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The densities for 3-ranks of tame kernels of cyclic cubic number fields

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Abstract Let F be a cubic cyclic field with t ($\geqslant 2$) ramified primes. For a finite abelian group G, let $r_3(G)$ be the 3-rank of G. If 3 does not ramify in F, then it is proved that $t-1 \leqslant r_3(K_2\mathcal{O}_F) \leqslant 2t$. Furthermore if t is fixed, then for any s satisfying $t-1 \leqslant s \leqslant 2t-1$, there is always a cubic cyclic field F with exactly t ramified primes such that $r_3(K_2\mathcal{O}_F) = s$. It is also proved that the densities for 3-ranks of tame kernels of cyclic cubic number fields satisfy a Cohen-Lenstra type formula

$$d_{\infty,\, r} = 3^{-r^2}\prod_{k=1}^{\infty}(1-3^{-k})/\prod_{k=1}^{r}(1-3^{-k})^2.$$

This suggests that the Cohen-Lenstra conjecture for ideal class groups can be extended to the tame kernels of cyclic cubic number fields.

Keywords densities, tame kernels, cyclic cubic fields

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1 Introduction

The structure of the tame kernels of algebraic number fields has been intensively studied by many authors. For quadratic number fields, the structure of 2-Sylow subgroup of the tame kernels can be effectively determined by the methods developed in [28], [29] and [30]. However for other number fields, the structure of the tame kernels remains far from clear.

Let F be a quadratic field, $\mathcal{O}_F$ the ring of integers of F. It is well known that the 2-rank of the class group $\mathrm{Cl}(\mathcal{O}_F)$ is determined by the genus theory. Similarly, the 2-rank of $K_2\mathcal{O}_F$ is essentially determined by the genus theory. One can see [5], [6], [7] and [35] for details. Qin

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gave in [28] and [29] an efficient method to compute the 4-ranks of $K_2\mathcal{O}_F$. Later Hurrelbrink and Kolster introduced in [19] a kind of sign matrices to compute the 4-rank of $K_2\mathcal{O}_F$ for relative quadratic extensions via the local Hilbert symbols. Computing the 8-rank of $K_2\mathcal{O}_F$ is much more difficult than computing the 4-rank. Qin ([30]) extended his original method in [28] and [29] to compute the 8-ranks of $K_2\mathcal{O}_F$ efficiently. The 4-rank density of $K_2\mathcal{O}_F$ has been intensively studied. One can see [9], [18], [25], [26], [31], [32], [36], [37], and [38] for results on the 4-rank density of $K_2\mathcal{O}_F$.

As for cubic number fields F, there are only few results on $K_2\mathcal{O}_F$. One can see [10] and [22] for the 3-rank of $K_2\mathcal{O}_F$ where F is pure cubic.

In 2004, Browkin proved the following theorem for cyclic cubic number fields.

Theorem 1.1 (Browkin, [8], Theorem 2.4 (iii))**.** *Let F be a cyclic cubic number field with only one ramified prime p. Then $3|\#K_2\mathcal{O}_F$ if and only if $p \equiv 1 \pmod{18}$.*

Browkin gave two different proofs in [8]. One proof is analytic and depends on deep results by Mazur and Wiles. While the other one is totally algebraic, using Quillen's long exact sequence in K-theory. Zhou generalized in 2006 the above theorem to the following result by considering the Galois action on K_2-groups.

Theorem 1.2 (Zhou, [39])**.** *Let F be a cyclic cubic field with t ramified primes and $t \geqslant 2$. Then $r_3(K_2\mathcal{O}_F) \leqslant 2t$. Moreover, if 3 does not ramify in F, then $r_3(K_2\mathcal{O}_F) \geqslant 1$.*

One can also see [33] and [34] for results on the 9-rank of $K_2\mathcal{O}_F$ by using the reflection theorems. It is interesting to study the exact lower and upper bound of $r_3(K_2\mathcal{O}_F)$. We shall prove the following result.

Theorem 1.3. *Let F be a cubic cyclic field with t $(\geqslant 2)$ ramified primes. Suppose that 3 does not ramify in F, then $t-1 \leqslant r_3(K_2\mathcal{O}_F) \leqslant 2t$. Furthermore if t is fixed, then for any s satisfying $t-1 \leqslant s \leqslant 2t-1$, there is a cubic cyclic field F with exactly t ramified primes such that $r_3(K_2\mathcal{O}_F) = s$.*

The above theorem gives the exact lower bound of $r_3(K_2\mathcal{O}_F)$. On the other hand, it is very difficult to study the exact upper bound. Numerical computation shows that there are many cubic cyclic fields with only one ramified prime and $r_3(K_2\mathcal{O}_F) = 2$. For general t, one can expect that there are infinitely many cubic cyclic fields F with t ramified primes such that $r_3(K_2\mathcal{O}_F) = 2t$. However it seems very difficult to prove this.

Let

$$A_t = \{F|\ F \text{ cyclic cubic with exactly } t \text{ rational primes ramified in } F\},$$
$$A_{t;\,x} = \{F \in A_t|\ \text{ the conductor of } F \leqslant x\}.$$

It is well known ([23]) that the conductor of F must have the form

$$f_F = 3^e p_1 \cdots p_t,$$

where $e = 0$ or $e = 2$, $t > 0$ and the p_i are pairwise distinct rational primes satisfying $p_i \equiv 1 (\text{mod } 3)$ for $i = 1, \ ..., t$. The discriminant of F is $d_F = f_F^2$. Now we assume that exactly t primes of $\mathbb{Q}$ ramify in F.

As pointed out by Gerth in [14], the number of cyclic cubic fields F with $f_F = 3^2 p_1 \cdots p_{t-1} \leqslant x$ is

$$O(\frac{x(\log\log x)^{t-2}}{\log x}),$$

while $|A_{t;\,x}| \gg \frac{x(\log\log x)^{t-1}}{\log x}$. Hence we can ignore those fields F with $f_F = 3^2 p_1 \cdots p_{t-1}$ in the counting arguments when $x \longrightarrow \infty$. So in this paper, we shall consider only the cyclic cubic fields with $f_F = p_1 \cdots p_t$, each $p_i \equiv 1 \pmod 3$.

By Theorem 1.3, we always have $r_3(K_2\mathcal{O}_F) \geqslant t-1$. Hence we shall study the density problem for $r_3(K_2\mathcal{O}_F) - (t-1)$. Let

$$\begin{aligned} A_{t,\ r;\ x} &= \{F \in A_{t;\ x} |\ r_3(K_2\mathcal{O}_F) = t-1+r\}, \\ d_{t,\ r} &= \lim_{x\to\infty} \frac{|A_{t,\ r;\ x}|}{|A_{t;\ x}|}, \\ d_{\infty,\ r} &= \lim_{t\to\infty} d_{t,\ r},\ \text{where } r \geqslant 0. \end{aligned}$$

In this paper, we shall prove the following Cohen-Lenstra type formula

Theorem 1.4. *With notations as above, we have*

$$d_{\infty,\ r} = \frac{3^{-r^2}\prod_{k=1}^{\infty}(1-3^{-k})}{\prod_{k=1}^{r}(1-3^{-k})^2}.$$

In particular, $d_{\infty,\ 0} \approx 0.56013$, $d_{\infty,\ 1} \approx 0.42009$, $d_{\infty,\ 2} \approx 0.01969$.

2 Main results

Let F be a cyclic cubic number field, ζ_3 a primitive cubic root of unity, $E = \mathbb{Q}(\zeta_3)$ and $K = EF$. Then we have the following theorem proved by Zhou. One can also see [17] for a simplified proof of it.

Theorem 2.1 (Theorem 3.3 of [39])**.**

$$r_3(K_2\mathcal{O}_F) = r_3(A_K) - r_3(A_F),$$

where $A_K,\ A_F$ *are the* 3*-class groups of* $K,\ F$ *respectively.*

As stated in the introduction, we can ignore those fields F with $f_F = 3^2 p_1 \cdots p_{t-1}$ in the counting arguments when $x \longrightarrow \infty$. So we shall consider only the cyclic cubic fields with $f_F = p_1 \cdots p_t$.

Since each $p_i \equiv 1 (\text{mod } 3)$, we know that p_i splits completely in E. We assume that $p_i = \pi_i \bar{\pi}_i$ in $\mathcal{O}_E$. The relative discriminant

$$D_{K/E} = (p_1 \cdots p_t)^2$$

and the absolute norm of the relative discriminant

$$N(D_{K/E}) = (p_1 \cdots p_t)^4.$$

Let A_F be the 3-part of the class group of $\mathcal{O}_F$. And let σ be the generator of the Galois group $\text{Gal}(F/\mathbb{Q})$. Then it is well known ([14], Proposition 4.2 of [15] and Satz 6 of [20]) that

$$r_3(A_F) = t - 1 + R_F,$$

where $R_F = r_3(A_F^{1-\sigma}/A_F^{(1-\sigma)^2})$. Let M_F be the $t \times t$ matrix whose (i,j) entries m_{ij} are defined as

$$\zeta_3^{m_{ij}} = \left(\frac{p_j,\ x_F}{(\pi_i)}\right),$$

where $x_F = \pi_1 \bar{\pi}_1{}^2 \cdots \pi_t \bar{\pi}_t{}^2$ and $\left(\frac{p_j,\ x_F}{(\pi_i)}\right)$ is the cubic Hilbert symbol.

Theorem 2.2 (Proposition 4.6 of [15])**.** *With notations as above,*

$$R_F = t - 1 - \text{rank} M_F.$$

Recall that $K = EF = F(\zeta_3)$. So σ can be extended to a generator of $\mathrm{Gal}(K/E)$. Let A_K be the 3-part of the class group of $\mathcal{O}_K$ and A_K^σ the subgroup fixed by the action of σ. Suppose that exactly t rational primes $p_1, \ldots, p_t$ ($p_i \equiv 1 \pmod 3$) ramify in F. Then by Equation (2.1) of [12],

$$r_3(A_K^\sigma) = \begin{cases} 2t-1, & \text{if each } \pi_i \equiv 1 \bmod \lambda^3; \\ 2t-2, & \text{otherwise;} \end{cases}$$

where $\lambda = 1 - \zeta_3$.

Let $H_K = A_K^{1-\sigma}$ and $R_K = r_3(H_K/H_K^{1-\sigma})$. By Equation (2.11) of [13] and Theorem 3.1 of [14], we have

$$r_3(A_K) = r_3(A_K^\sigma) + R_K.$$

Let τ be the nontrivial element of $\mathrm{Gal}(K/F)$.

Lemma 2.3 (Lemma 2.1 of [11])**.** *For any $\mathbb{Z}_3[\mathrm{Gal}(K/F)]$-module S, let $S^+ = \{a \in S | a^\tau = a\}$ and $S^- = \{a \in S | a^\tau = a^{-1}\}$. Then $S = S^+ \times S^-$.*

Corollary 2.4. *Let $R_K^+ = r_3((H_K/H_K^{1-\sigma})^+)$ and $R_K^- = r_3((H_K/H_K^{1-\sigma})^-)$. Then $R_K = R_K^+ + R_K^-$.*

Proof of Theorem 1.3. By Equation 2.24 of [13],

$$R_K = 2t - 1 - \text{rank } M_K - \varepsilon,$$

where

$$\varepsilon = \begin{cases} 0 \text{ or } 1, & \text{if each } \pi_i \equiv 1 \bmod \lambda^3; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$M_K = \begin{pmatrix} M_K^- & 0 \\ 0 & M_K^+ \end{pmatrix},$$

where

$$M_K^- = (m_{ij}^-)_{t\times(t+1)}$$

with

$$\zeta_3^{m_{ij}^-} = \begin{cases} \left(\frac{\zeta_3,\ x_F}{\pi_i}\right), & \text{if } 1 \leqslant i \leqslant t \text{ and } j = t+1; \\ \left(\frac{\pi_j \bar{\pi}_j^{\,2},\ x_F}{\pi_i}\right), & \text{if } 1 \leqslant i \leqslant t \text{ and } 1 \leqslant j \leqslant t \end{cases}$$

and

$$M_K^+ = (m_{ij}^+)_{t\times t}$$

with

$$\zeta_3^{m_{ij}^+} = \left(\frac{p_j,\ x_F}{(\pi_i)}\right), \text{ for } 1 \leqslant i \leqslant t \text{ and } 1 \leqslant j \leqslant t.$$

Hence $M_F = M_K^+$. So we have

$$\begin{aligned} r_3(A_K) - r_3(A_F) &= r_3(A_K^\sigma) + R_K - r_3(A_F) \\ &= r_3(A_K^\sigma) + 2t - 1 - \text{rank } M_K - \varepsilon - (2t - 2 - \text{rank} M_F) \\ &= r_3(A_K^\sigma) + 1 - \varepsilon - \text{rank } M_K^-. \\ &= \begin{cases} 2t - \varepsilon - \text{rank } M_K^-, & \text{if each } \pi_i \equiv 1 \bmod \lambda^3; \\ 2t - 1 - \text{rank } M_K^-, & \text{otherwise} \end{cases} \end{aligned}$$

Since rank $M_K^- \leqslant t$ and $\varepsilon \leqslant 1$. We have that

$$r_3(A_K) - r_3(A_F) \geqslant t - 1$$

which implies that

$$r_3(K_2(\mathcal{O}_F)) \geqslant t - 1.$$

If t is fixed, then for any $t-1 \leqslant s \leqslant 2t-1$, there is a cubic cyclic field F with exactly t ramified primes such that rank $M_K^- = 2t - 1 - s$ by the Section 3 of [13]. Hence for this cubic cyclic field F, we have $r_3(K_2\mathcal{O}_F) = s$. □

As pointed out by Gerth in [13], the case that each $\pi_i \equiv 1 \bmod \lambda^3$ can be omitted when considering $d_{\infty,\ r}$.

Proof of Theorem 1.4. We have

$$r = r_3(K_2\mathcal{O}_F) - t + 1 = t - \text{rank } M_K^-.$$

By the equation 4.42 of [13], we know that $d_{t,\ r}$ is the probability that a randomly selected $t \times t$ matrix over $\mathbb{F}_3$ has rank equal to $t - r$. Hence the theorem follows from the equation 4.43 and Theorem 4.1 of [13]. □

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