# DIAGONABILITY OF IDEMPOTENT MATRICES OVER APT RINGS

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Abstract.

**APT ring, Idempotent matrix.**
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## 0. Introduction

## 1. Idempotent matrix

We will call $R$ an abelian ring, if $R$ is a ring with identity and all idempotents of $R$ lie in the center of $R$.

**Theorem 1** Let $R$ be an abelian ring and $A$ be an $n \times n$ idempotent matrix over $R$. If there exist invertible matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix, then there is an invertible matrix $U$ such that $UAU^{-1}$ is a diagonal matrix.

*Proof.* Suppose that there exist invertible matrices $P$ and $Q$ such that $PAQ = diag(b_1, b_2, ..., b_n) = B$. Set $U = Q^{-1}P^{-1}$, then $(BU)^2 = BU$ and $BUB = B$. Therefore, if $U = [u_{ij}]$, we have $b_i = b_i u_{ii} b_i$, $b_i u_{ii}$ and $u_{ii} b_i$ are idempotents of $R$. Set $e = b_i u_{ii}$. Then $b_i = eb_i$ and thus $b_i = e(1 - e + b_i)$. Note that

$$\begin{aligned}(1-e+b_i)(1-e+eu_{ii}) &= 1-e+b_i e u_{ii} + (1-e)(b_i + eu_{ii}) \\ &= 1-e+e+(1-e)b_i = 1+(1-e)eb_i = 1\end{aligned}$$

and

$$\begin{aligned}(1-e+eu_{ii})(1-e+b_i) &= 1-e+eu_{ii}b_i + (1-e)(b_i+eu_{ii}) \\ &= 1-e+eu_{ii}b_i = 1-e+b_i u_{ii} u_{ii} b_i \\ &= 1-e+b_i u_{ii} b_i u_{ii} = 1-e+e^2 = 1\end{aligned}$$

So $1-e+b_i$ is a unit, $b_iu_{ii}$ and $b_i$ differ by a unit factor. Thus we may assume that $Q$ has been adjusted so that $b_i^2 = b_i, i = 1, 2, ..., n$. The matrix equality $BUB = B$ will now give

$$(a) \quad b_iu_{ii} = b_i, i = 1, 2, ..., n.$$
$$(b) \quad b_ib_ju_{ij} = 0, i \neq j, i, j = 1, 2, ..., n.$$

Thus

$$BU = \begin{pmatrix} b_1 & b_1u_{12} & \cdots & b_1u_{1n} \\ b_2u_{21} & b_2 & \cdots & b_2u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_nu_{n1} & b_nu_{n2} & \cdots & b_n \end{pmatrix}$$

Set

$$D = \begin{pmatrix} 1 & b_1u_{12} & b_1u_{13} & \cdots & b_1u_{1n} \\ b_2u_{21} & 1 & b_2u_{23} & \cdots & b_2u_{2n} \\ b_3u_{31} & b_3u_{32} & 1 & \cdots & b_3u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_nu_{n1} & b_nu_{n2} & b_nu_{n3} & \cdots & 1 \end{pmatrix}$$

then $D^2 = 2D - I_n$, so $D$ is invertible . Thus

$$(DP)A(DP)^{-1} = D(PAP^{-1})D^{-1} = D(PAQQ^{-1}P^{-1})D^{-1} = DBUD^{-1}$$
$$= diag(b_1, b_2, ..., b_n)$$

Q.E.D.

Recall that a commutative ring $R$ is a PT(projective trivial) ring if every idempotent matrix over $R$ is similar to a diagonal matrix. From the above theorem, we see that the commutativity of ring $R$ can be weakened to all idempotents of $R$ lie in the center of $R$. We call such rings APT(abel projective trivial) rings.

**Theorem 2** Let $R$ be an APT ring. Then any unimodular vector $(a_1, a_2, ..., a_n)$ in $R^n$ is completable(i.e. can be seen as the first row of some invertible matrix).

*Proof.* Suppose $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is unimodular and $\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n = 1$. Set $A = (\beta_i\alpha_j)$. Then $A^2 = A$. Since $R$ is an APT ring, there exists an invertible matrix $P$ with $PAP^{-1} = B = diag(e_1, e_2, ..., e_n$. Let $X = (x_1, ..., x_n) = \alpha P^{-1}$, $Y = (y_1, ..., y_n) = P\beta$, then $XY = \alpha\beta = 1$, $YX = PAP^{-1} = diag(e_1, ..., e_n)$ . So $\alpha$ is completable iff $X$ is completable. Since $\sum_{i=1}^n x_iy_i = 1$ and $y_ix_i = e_i, y_ix_j = 0$ $(i \neq j)$, so $y_ix_iy_i = y_i, x_iy_ix_i = x_i$. So $e_i = y_ix_i$ and $f_i = x_iy_i$ are idempotents. Since $R$ is an abelian ring, $e_i$, $f_i$ are in the center of $R$, $e_i = e_i^2 = y_ix_iy_ix_i = y_if_ix_i = f_ie_i$, $f_i = f_i^2 = x_iy_ix_iy_i = x_ie_iy_i = f_ie_i$, so $e_i = f_i$, i.e., $x_iy_i = y_ix_i$. So $(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i) =$

$(\sum_{i=1}^{n} y_i)(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} x_i y_i = 1$, this means $\sum_{i=1}^{n} x_i$ is an unit of $R$. Let

$$D = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Observe that $D$ is an elementary matrix, so $D$ is an invertible matrix and $X$ is completable. This implies $\alpha$ is completable. Q.E.D.

The following theorem generalized Foster's Theorem. The proof of Foster's Theorem in [3] can be generalized to the abelian ring.

**Theorem 3**. The following are equivalent for an abelian ring $R$:

(a) Each idempotent matrix over $R$ is diagonalizable under a similarity transformation.

(b) Each idempotent matrix over $R$ has a charateristic vector.

*Proof.* Suppose that we have (a) and $A$ be an $n \times n$ idempotent matrix. then there is an invertible matrix $Q$ with $QAQ^{-1} = diag(e_1, ..., e_n)$. Let $\alpha = (1, 0, ..., 0)^t$, then $\alpha$ is both unimodular and completable. Further, $QAQ^{-1}\alpha = e_1\alpha$. Set $\beta = Q^{-1}\alpha$, $\beta$ is completable. Then $A\beta = e_i\beta$ and $\beta$ is completable. Hence $A$ has a characteristic vector.

Suppose that we have (b) and $A$ be an $n \times n$ idempotent matrix. We will use a proof based on induction on $n$, the size of $A$. Assume that (a) is true for all idempotent matrices of size ¡ n. if $A = 0$, there is nothing to prove. Assume that $A \neq 0$. Let $\alpha$ be a characteristic vector of $A$. Setting $\beta_1 = \alpha$, Let $\beta_1, \beta_2, ..., \beta_n$ be a basis of $R^n$. Let $A\alpha = e_1\alpha$. Employing the basis $\beta_1, \beta_2, ..., \beta_n$ of $R^n$, the matrix $A$ has the form

$$A_1 = \begin{pmatrix} e_1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Since $A_1^2 = A_1$, we must have $e_1^2 = e_1$. Let

$$B = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

we have $B^2 = B$. By the induction hypothesis, the matrix $B$ may be diagonalized under a suitable similarity transformation. Thus by a suitable change of of basis, we may assume that we have chosen a new basis $\alpha_1,\ \alpha_2,\ ...,\alpha_n$ for $R^n$ such that, relating to this basis, $A$ has the form

$$A_2 = \begin{pmatrix} e_1 & b_2 & b_3 & \cdots & b_n \\ 0 & e_2 & 0 & \cdots & 0 \\ 0 & 0 & e_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_n \end{pmatrix}$$

Since $A_2^2 = A_2$, we have $e_1^2 = e_1$,..., $e_n^2 = e_n$, and $b_2(e_1 + e_2 - 1) = 0$,..., $b_n(e_1 + e_n - 1) = 0$. Let

$$P = \begin{pmatrix} 1 & r_2 & \cdots & r_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$P$ is invertible, we will choose suitable $r_2, r_3, ..., r_n$, such that $PA_2P^{-1} = diag(e_1, e_2, ..., e_n)$ which is satiafied if $r_i(e_i - e_1) = b_i$. Since $b_i(e_1 + e_i - 1) = 0$, so $b_i = b_i(1 - 2e_i)^2 = b_i(e_1 - e_i)(1 - 2e_i)$. So let $r_i = b_i(1 - 2e_i)$, then $PA_2P^{-1} = diag(e_1, e_2, ..., e_n)$. Q.E.D.

Next Theorem generalized Steger's Uniqueness Theorem

**Theorem 4** Let $R$ be an APT ring and $A$ be an $n \times n$ idempotent matrix over $R$. Then

(a) There is an invertible matrix $P$ with $PAP^{-1} = diag(a_1, a_2, ..., a_n)$ where $a_i$ divides $a_{i+1}$ for $1 \leq i \leq n - 1$.

(b) If $Q$ is another invertible matrix with $QAQ^{-1} = diag(b_1, b_2, ..., b_n)$ where $b_i$ divides $b_{i+1}$ for $1 \leq i \leq n - 1$, then $b_i = a_i$ for $1 \leq i \leq n$.

*Proof.* Suppose that $P$ is an invertible matrix with $PAP^{-1} = diag(e_1, ..., e_n)$, Let $a_1 = 1-(1-e_1)(1-e_2)\cdots(1-e_n)$ and $x_i = e_i + (1-e_1)(1-e_2)\cdots(1-e_n)$, then $a_1x_i = e_i$ and $I(x_1, ..., x_n) = R$, i.e., $x_1, ..., x_n$ generate $R$. By theorem 2, $X = (x_1, ..., x_n)$ is completable, so $X$ is a characteristic vector. Then in a fashion analogue to the proof of theorem 3, $A$ is similar to $diag(a_1, e_2', ..., e_n')$, by induction on $n$ (the size of the matrix), assume that $A$ is similar to $diag(a_1, a_2, ..., a_n)$ where $a_i$ divides $a_{i+1}$ for $2 \leq i \leq n - 1$. Since $a_1$ divides each entry of $diag(e_1, ..., e_n)$, and $diag(a_1, a_2, ..., a_n)$ is similar to $diag(e_1, ..., e_n)$, we have that $a_1$ divides $a_2$. This completes part (a).

To show (b), observe that $a_r$ divides the products of arbitrary $r$ entries of $diag(a_1, a_2, ..., a_n)$, so $a_r$ divides the products of arbitrary $r$ entries of $diag(b_1, b_2, ..., b_n)$. Since $b_i$ is idempotent and $b_i|b_{i+1}$, $b_r = b_1b_2\cdots b_r$, so $a_r|b_r$. Similarly, $b_r|a_r$. Since $a_r$ and $b_r$ are idempotents, we have $a_r = b_r$, $1 \leq r \leq n$. Q.E.D.

**Corollary 5** If $R$ is an APT ring, then for an arbitrary projective $R$-module $P$, there exist idempotents $e_1, e_2, ..., e_n$ of $R$ such that $e_i$ divides $e_{i+1}$ for $1 \leq i \leq n-1$ and $P \simeq Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$.

**Lemma 6** If $R$ is an APT ring, then $R^m \simeq R^m \oplus K$ implies $K = 0$.

*Proof.* If $R^m \simeq R^m \oplus K$, by Corollary 5, $K \simeq Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$ for suituable idempotents $e_1, e_2, ..., e_n$ of $R$. So

$$R^m \simeq R^m \oplus Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$$

By the uniqueness of decomposition in Theorem 4, we know that $e_i = 0$ for $1 \leq i \leq n$, so $K = 0$. Q.E.D.

**Theorem 7** If $R$ is an APT ring, then every invertible $R$-$R$-bimodule $P$ is a cyclic module.

*Proof.* Suppose $P \in Pic(R)$, then there exist idempotents $e_1, e_2, ..., e_n$ of $R$ such that $e_i$ divides $e_{i+1}$ for $1 \leq i \leq n-1$ and $P \simeq Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$. Since $P$ is an invertible $R$-$R$-bimodule, so $End_R({}_RP) \simeq R$ i.e.

$$R = End(Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n) \simeq \oplus_{i,j=1}^n Hom(Re_i, Re_j) \simeq \oplus_{i,j=1}^n R(e_ie_j)$$

. Since $e_i$ divides $e_j$ for $i \leq j$, so $e_ie_j = e_j$. Since $R \otimes_R P \simeq P$, so

$$(\oplus_{i,j=1}^n Re_ie_j) \otimes (Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n) \simeq Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$$

i.e.

$$Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n \oplus Re_2 \oplus \cdots \oplus Re_n \oplus K = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$$

for suituable projective $R$-module $K$. So $Re_2 \oplus \cdots \oplus Re_n \oplus K = 0$ and $e_2 = e_3 = \cdots = e_n = 0$. So $P \simeq Re_1$ and $P$ is a cyclic module. Q.E.D.

**Theorem 8** Let $R$ be an abelian ring, $N$ be the set of nilpotents in $R$, and $I$ be an ideal in $R$ with $I \subseteq N$. Then $R/I$ is an APT ring, if and only if $R$ is an APT ring.

*Proof.* "$\Longrightarrow$". Suppose that $R/I$ is an APT ring. Let $f : R \longrightarrow R/I$ denote the natural morphism. If $r$ is in $R$, Then $f(r)$ will be denoted by $\overline{r}$. The "bar" notation will also be used for all $n$ dimensional vectors $(R)_n$ and all $n \otimes n$ matrices $M_n(R)$.

Suppose that $A$ is an idempotent matrix in $M_n(R)$. Let $\overline{A} = f(A)$. Then $\overline{A}$ is idempotent in $R/I$. So $\overline{A}$ is similar to $diag(\overline{a}_1, \overline{a}_2, ..., \overline{a}_n)$ where $\overline{a}_i$ divides $a_{i+1}$. Since $I \subseteq N$, by 27.1 in [AF], all the idempotents in $R/I$ can be lifted modulo I. So there is an idempotent $d$ in $R$ such that $f(d) = \overline{a}_1$. By Theorem 3, $\overline{A}$ has a charateristic vector $\overline{x} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)'$ corresponding to $\overline{a}_1 = \overline{d}$. Let $x_i$ be in $R$ with $f(x_i) = \overline{x}_i$, $1 \leq i \leq n$. Set $x = (x_1, x_2, ..., x_n)'$, then since $x$ is completable to $\overline{X}$ in $GL_n(R/I)$ and $f: GL(R) \longrightarrow GL(R/I)$ is sujective, $x$ is unimodular and completable. Then $Ax = dx + r$ where $r = (r_1, r_2, ..., r_n)'$ with the $r_i$ in $I$. Since $A^2 = A$ and $d^2 = d$, $Ax = dAx + Ar$ and thus

$$Ar = (1-d)Ax = (1-d)(dx+r) = (1-d)r$$

since $(1-d)d = 0$. Thus

$$\begin{aligned} A(x + (2d-1)r) &= Ax + (2d-1)Ar = dx + r + (2d-1)(1-d)r \\ &= dx + dr = d(x + (2d-1)r). \end{aligned}$$

Further, $x + (2d-1)r \equiv \overline{x}(modI)$. Hence as above, $x + (2d-1)r$ is unimodular and completable. Thus $A$ has a characteristic vector and the proof follows from Theorem 3.

"$\Longleftarrow$" Assume that $R$ is an APT ring and $\overline{A} = (\overline{A})^2 = (\overline{a}_{ij}) \in M_n(R/I)$. It will suffice to show that there exist an idempotent matrix $F = (f_{ij}) \in M_n(R)$ such that $\overline{F} = \overline{A}$. If $A = (a_{ij})$ then $A^2 = A + B$ where the entries of $B$ are in $I$. Thus $B$ is nilpotent. let $k$be the least natural number such that $B^k = 0$. If k=1, ther is nothing left to prove. hence, assume that $k > 1$ and let $C = A + (I - 2A)B$. Then teh entries of $C - A$ are in $I$ and, Since $AB = BA = A^3 - A^2$,

$$C^2 = A^2 + 2A(I - 2A)B + (I - 2A)^2 B^2.$$

Therfore, $C^2 - C = B + (I - 2A)^2(B^2 - B)$. Since $(I - 2A)^2 = I + 4B$, $C^2 = C + B^2(4B - 3I)$. If we let $D = B^2(4B - 3I)$, we have $c^2 = C + D$ where the entries of $D$ are in $I$ and, for some natural integer $l < k$, $D^l = 0$. Repeating this process, we arrive in a finite number of steps at the required matrix $F$. Q.E.D.

**Corollary 9** Let $N$ be an ideal whose elements are nilpotent in an APT ring $R$ and let $x_1, x_2, ..., x_k$ be indeterminates. Then $R[x_1, x_2, ..., x_n]$ is an APT ring if and only if $R/N)[x_1, x_2, ..., x_k]$ is an APT ring.

sl Proof. corollary follows by observing that $N[x_1, x_2, ..., x_k]$ is the ideal of nilpotents in $R[x_1, x_2, ..., x_k]$ and that

$$R[x_1, x_2, ..., x_k]/N[x_1, x_2, ..., x_k] \simeq (R/N)[x_1, x_2, ..., x_k].$$

Q.E.D.

**Theorem 10** Let $R$ be an abelian regular ring, and for any finitely generated projective $R$-modules $A$ and $B$, $2R \oplus A \simeq R \oplus B$ implies $R \oplus A \simeq B$, then $R$ is an APT ring.

*Proof.* By Theorem 2.5 in [AG], every square matrix over $R$ admits a diagonal reduction (i.e., there exist invertible matrix $P$ and $Q$ such that $PAQ$ is a diagonal matrix). Suppose $A$ is an idempotent matrix, by Theorem 1, $A$ is similar to a diagonal matrix whose diagonal entries are idempotents of $R$. So $R$ is an APT ring Q.E.D.

## References

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