \varepsilon_1 A_E.$$

(3) *If* $q = 3$ *and* $a^2 \equiv b^2 \pmod 9$, *then*

$$3\text{-rank } K_2\mathcal{O}_F \leq 3\text{-rank } \varepsilon_1 A_E + 1.$$

Proof By Lemma 4.4, in Cases (1), (2), $S' = \emptyset$. The proof is the same as that for Theorem 4.3 of Browkin [1].

The proof of (3) is similar, except that λ is only surjective in this case. So

$$(\mu_3 \otimes Cl(\mathcal{O}_E[1/3]))^G = (\mu_3 \otimes \lambda(A))^G = \varepsilon_0(\mu_3 \otimes \lambda(A)) = \mu_3 \otimes \varepsilon_1\lambda(A) \subset \mu_3 \otimes \varepsilon_1 A.$$

The result follows by Keune's exact sequence; see below. □

5 3-primary Part of the Tame Kernel

If $q = 3$, we can give a more precise description of the pure cubic fields.

In [6], Keune gives the following

Lemma 5.1 *Let* p *be an odd prime,* F *a number field,* ζ_p *a primitive* p*-th root of unity, and suppose that* $\zeta_p \notin F$. *Then we have a short exact sequence*

$$0 \to \left(\mu_p \otimes Cl\left(O_{F(\zeta_p)}\left[\frac{1}{p}\right]\right)\right)^\Gamma \to K_2\mathcal{O}_F/p \to \bigoplus_{\mathfrak{p}\in S'} \mu_p \to 0,$$

where $\Gamma = \mathrm{Gal}(F(\zeta_p)/F)$ *and* S' *is the set of* p*-adic primes of* F *which split completely in* $F(\zeta_p)$.

Now let us recall a result on 3-class groups of non-Galois cubic fields by Iimura in [7].

Lemma 5.2 (Iimura [7]) *Let* G *be any finite elementary abelian* 3*-group. Then there exist infinitely many pure cubic fields whose* 3*-class groups are isomorphic to* G.

In fact suppose $m = \mathrm{rank}(G)$. Let $p_1, p_2, \ldots, p_m, q$ be rational primes satisfying the following conditions:

(i) $p_i \equiv 1 \pmod 9$ for $1 \leq i \leq m, q \equiv 2 \pmod 9$;

(ii) p_i is a cubic residue modulo p_j if $i < j$;

(iii) $p_1 \cdots p_{i-1}q$ is a cubic nonresidue modulo p_i for each i.

Let $n = p_1p_2\cdots p_m q$ and $F = \mathbb{Q}(\sqrt[3]{n})$. Then $E = F(\zeta_3) = \mathbb{Q}(\sqrt[3]{n}, \zeta_3)$ is the normal closure of F. For any algebraic number field K, let A_K be the 3-Sylow subgroup of the ideal class group $Cl(\mathcal{O}_K)$. In [7], it is proved that A_F is an elementary abelian 3-group of rank m and A_E is an elementary 3-group of rank $2m$. Let σ be the generator of $\mathrm{Gal}(E/F)$. Then

$$A_E = A_F \times B,$$

where $B = \{y \in A_E | \sigma(y) = y^{-1}\} \cong (\mathbb{Z}/3\mathbb{Z})^m$.

Now we can prove the following

Theorem 5.3 *Let* m *be a fixed positive integer. Then there exist infinitely many pure cubic fields whose* 3*-rank of the tame kernels is equal to* m.

Proof Let $p = 3$ in Lemma 5.1, then we have the short exact sequence

$$0 \to \Big(\mu_3 \otimes Cl\Big(\mathcal{O}_{F(\zeta_3)}\Big[\frac{1}{3}\Big]\Big)\Big)^{\Gamma} \to K_2\mathcal{O}_F/3 \to \bigoplus_{\mathfrak{p}\in S'} \mu_3 \to 0,$$

where $\Gamma = \mathrm{Gal}(F(\zeta_3)/F)$ and S' is the set of 3-adic primes of F which split completely in $F(\zeta_3)$.

Let $p_1, \dots, p_m, q$ and n be defined as above, and $F = \mathbb{Q}(\sqrt[3]{n})$. It is easy to see that $(p_1 p_2 \cdots p_m q)^2 \equiv 4 \not\equiv 1^2 \pmod 9$. By Lemma 4.4, S' is an empty set. Hence we have

$$\Big(\mu_3 \otimes Cl\Big(\mathcal{O}_{F(\zeta_3)}\Big[\frac{1}{3}\Big]\Big)\Big)^{\Gamma} \cong K_2\mathcal{O}_F/3.$$

On the other hand, let $\mathfrak{p}$ be the 3-adic prime of $\mathbb{Q}(\zeta_3)$, then $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}^3$, where $\mathfrak{P} \in E = F(\zeta_3)$. It is easy to see that $[\mathfrak{p}]^3 = 1$ since $\mathbb{Z}(\zeta_3)$ is a principal ideal domain. As a result, $\mathfrak{P}$ or $\mathfrak{P}^3$ is a principal ideal.

Suppose that $\mathfrak{P}$ is a principal ideal. Then $Cl(\mathcal{O}_E[\frac{1}{3}]) = Cl(\mathcal{O}_E)$. Recall that σ is the generator of Γ, then $\sigma(\zeta_3) = \zeta_3^{-1}$. Choose an element $\alpha = \zeta_3 \otimes x \in \mu_3 \otimes A_E = \mu_3 \otimes Cl(O_E)$. Then

$$\sigma(\zeta_3 \otimes x) = \sigma(\zeta_3) \otimes \sigma(x) = \zeta_3^{-1} \otimes \sigma(x) = \zeta_3 \otimes \sigma(x)^{-1}.$$

Since $x \in A_E$, we can write $x = (x_1, x_2)$, where $x_1 \in A_F, x_2 \in B$. Then

$$\sigma(x) = (\sigma(x_1), \sigma(x_2)) = (x_1, \sigma(x_2)).$$

Then $\sigma(x)^{-1} = (x_1^{-1}, \sigma(x_2)^{-1})$. So if $\zeta_3 \otimes x \in (\mu_3 \otimes A_E)^{\Gamma}$, then $x = \sigma(x)^{-1}$. Consequently, $(x_1, x_2) = (x_1^{-1}, \sigma(x_2)^{-1})$. So we have

$$x_1 = x_1^{-1}, \quad x_2 = \sigma(x_2)^{-1}.$$

Since $x_1^2 = 1$ and $Cl(\mathcal{O}_F)$ is a 3-group, we have $x_1 = 1$. Recall that $B = \{y \in A_E | \sigma(y) = y^{-1}\} \cong (\mathbb{Z}/3\mathbb{Z})^m$. Hence $x_2 = \sigma(x_2)^{-1}$ holds for all $x_2 \in B$.

Hence $a \otimes x = a \otimes (x_1, x_2) \in (\mu_3 \otimes Cl(\mathcal{O}_E))^{\Gamma}$ if and only if $x_1 = 1$. Hence we have $(\mu_3 \otimes Cl(\mathcal{O}_E))^{\Gamma} \cong (\mathbb{Z}/3\mathbb{Z})^m$.

Now suppose that $\mathfrak{P}|3$ is not a principal ideal. Since 3 is totally ramified in E in our case, we have $\sigma(\mathfrak{P}) = \mathfrak{P}$. So $Cl(\mathcal{O}_E)[\frac{1}{3}] = Cl(\mathcal{O}_E)/\langle[\mathfrak{P}]\rangle$ and $\zeta_3 \otimes [\mathfrak{P}] \notin (\mu_3 \otimes Cl(\mathcal{O}_E)[\frac{1}{3}])^{\Gamma}$ since $(\zeta_3 \otimes \mathfrak{P})^{\sigma} = \sigma(\zeta_3) \otimes \sigma([\mathfrak{P}]) = \zeta_3^{-1} \otimes [\mathfrak{P}]$. Hence $(\mu_3 \otimes Cl(\mathcal{O}_E)[\frac{1}{3}])^{\Gamma} = (\mu_3 \otimes Cl(\mathcal{O}_E))^{\Gamma}$. The same argument follows. □

Remark 5.4 Let $F = \mathbb{Q}(\sqrt[3]{n})$, where $n = 3^e q_1^{f_1} \cdots q_J^{f_J}$, each q_i is a rational prime $\equiv -1 \pmod 3$, $f_i = 1$ or 2 for $1 \le i \le J$, and $e = 0, 1$, or 2. Let $K = \mathbb{Q}(\zeta_3)$. And let $E = K(\sqrt[3]{n})$. By Theorem 5.1 of [8], the rank of the 3-class group A_F (resp. A_E) of F (resp. E) is t (resp. $2t$), where

$$t = \begin{cases} J, & \text{if each } q_i \equiv -1 \pmod 9 \text{ and } n \not\equiv \pm 1 \pmod 9, \\ J-1, & \text{if each } q_i \equiv -1 \pmod 9 \text{ and } n \equiv \pm 1 \pmod 9, \\ J-1, & \text{if some } q_i \equiv 2 \text{ or } 5 \pmod 9 \text{ and } n \not\equiv \pm 1 \pmod 9, \\ J-2, & \text{if some } q_i \equiv 2 \text{ or } 5 \pmod 9 \text{ and } n \equiv \pm 1 \pmod 9. \end{cases}$$

Similarly to the previous discussion, we can determine the 3-rank of the tame kernel of some special pure cubic fields.

Examples (1) Let us consider the pure cubic fields $F=\mathbb{Q}(\sqrt[3]{q}),\mathbb{Q}(\sqrt[3]{3q}),\mathbb{Q}(\sqrt[3]{9q})$, where $q\equiv 2 \pmod 9$ or $q\equiv 5 \pmod 9$. By the previous remark, the 3-rank of the class group of the normal closure $E=F(\zeta_3)$ of F is $2(1-1)=0$. Moreover, it is easy to see that the 3-adic prime of F ramifies totally in E, so by Keune's exact sequence, the 3-rank of the tame kernel of the number field F is 0.

(2) Consider the pure cubic fields $F=\mathbb{Q}(\sqrt[3]{q_1q_2})$, where $q_1\equiv 2 \pmod 9, q_2\equiv 5 \pmod 9$. It is easy to see that $q_1q_2\equiv 1 \pmod 9, q_1^2q_2^2\equiv 1 \pmod 9$. By the previous remark, the rank of the 3-class group of the normal closure E of F is $2(2-2)=0$, so by Keune's short exact sequence, the 3-rank of $K_2\mathcal{O}_F$ is equal to the number of 3-adic primes of F which splits completely in E, which is 1. In fact, by Proposition 2.1, we have $3\mathcal{O}_F=\mathfrak{p}_1^2\mathfrak{p}_2$, so $f(\mathfrak{p}_1/3)=f(\mathfrak{p}_2/3)=1$, where $f(\cdot)$ denotes the residue degree. Since $E/\mathbb{Q}$ is a Galois extension and E/F is a quadratic extension, $\mathfrak{p}_2$ ramifies totally in E/F. Let $\mathfrak{p}_2\mathcal{O}_E=\mathfrak{P}_2^2$, we have $f(\mathfrak{P}_2/3)=f(\mathfrak{P}_2/\mathfrak{p}_2)f(\mathfrak{p}_2/3)=1$. Hence $f(\mathfrak{P}_1/\mathfrak{p}_1)=1$ in the extension E/F, where $\mathfrak{P}_1$ is the prime ideal of E over $\mathfrak{p}_1$ since $E/\mathbb{Q}$ is a Galois extension and $f(\mathfrak{P}_1/3)=f(\mathfrak{P}_2/3)=1$. So $\mathfrak{p}_1$ splits completely in E/F since $\mathfrak{p}_1$ does not ramify in the extension E/F.

Proposition 5.5 (1) *If* $a^2\not\equiv b^2 \pmod 9$, *then*

$$\text{3-rank}\,K_2\mathcal{O}_F=\text{3-rank}\,A_E-\text{3-rank}\,A_F.$$

(2) *If* $a^2\equiv b^2 \pmod 9$, *then*

$$\text{3-rank}\,K_2\mathcal{O}_F\le\text{3-rank}\,A_E-\text{3-rank}\,A_F+1.$$

Proof Since $E=F(\zeta_3)$, we have $G=\mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})=\langle\sigma\rangle$, where σ is the complex conjugation. So we have

$$\varepsilon_0=\frac{1}{2}(1+\sigma),\quad \varepsilon_1=\frac{1}{2}(1-\sigma).$$

By Theorem 4.6, if $a^2\not\equiv b^2 \pmod 9$, then 3-rank $K_2\mathcal{O}_F=$ 3-rank ε_1A_E. By Lemma 2.1 of [9], the norm map $N_{E/F}:A_E\to A_F$ is surjective and $\ker N_{E/F}=\varepsilon_1A_E$. The result follows.

The proof of (2) is similar. □

Let h_E (resp. h_F) denote the class number of E (resp. F). By Theorems 12.1 and 14.1 of [10], $h_E=q\cdot h_F^2/3$, where $q=1$ or 3.

Corollary 5.6 (1) *If* $a^2\not\equiv b^2 \pmod 9$, *then* 3-rank $K_2\mathcal{O}_F\le$ 3-rank A_F.

(2) *If* $a^2\not\equiv b^2 \pmod 9$, *then* 3-rank $K_2\mathcal{O}_F\le$ 3-rank A_F+1.

So by Remark 5.4 and Proposition 5.5, we can have the following

Corollary 5.7 *Let* $F=\mathbb{Q}(\sqrt[3]{n})$, *where* $n=3^eq_1^{f_1}\cdots q_J^{f_J}=ab^2$, *each* q_i *is a rational prime* $\equiv -1 \pmod 3), f_i=1$ *or* 2 *for* $1\le i\le J$, *and* $e=0,1,$ *or* 2. *Let* $K=\mathbb{Q}(\zeta_3)$, *and* $E=K(\sqrt[3]{n})$. *If* $a^2\not\equiv b^2 \pmod 9$, *then the* 3*-rank of the tame kernel of* F *is* t, *where*

$$t=\begin{cases} J, & \text{if each } q_i\equiv -1 \pmod 9 \text{ and } n\not\equiv\pm1 \pmod 9,\\ J-1, & \text{if each } q_i\equiv -1 \pmod 9 \text{ and } n\equiv\pm1 \pmod 9,\\ J-1, & \text{if some } q_i\equiv 2 \text{ or } 5 \pmod 9 \text{ and } n\not\equiv\pm1 \pmod 9,\\ J-2, & \text{if some } q_i\equiv 2 \text{ or } 5 \pmod 9 \text{ and } n\equiv\pm1 \pmod 9.\end{cases}$$

Let us now recall the Reflection Theorem of [11]. Let L be the maximal unramified elementary (i.e., isomorphic to $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\times\cdots\times\mathbb{Z}/3\mathbb{Z}$) abelian 3-extension of E with the Galois group $H := \mathrm{Gal}(L/E)$. The Artin reciprocity map establishes an isomorphism

$$\alpha : A_E/3 \to H$$

of Γ-modules, where $\Gamma := \mathrm{Gal}(E/\mathbb{Q})$ and Γ acts on H by conjugation. Since $\zeta_3 \in E, L/E$ is a Kummer extension, so there is a subgroup B of E^* containing $(E^*)^3$ such that $L = E(\sqrt[3]{B})$. Let $B_0 := B/E^3$. There is a nondegenerate ($\langle h, B_0\rangle = 1 \Leftrightarrow h = 1$, and $\langle H, b_0\rangle = 1 \Leftrightarrow b_0 = 1$) and bilinear pairing

$$H \times B_0 \to \mu_3$$

given by $\langle h, b_0\rangle = h(b^{1/3})/b^{1/3}$, where $h \in H, b_0 = bE^{*3} \in B_0$, which satisfies

$$\langle h, b_0\rangle^\tau = \langle h^\tau, b_0^\tau\rangle, \quad \text{for } \tau \in \Gamma.$$

Consequently, $B_0 \cong \hat{H}$ as G-modules, and $\hat{H} \cong H$ as abelian groups. Since L/E is unramified, we can see that every principal ideal (b), where $b \in B$, is the 3rd power of an ideal in E. Then there is a homomorphism of Γ-modules

$$\phi : B_0 \to {}_3A_E$$

such that $\varphi(bE^{*3}) = Cl(\mathfrak{a})$, where the ideal $\mathfrak{a}$ of $\mathcal{O}_E$ is defined by the condition $(b) = \mathfrak{a}^3$. It is easy to prove that $\ker\phi \cong$ subgroup of U_E/U_E^3 as Γ-modules. Since $\zeta_3 \notin F$ and $E = F(\zeta_3)$, we can see that $G = \mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) = \mathrm{Gal}(E/F)$ is a subgroup of Γ. Thus all Γ-modules are G-modules.

Let σ be a generator of G. Then $\varepsilon_0 = \frac{1}{2}(1+\sigma)$ and $\varepsilon_1 = \frac{1}{2}(1-\sigma)$ are two idempotents of the group ring $\mathbb{Z}_3[G]$. Note that $H \cong A_E/3$ as G-modules. So we have $\varepsilon_i H \cong \varepsilon_i A_E/3$ for $i = 0, 1$. Let $h \in \varepsilon_i H$. Then $\sigma_a h = h^{\omega^i(a)}$ for all $a \in (\mathbb{Z}/3\mathbb{Z})^*$, where $\sigma_1 = 1, \sigma_2 = \sigma$ and ω is the Teichmuller character of the group $(\mathbb{Z}/3\mathbb{Z})^*$. Let $b \in \varepsilon_k B_0$. Then

$$\langle h, b\rangle^{\omega(a)} = \langle h, b\rangle^{\sigma_a} = \langle h^{\omega^i(a)}, b^{\omega^k(a)}\rangle = \langle h, b\rangle^{\omega^{i+k}(a)}$$

for all a. If $i + k \not\equiv 1 \pmod 2$, then $\langle h, b\rangle = 1$. Since the pairing between $B = \varepsilon_0 B \oplus \varepsilon_1 B$ and $H = \varepsilon_0 H \oplus \varepsilon_1 H$ is nondegenerate, it follows easily that the induced pairing

$$\varepsilon_i H \times \varepsilon_j B \to \mu_3, \quad i = 0, j = 1 \text{ or } i = 1, j = 0$$

is nondegenerate. Hence we have

$$\varepsilon_0 B_0 \cong \varepsilon_1 H \cong \varepsilon_1(A/3), \quad \text{as abelian groups.}$$

Note that the reflection map $\phi : B_0 \to {}_3A_E$ is G-linear, so we have the homomorphism

$$\phi : \varepsilon_0 B_0 \to \varepsilon_0({}_3A_E).$$

Thus we have the following exact sequence

$$0 \longrightarrow \ker(\phi) \cap \varepsilon_0 B_0 \longrightarrow \varepsilon_0 B_0 \xrightarrow{\phi} A_F.$$

We also have

$$\ker(\phi) \cap \varepsilon_0 B_0 \cong \text{subgroup of } \varepsilon_0(U_E/U_E^3).$$

Also we have

$$\phi_1 : \varepsilon_1 B_0 \to \varepsilon_1 A_E$$

and

$$(\ker\phi_1) \cap \varepsilon_1 B_0 \cong \text{subgroup of } \varepsilon_1(U_E/U_E^3).$$

Now we have

Theorem 5.8 *Let $E = F(\zeta_3)$, where F is a pure cubic field defined above. Then*

$$3\text{-rank}\, A_F - 3 \leq 3\text{-rank}\, K_2\mathcal{O}_F \leq 3\text{-rank}\, A_F + 1.$$

Proof By Theorem 4.6 and the above arguments, we have

$$\begin{aligned} 3\text{-rank} K_2\mathcal{O}_F &= 3\text{-rank}\, \varepsilon_1 A_E = 3\text{-rank}\, \varepsilon_0 B_0 \\ &\leq 3\text{-rank}\, A_F + \ker\phi \cap \varepsilon_0 B_0 \\ &\leq 3\text{-rank}\, A_F + 3\text{-rank}\, U_F/U_F^3 \\ &\leq 3\text{-rank}\, A_F + 1. \end{aligned}$$

The last statement comes from the Dirichlet's unit theorem.

Since $E/\mathbb{Q}$ is a Galois extension and $\zeta_3 \in E, E$ is a totally imaginary number field. By Dirichlet's unit theorem, $3\text{-rank}\, \varepsilon_1(U_E/U_E^3) \leq 2$.

Similarly, we have

$$\begin{aligned} 3\text{-rank}\, A_F &= 3\text{-rank}\, \varepsilon_0 A_E = 3\text{-rank}\, \varepsilon_0 H = 3\text{-rank}\, \varepsilon_1 B_0 \\ &\leq 3\text{-rank}\, \varepsilon_1 A_E + \ker\phi \cap \varepsilon_0 B_0 \\ &\leq 3\text{-rank}\, \varepsilon_1 A_E + 3\text{-rank}\, \varepsilon_1(U_E/U_E^3) + 1 \\ &\leq 3\text{-rank}\, K_2\mathcal{O}_F + 3. \end{aligned}$$

So $3\text{-rank}\, K_2\mathcal{O}_F \geq 3\text{-rank}\, A_F - 3$. □

Combining Corollary 5.6 with Theorem 5.8, we have

Corollary 5.9 *Let $E = F(\zeta_3)$, where F is a pure cubic field defined above. If $a^2 \not\equiv b^2 \pmod 9$, then*

$$3\text{-rank}\, A_F - 3 \leq 3\text{-rank}\, K_2\mathcal{O}_F \leq 3\text{-rank}\, A_F.$$

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