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On the 2-primary Part of Tame Kernels of Real Quadratic Fields

Xiao Yun CHENG
School of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016*, P. R. China*
Email: *xycheng@nuaa.edu.cn*

Xue Jun GUO
Department of Mathematics, Nanjing University, Nanjing 210093*, P. R. China*
Email: *guoxj@nju.edu.cn*

Abstract Let $F = \mathbb{Q}(\sqrt{p})$, where $p = 8t + 1$ is a prime. In this paper, we prove that a special case of Qin's conjecture on the possible structure of the 2-primary part of $K_2\mathcal{O}_F$ up to 8-rank is a consequence of a conjecture of Cohen and Lagarias on the existence of governing fields. We also characterize the 16-rank of $K_2\mathcal{O}_F$, which is either 0 or 1, in terms of a certain equation between 2-adic Hilbert symbols being satisfied or not.

Keywords Tame kernels, class groups, diophantine equations

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1 Introduction

Let $F = \mathbb{Q}(\sqrt{d})$, $d > 2$ square free having k odd prime factors, $K_2\mathcal{O}_F$ the tame kernel of F. By Theorem 1 of [2], the 2-rank of $K_2\mathcal{O}_F = s + k$, where 2^s is the number of elements of the set $\{\pm 1, \pm 2\}$ that are norms of an element of F. The formulas for the 4-rank of $K_2\mathcal{O}_F$ is much more complicated. One can see [2–4, 7, 8, 13, 14]. By Qin's methods in [20–22, 24], one can determine the 2^n-rank of $K_2\mathcal{O}_F$ for $n = 2, 3$ by calculating the Legendre symbols. One can see the explicit table of the structure of tame kernels of quadratic fields F whose discriminants have few prime divisors in [19–24]. Qin's method is generalized to relatively quadratic extensions in [12]. The 4-rank density of tame kernels of quadratic fields whose discriminant have less than 3 prime divisors can be found in [5, 16–18]. One can see the explicit formulas on the 4-rank of general quadratic number fields in [28, 29]. The 4-rank density for general quadratic fields can be found in [9, 11].

In [24], Qin made the following conjecture.

Conjecture 1.1 (Qin) *Let $k \geq 2$ and $n \in \mathbb{N}$. Given $k - 1$ integers $r_4, r_8, \dots, r_{2^k}$ satisfying $n \geq r_4 \geq r_8 \geq \cdots \geq r_{2^k} \geq 0$. Then there exist infinitely many quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ such that $d > 0$ square-free has exactly n prime divisors, any of which $\equiv 1 \pmod 8$ and the 2^j-rank of $K_2\mathcal{O}_F = r_{2^j}$ $(2 \leq j \leq k)$.*

The same assertion should be true for $F = \mathbb{Q}(\sqrt{d})$ with $d = -d'$ or $d = 2d'$ or $d = -2d'$, where $d' > 0$ has exactly n prime divisors, any of which $\equiv 1 \pmod 8$.

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In [23], Qin proved that above conjecture is true for $k = 2$ and $n-1 \geq r_4 \geq 0$. In [10], it is proved that for any finite abelian group G of exponent 8, there are infinitely many imaginary quadratic fields E such that $K_2(\mathcal{O}_E)/(K_2(\mathcal{O}_E))^8 \simeq G$; and for any finite abelian group H of exponent 8 with $\mathrm{rk}_2 H \geq 2 + \mathrm{rk}_4 H$, there are infinitely many real quadratic fields F such that $K_2(\mathcal{O}_F)/(K_2(\mathcal{O}_F))^8 \simeq H$. Note that there is a prime divisor q of d with $q \equiv 3$ or $5 \pmod 8$, while each prime divisors q of d should be $q \equiv 1 \pmod 8$ in Qin's conjecture. Hence Qin's conjecture remains open. We will see that when $n = 1$ and $k = 3$, Qin's conjecture is connected to the following long-standing conjecture in classical number theory.

Conjecture 1.2 (Cohn and Lagarias [6]) *There is a field K with the property that if p_1 and p_2 are primes such that the Artin symbols $[(K/\mathbb{Q})/(p_1)] = [(K/\mathbb{Q})/(p_2)]$, then 2^i-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p_1})) = 2^i$-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p_2}))$ for $1 \leq i \leq 4$.*

In [6], the smallest K with the above property is denoted by $\Omega_4(-4)$. In fact, we will prove that Conjecture 1.2 implies Qin's conjecture 1.1 for $n = 1$ and $k = 3$ by Theorem 1 of [2].

We will briefly explain our strategy. Recall that if $p \equiv 1 \pmod 8$ and $F = \mathbb{Q}(\sqrt{p})$, then 2-rank of $K_2(\mathcal{O}_F) = 3$. By [27, Corollary (25.12)], the 4-rank of $K_2(\mathcal{O}_F) = 1$ if and only if $p = x^2 + 32y^2$ for some rational integers $x,\ y$. By the Preliminary Theorem of [1], we have that $p = x^2 + 32y^2$ for some rational integers $x,\ y$ if and only if $p = e^2 - 32f^2$ for some rational integers $e,\ f$. For any $\alpha, \beta \in \mathbb{Z}$ and prime ℓ, let $(a, \beta)_\ell$ be the Hilbert symbol at ℓ.

Theorem 1.3 (Qin [22]) *Let p be a prime such that $p = e^2 - 32f^2$ for some rational integers $e,\ f$. Then there are rational integers $X_1,\ Y_1,\ Z_1$, such that $(e + 2f)Z_1^2 = X_1^2 + pY_1^2$. Then the 8-rank of $K_2(\mathcal{O}_F) = 1$ if and only if $(e, 2)_2(Z_1, -1)_2 = 1$.*

If $e \equiv 1 \pmod 8$ (which is equivalent to $p \equiv 1 \pmod{16}$), then $(e, 2)_2 = 1$ which implies that 8-rank of $K_2(\mathcal{O}_F) = 1$ if and only if $(Z_1, -1)_2 = 1$. Hence $(Z_1, -p)_2 = 1$. So there are rational integers X_2, Y_2, Z_2 such that $Z_1 Z_2^2 = X_2^2 + pY_2^2$. In this case, we have $e \in \mathrm{Norm}_{\mathbb{Q}(\sqrt{-2p})/\mathbb{Q}}(\mathbb{Q}(\sqrt{-2p})$. Hence there are rational integers $\tilde{X},\ \tilde{Y},\ \tilde{Z}$ such that $e\tilde{Z}^2 = \tilde{X}^2 + 2p\tilde{Y}^2$.

Our main theorem is the following.

Theorem 1.4 *We use the same notations as above. If $p \equiv 1 \pmod{16}$ and 8-rank of $K_2(\mathcal{O}_F) = 1$, then 16-rank of $K_2(\mathcal{O}_F) = 1$ if and only if*

$$(Z_2, -p)_2(\tilde{Z}, -2p)_2 = 1.$$

2 Main Results

For a real quadratic field F with discriminant D, let $h(D)$ be the class number of F, $k_2(D)$ the cardinality of $K_2(\mathcal{O}_F)$.

In [25] and [26], Urbanowicz extended the Hardy–Williams congruence to a linear congruence involving the orders of K_2-groups of the integers of real quadratic fields. He proved the following theorem. One should note that Urbanowicz's original theorems contain many more statements.

Theorem 2.1 (Urbanowicz [27, Theorems 32, 34]) *We use the notations as above. Then*

(1) $k_2(p) \equiv 2h(-4p) + 16t \pmod{32}$;

(2) $64|k_2(p)$ *if and only if* $8|h(-4p)$ *and* $(h(-4p)/8) + (h(-8p)/4) \equiv 0 \pmod 4$.

In the following context, $F = \mathbb{Q}(\sqrt{p})$, where $p = 8t + 1$ is a prime.

Theorem 2.2 *We assume that there is a governing field K having the property that if p_1 and p_2 are primes such that the Artin symbols $[(K/\mathbb{Q})/(p_1)] = [(K/\mathbb{Q})/(p_2)]$, then 2^i-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p_1})) = 2^i$-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p_2}))$ for $1 \le i \le 4$. Then for any non-negative integer r_4 and r_8 satisfying $1 \ge r_4 \ge r_8 \ge 0$, there are infinitely many primes p such that*

(1) $p \equiv 1 \pmod 8$;

(2) 4*-rank of* $K_2(\mathcal{O}_F) = r_4$;

(3) 8*-rank of* $K_2(\mathcal{O}_F) = r_8$.

Proof By Theorem 2.1(1), we have

$$k_2(p) \equiv \begin{cases} 2h(-4p) \pmod{32}, & \text{if } p \equiv 1 \pmod{16}; \\ 2h(-4p) + 16 \pmod{32}, & \text{if } p \equiv 9 \pmod{16}. \end{cases} \tag{2.1}$$

By Browkin and Schinzel's 2-rank formula of $K_2(\mathcal{O}_F)$ in the first paragraph of Introduction, we know that the 2-rank of $K_2(\mathcal{O}_F) = 3$. By [7, Corollary (25.12)], we have the following formula:

$$\text{4-rank of } K_2(\mathcal{O}_F) = \begin{cases} 1, & \text{if } p = x^2 + 32y^2 \text{ for some } x, y \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Hence by Proposition 5.3 of [15], 4-rank of $K_2(\mathcal{O}_F) = 1$ if and only if $8|h(-4p)$ which implies that there are infinitely many p such that 4-rank of $K_2(\mathcal{O}_F) = 0$. Hence, we only need to consider the case when $r_4 = 1$.

Now $r_2 = 3$ and $r_4 = 1$. By (2.1), we have

$$\begin{aligned} \text{8-rank of } K_2(\mathcal{O}_F) = 1 &\Longleftrightarrow 32|k_2(p) \\ &\Longleftrightarrow \begin{cases} 16|h(-4p), & \text{if } p = x^2 + 32y^2 \equiv 1 \pmod{16}; \\ 16 \nmid h(-4p), & \text{if } p = x^2 + 32y^2 \equiv 9 \pmod{16}. \end{cases} \end{aligned} \tag{2.3}$$

We assume that there is a governing field K having the property that if p_1 and p_2 are primes such that the Artin symbols $[(K/\mathbb{Q})/(p_1)] = [(K/\mathbb{Q})/(p_2)]$, then 2^i-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p_1})) = 2^i$-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-p_2}))$ for $1 \le i \le 4$. If we do not put K to be the smallest one, we can assume the 16-th primitive root of unity $\zeta_{16} \in K$, otherwise we can add ζ_{16} to K. Let $p_1 = 41$, $p_2 = 257$. Then $p_1 \equiv 9 \pmod{16}$ and $8||h(-4p_1)$, $p_2 \equiv 1 \pmod{16}$ and $16|h(-4p_2)$. Hence those primes p such that $[(K/\mathbb{Q})/(p_1)] = [(K/\mathbb{Q})/(p)]$ will satisfy the following condition:

$$p \equiv 9 \pmod{16} \quad \text{and} \quad 16 \nmid h(-4p).$$

And those primes p such that $[(K/\mathbb{Q})/(p_2)] = [(K/\mathbb{Q})/(p)]$ will satisfy the following condition:

$$p \equiv 1 \pmod{16} \quad \text{and} \quad 16|h(-4p).$$

Hence by Chebotarev's density theorem, there are infinitely many primes p such that $p \equiv 9 \pmod{16}$ and $16 \nmid h(-4p)$, and there are also infinitely many primes p such that $p \equiv 1 \pmod{16}$ and $16|h(-4p)$. Hence by (2.3), there are infinitely many p such that 8-rank of $K_2(\mathcal{O}_F) = 1$. In fact, the density of these p is positive.

Similarly, one can prove that there are infinitely many p such that 8-rank of $K_2(\mathcal{O}_F) = 0$ and 4-rank of $K_2(\mathcal{O}_F) = 1$.

By the same argument, we can prove the following theorem for $\mathbb{Q}(\sqrt{2p})$.

Theorem 2.3 *We assume that there is a governing field K having the property that if p_1 and p_2 are primes such that the Artin symbols $[(K/\mathbb{Q})/(p_1)] = [(K/\mathbb{Q})/(p_2)]$, then 2^i-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-2p_1})) = 2^i$-rank of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-2p_2}))$ for $1 \le i \le 4$. Then for any non-negative integer r_4 and r_8 satisfying $1 \ge r_4 \ge r_8 \ge 0$, there are infinitely many primes p such that*

(1) $p \equiv 1 \pmod 8$;

(2) 4-*rank of* $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{2p})} = r_4$;

(3) 8-*rank of* $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{2p})} = r_8$.

Proposition 2.4 *Let p be a prime such that $p \equiv 1 \pmod{16}$ and 8-rank of $K_2(\mathcal{O}_F) = 1$. Then 16-rank of $K_2(\mathcal{O}_F) = 1$ if and only if $32|h(-4p)$ and $16|h(-8p)$, or $16||h(-4p)$ and $8||h(-8p)$.*

Proof By Theorem 2.1(2), $64|k_2(p)$ if and only if $8|h(-4p)$ and $(h(-4p)/8) + (h(-8p)/4) \equiv 0 \pmod 4$. Since we have assumed that $p \equiv 1 \pmod{16}$ and 8-rank of $K_2(\mathcal{O}_F) = 1$, one can see that $64|k_2(p)$ if and only if $32|h(-4p)$ and $16|h(-8p)$, or $16||h(-4p)$ and $8||h(-8p)$. Since the 2-rank of $K_2(\mathcal{O}_F) = 3$ and 4-rank of $K_2(\mathcal{O}_F) = 1$, 16-rank of $K_2(\mathcal{O}_F) = 1$ if and only if $64|k_2(p)$. □

Lemma 2.5 *Let p be a prime such that $p \equiv 1 \pmod{16}$ and 8-rank of $K_2(\mathcal{O}_F) = 1$. Then there are rational integers $e, f, X_1, Y_1, Z_1,\ X_2, Y_2, Z_2$ such that*

$$\begin{cases} p = e^2 - 32f^2, \\ (e + 2f)Z_1^2 = X_1^2 + pY_1^2, \\ Z_1Z_2^2 = X_2^2 + pY_2^2. \end{cases}$$

Proof By the proof of Proposition 5.3(1) of [15], we know that there are rational integers e, f, X_1, Y_1, Z_1 such that

$$p = e^2 - 32f^2 \quad \text{and} \quad (e + 2f)Z_1^2 = X_1^2 + pY_1^2.$$

Since $p \equiv 1 \pmod{16}$, by (2.3), we have $16|h(-4p)$. Hence by [15, Theorem 4.1], $Z_1 \in \mathrm{Norm}_{F/\mathbb{Q}}(F)$ which implies that there are rational integer X_2, Y_2, Z_2 such that

$$Z_1Z_2^2 = X_2^2 + pY_2^2.$$ □

Lemma 2.6 *We use the same notations and assumption as in Lemma 2.5. Then*

(1) $32|h(-4p)$ *if and only if* $(Z_2, -p)_2 = 1$;

(2) $16||h(-4p)$ *if and only if* $(Z_2, -p)_2 = -1$.

Proof The lemma follows from Theorem 4.1 of [15] and (2.3). □

Lemma 2.7 *We use the same notations and assumption as in Lemma 2.5. Then there exist rational integers $\tilde{X}, \tilde{Y}, \tilde{Z}$ such that*

$$e\tilde{Z}^2 = \tilde{X}^2 + 2p\tilde{Y}^2.$$

(1) $16|h(-8p)$ *if and only if* $(\tilde{Z}, -2p)_2 = 1$;

(2) $8||h(-8p)$ *if and only if* $(\tilde{Z}, -2p)_2 = -1$.

Proof Since $2e^2 = (8f)^2 + 2p$ and the assumption that 8-rank of $K_2(\mathcal{O}_F) = 1$, we have that $8|h(-8p)$ by Proposition 5.3(3) of [15]. Hence by Theorem 4.1 of [15], e is the norm of some element of $\mathbb{Q}(\sqrt{-2p})$ which implies that there exist rational integers $\tilde{X},\ \tilde{Y},\ \tilde{Z}$ such that

$$e\tilde{Z}^2 = \tilde{X}^2 + 2p\tilde{Y}^2.$$

Again by Theorem 4.1 of [15], we know that $16|h(-8p)$ if and only if $(\tilde{Z}, -2p)_2 = 1$. □

Now we can prove Theorem 1.4.

Proof of Theorem 1.4 By Proposition 2.4, 16-rank of $K_2(\mathcal{O}_F) = 1$ if and only if $32|h(-4p)$ and $16|h(-8p)$, or $16||h(-4p)$ and $8||h(-8p)$. By Lemmas 2.6 and 2.7, $32|h(-4p)$ and $16|h(-8p)$, or $16||h(-4p)$ and $8||h(-8p)$ if and only if $(Z_2, -p)_2(\tilde{Z}, -2p)_2 = 1$. □

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