# THE DEFINITION OF ELLIPTIC CURVES

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**Abstract**: All the definitions can be found in [1], [2].

## 1. **The definition**

**Definition 1.1.** An elliptic curve $E$ over a scheme $S$ is a diagram

$$\begin{tikzcd} E \arrow[d, shift right, "f"'] \\ S \arrow[u, shift right, bend right, "e"'] \end{tikzcd}$$

where

(1) $f$ is a smooth and proper morphism,
(2) $e$ is a section, i.e., $e$ is a morphism such that $fe$ is an identity,
(3) the fibre of $f$ are geometrically connected genus 1 curves.

Since $f$ is smooth, it is also flat. And the fibre of $f$ is also smooth for the smoothness of a morphism is invariant under base change.

Let $X$ be a scheme over a field $k$, i.e., over Spec$k$. Then the following statements are equivalent,

(1) $X$ is geometrically connected,
(2) $X_{k'}$ is geometrically connected for any base change Spec$k' \longrightarrow$ Spec$k$.
(3) $X_{\overline{k}}$ is geometrically connected, where $\overline{k}$ is the separable closure of $k$.

If $X$ is quasi-compact, then $X$ is geometrically connected if and only if for every finite separable field extension $k \hookrightarrow k'$, the scheme $X_{k'}$ connected.

We know that properness is preserved under base change. Hence the fibre of $f$ in the definition of elliptic curve over $S$ is a smooth proper geometrically connected curve with genus 1. Any it has also a "zero section" which is the restriction of $e$ to the point $s$.

In this paragraph, we will show that $f$ is finitely presented. Since $f$ is proper, by the definition of properness, we know that $f$ is of finite type. By the definition of finte type, we know that $f$ is quasi-compact. Also by the definition of properness, we know that $f$ is separated. Hence $f$ is quasi-separated. By De Jong's book, f is of finite presentation if it is locally of finite presentation, quasi-compact and quasi-separated. Hence we need only to show that $f$ is locally of finite presentation. While this fact is contained in the definition of smoothness of $f$

By definition, $f$ is locally of finite presentation if it is of finite presentation at every point of $E$. We say that $f$ is of *finite presentation at $x \in E$* if there exists a affine open neighbourhood $\mathrm{Spec}(A) = U \subset E$ of $x$ and and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation. Recall that a ring map $R \to A$ is of finite presentation if $A$ is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ as an $R$-algebra for some $n, m$ and some polynomials $f_j$.

If $E \longrightarrow S$ is an elliptic curve, then any base change $E \times_S T \longrightarrow T$ is also an elliptic curve. This is because of properness, smoothness, and the geometric fibre being connected of genus 1 are preserved under base change. If fact properness, smoothness and the property being geometrically connected are preserved by definition. The genus 1 property is also preserved by base change can be gotten by the fact that $\mathrm{H}^1(E_s = E \times_S \mathrm{Spec}k(s), \mathcal{O}_{E_s})$ is preserved under base change. In fact if $E_s$ is an elliptic curve over $\mathrm{Spec}k$, then it can be defined by a cubic equation and its discriminant of its cubic equation is not zero. Obviously the discriminant is not zero in any field extension either.

## References


[1] J. De Jong, The Stacks Project, http://math.columbia.edu/ dejong/

[2] P. Bruin, Moduli of elliptic curves, http://www.math.leidenuniv.nl/ streng/modular/peter.ps