ON K_2 OF DIVISION ALGEBRAS

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Abstract: In this paper, it is proved that if F is a global field, then for any integer $n > 3$, there is an extension field E over F of degree n such that K_2E is not generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$. However if F is a number field and D is a finite dimensional central division F-algebra with square free index, then K_2D is always generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in D^*$. Finally the tame kernels of central division algebras over F are expressed explicitly.

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1. Introduction

Let E/F be a field extension of degree n. If $n \leq 3$, it is well known that K_2E is generated by symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$. The proof is elementary and can be found in [7] and [13]. A very explicit version of this proposition can be found in [20] (Lemma 12.1). If $n > 3$, then in general K_2E is not generated by symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$. A counterexample can be found in Proposition 3.1 of [2].

Let F be a global field. In section 2, we give a very simple counterexample. In fact, we prove (Proposition 2.3) that for any integer $n > 3$ and any global field F, there is an extension field E over F of degree n such that K_2E is not generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$.

Although K_2E is not generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$ in general, these symbols generate a subgroup of finite index (Proposition 2.4).

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In section 3, we consider the noncommutative version of this problem in case F is a number field or a local field. Let D be a finite dimensional central division F-algebra. If F is a number field and the index of D is square free, then K_2D is generated by symbols of the form $\{a,\ b\}$ with $a \in F^*,\ b \in D^*$ (Theorem 3.1). If F is a local field and the characteristic of F is prime to the index of D or the index of D is square free, then K_2D is generated by symbols of the form $\{a,\ b\}$ with $a \in F^*,\ b \in D^*$ (Theorem 3.1). The proofs depend on Merkurjev and Suslin's results on injectivity of reduced norm for K_2.

Next, we study the tame kernels of central division algebras over number fields. Let D be a central division algebra over a number field F and Λ a maximal R-order in D, where R is the ring of integers of F. It is proved (Theorem 3.4) that

$$K_2\Lambda \simeq K_2^+R = \ker(K_2R \longrightarrow \bigoplus_{\text{real ramified } \wp} \{\pm 1\}\ ).$$

Lenstra proved in [5] that every element of K_2F is simply a symbol if F is a global field. At the end of this paper, we prove that every element of K_2D is a symbol of form $\{a,\ b\}$ with $a \in F^*,\ b \in D^*$ if F is a number field and the index of D is square free.

2. Examples of K_2E not generated by $\{F^*,\ E^*\}$.

Proposition 2.1. *Let $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{-1},\ \sqrt{-3})$. Then K_2E can not be generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$.*

Proof. Let $\mathcal{O}_E$ be the ring of integers of E. Since 3 is ramified in $\mathbb{Q}(\sqrt{-3})$ and inert in $\mathbb{Q}(\sqrt{-1})$, we know that $3\mathcal{O}_E = \mathcal{P}^2$, where $\mathcal{P}$ is a prime ideal of $\mathcal{O}_E$ satisfying $[\mathcal{O}_E/\mathcal{P} : \mathbb{Z}/3\mathbb{Z}] = 2$. Considering the tame mapping at $\mathcal{P}$

$$\partial_{\mathcal{P}} :\ K_2E \longrightarrow (\mathcal{O}_E/\mathcal{P})^*, \qquad \partial_{\mathcal{P}}(\{a,\ b\}) = (-1)^{v(a)v(b)}\frac{a^{v(b)}}{b^{v(a)}} \pmod{\mathcal{P}},$$

obviously $\partial_{\mathcal{P}}$ is surjective. Hence the images of the elements $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$ would generate $(\mathcal{O}_E/\mathcal{P})^*$ if K_2E can be generated by symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$. Suppose $(\mathcal{O}_E/\mathcal{P})^*$ is generated by g. Let $E_{\mathcal{P}}$ be the $\mathcal{P}$-adic completion of E. For any $a \in F^*$, assume that $a = u_a\pi^{v_{\mathcal{P}}(a)}$, where π is the uniformizer of $E_{\mathcal{P}}$. Then the image $\overline{u_a}$ of u_a in $(\mathcal{O}_E/\mathcal{P})^*$ belongs to $(\mathbb{Z}/3\mathbb{Z})^* \subset (\mathcal{O}_E/\mathcal{P})^*$. But $(\mathbb{Z}/3\mathbb{Z})^*$ is generated by g^4, so $\bar{a}$ is a square in $(\mathcal{O}_E/\mathcal{P})^*$. Since $e(\mathcal{P}/3) = f(\mathcal{P}/3) = 2$, we have $2|v_{\mathcal{P}}(a)$ for all $a \in F^*$. So $(-1)^{v(a)v(b)}a^{v(b)}/b^{v(a)}$ is always a square in $(\mathcal{O}_E/\mathcal{P})^*$. So K_2E can not be generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$. □

Proposition 2.2. *Let F be a local field with odd residue characteristic, E be a field extension of F of finite degree. Let $\mathcal{P}$ be the maximal ideal of the integers ring of E. Suppose that the residue class field index and the ramification index are all even. Then* *the images of the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$ can not generate $k(E)^*$ under the tame mapping $\partial_{\mathcal{P}}$, where $k(E)$ is the residue class field of E. So K_2E can not be generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$.*

Proof. By the same arguments as in the above proposition, it can be proved that $\partial_{\mathcal{P}}(\{a,\ b\})$ is always a square in $k(E)^*$. Since $\partial_{\mathcal{P}}:\ K_2E \longrightarrow k(E)^*$ is surjective and $k(E)^{*2} \neq k(E)^*$, we know K_2E is not generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$. □

Proposition 2.3. *Let F be a global field. Then for any integer $n > 3$, there is an extension field E over F of degree n such that K_2E is not generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$.*

Proof. By the above proposition and the fact that the tame mapping in Quillen's localization sequence is surjective, it suffices to find a field E with $[E : F] = n$ such that the residue class field index and the ramification index are all even at some prime ideal $\mathcal{P}$ of $\mathcal{O}_E$. Let $\wp = \mathcal{P} \cap \mathcal{O}_F$. Finding such a field E is equivalent to finding an irreducible polynomial $h(x) \in F[x]$ of degree n such that $h(x) = h_1(x) \cdots h_r(x)$, where irreducible polynomial $h_i(x) \in F_\wp[x]$ and the residue class field index and the ramification index of $F_\wp[x]/h_1(x)$ over $F_\wp$ are both even, where we have used $h_1(x)$ without loss of generality.

First, we can find a polynomial $f(x) = f_1(x) \cdots f_r(x)$ of degree n without multiple roots, where the irreducible polynomial $f_i(x) \in F_\wp[x]$ for all $1 \le i \le r$, such that the residue class field index and the ramification index of $F_\wp[x]/f_1(x)$ over $F_\wp$ are all even.

Second, we can find a global polynomial $g(x) \in F[x]$ of degree n which is very close to $f(x) \in F_\wp[x]$ in the $\wp$-adic topology. This is due to the Weak Approximation Theorem. Since $g(x)$ is very close to $f(x)$, we know that every root of $g(x)$ is very close to some root of $f(x)$. By Krasner's Lemma, $f(x)$ and $g(x)$ must have the same decomposition form in $F_\wp[x]$. So we get a global polynomial $g(x)$ such that

$$F_\wp[x]/f(x) \simeq F_\wp[x]/g(x).$$

Note that $f(x)$ may not be irreducible. So we need to find another global polynomial $h(x)$ such that $h(x)$ is irreducible and $h(x)$ is very close to $f(x)$ in $F_\wp[x]$. By the Density Theorem, we can find a global prime element q such that the norm of q is sufficiently large and q is close enough to 1 in $F_\wp$. Assume

$$g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

Let

$$h(x) = x^n + qa_{n-1}x^{n-1} + \cdots + qa_1x + qa_0.$$

Then $h(x)$ is an Eisenstein polynomial. So $h(x)$ is irreducible and $h(x)$ is very close to $g(x)$ in $F_\wp[x]$. In $F_\wp[x]$, $h(x) = h_1(x) \cdots h_r(x)$, $h_i(x) \in F_\wp[x]$ and the residue class field index and the ramification index of $F_\wp[x]/h_1(x)$ over $F_\wp$ are all even. Let $E = F[x]/h(x)$, then K_2E is not generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$. □

Proposition 2.4. *Let F be a global field. Then for any extension field E over F of degree n, the subgroup H of K_2E generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*,\ b \in E^*$ is of finite index.*

Proof. Let S_∞ denote the set of archimedean places of E. If S is a non empty set of places containing S_∞. we put $\mathcal{O}_S = \{a \in E|\ v(a) \geq 0,\ for\ all\ v \notin S\}$ be the ring of S-integers. We shall put $K_2^S E$ the subgroup of K_2E generated by $\{x,\ y\}$, where $x,\ y \in \mathcal{O}_S^*$. We can list the the finite places of E, $v_1,\ v_2,\ ...,\ v_n,\ ...$ so that $N(v_i) \leq N(v_{i+1})$ for all i. Put $S_m = S_\infty \cup \{v_1,\ ...,\ v_m\}$. Let $S = S_m$, $v = v_{m+1} \notin S$, $S' = S_{m+1} = S \cup \{v\}$, $U = \mathcal{O}_S^*$. We can find an integer N_0 such that for all v satisfying $N(v) > N_0$, the following conditions hold(cf [19] Proposition 1)

(1) v is unramified,

(2) the natural quotient homomorphism $U \longrightarrow k_v^*$ is surjective,

(3) $\partial_v :\ \ K_2^{S'}E/K_2^S E \longrightarrow k_v^*$ is an isomorphism.

For such places, we have

$$\partial_v :\ H \cap K_2^{S'}E/H \cap K_2^S E \longrightarrow k_v^*, \qquad \{a,\ b\} \mapsto (-1)^{v(a)v(b)}\frac{a^{v(b)}}{b^{v(a)}} \mod v$$

is an isomorphism. So

$$(\partial_v) :\ H/H \cap K_2^S E \longrightarrow \bigoplus_{N(v)>N_0} k_v^*$$

is an isomorphism which implies $HK_2^S E = K_2E$. Since $K_2^S E$ is a finite group, we know H is a subgroup of K_2E of finite index. □

3. K_2D is generated by symbols of form $\{a,\ b\}$ with $a \in F^*,\ b \in D^*$

Let D be a finite dimensional central division algebra over F. We have proved in this section that K_2D is generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*$ and $b \in D^*$.

At first, let us recall the definition of reduced norm. A reduced norm homomorphism

$$\mathrm{Nrd}_2 : \qquad K_2D \longrightarrow K_2F$$

is a homomorphism such that the following diagram commutes for any subfield L in D containing F such that $[L : F]$ is equal to the index of D.

$$\begin{array}{ccc} K_2L & \xrightarrow{\ i\ } & K_2D \\ & {\scriptstyle N_{L/F}} \searrow \quad \swarrow {\scriptstyle \mathrm{Nrd}_2} & \\ & K_2F & \end{array}$$

The existence of Nrd_2 can be deduced from the Brown-Gersten-Quillen Spectral Sequence(cf [6] for details).

We now recall some basic properties of the K_2 group of central division algebras over global field F. Let R be a Dedekind domain whose field of quotients is F. For each prime $\wp$ of R, let $k(\wp)$ be the residue class field. Let D be a central division algebra over F and let $\mathcal{O}$ be an maximal R-order in D. Each prime of R extends to a unique prime ideal of $\mathcal{O}$ and the corresponding residue ring is a full matrix ring over a finite extension $d(\wp)$ of $k(\wp)$([12], IV, Theorem 5.9). By Quillen's localization sequence, we have the following exact sequence:

$$1 \longrightarrow K_2(\Lambda) \longrightarrow K_2(D) \xrightarrow{\partial_D} \bigoplus_{\text{finite } \wp} K_1(d(\wp)) \longrightarrow K_1(\Lambda) \longrightarrow K_1(D)$$

It is well known that K_2D is generated by the Steinberg symbols $\{\alpha,\ \beta\}$ with $\alpha,\ \beta \in D^*$ (cf Theorem 9.11 and Theorem 9.12 of [10] or Theorem 138 of [15]).

Theorem 3.1. *Let D be a division algebra over a field F. Then K_2D is generated by the Steinberg symbols $\{a,\ d\}$ with $a \in F^*$ and $d \in D^*$ in the following cases;*

(1) F is a number field and the index of D is square free;

(2) F is a non-archimedean local field and the index of D is square free;

(3) F is a non-archimedean local field and the character of the residue field is prime to the index of D;

Proof. In these four cases, the reduced norm $\mathrm{Nrd}_2 : \ K_2D \longrightarrow K_2F$ is injective (cf Proposition 26.6, Theorem 26.7 of [16]; [9] and [17]; Theorem 3 of [18]; [8] and [14]).

First let F be a global field. Let K_2^+F be the subgroup of K_2F generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*$ and $b \in F^+ = \{b \in F | v(b) > 0\}$ for all real places v such that D is ramified at $v\}$. Since every element of F^+ is a norm of some element of D^*, K_2^+F is generated by the Steinberg symbols $\{a,\ n(d)\}$ with $a \in F^*$ and $d \in D^*$. By Theorem 1 of [1] and the Theorem 2.2 of [3], the image of the reduced norm Nrd_2 is K_2^+F which is generated by $\{a,\ b\}$ with $a \in F^*$ and $b \in F^+$.

Since the reduced norm Nrd_2 is injective, we have K_2D is generated by the Steinberg symbols $\{a,\ d)\}$ with $a \in F^*$ and $d \in D^*$.

If F is a non archimedean local field, then D contains a maximal subfield which is unramified over F (Proposition 17.7 of [11]). So the norm homomorphism from D to F is surjective. Obviously K_2D is generated by the Steinberg symbols $\{a,\ d\}$ with $a \in F^*$ and $d \in D^*$ by the injectivity of reduced norm.

$\square$

Keating had proved in [4] that

$$\mathrm{Image}(K_2(D) \xrightarrow{\partial_D} \bigoplus_{\text{finite } \wp} d(\wp)^*) = \bigoplus_{\text{finite } \wp} k(\wp)^*,$$

where ∂_D is the tame mapping in Quillen's localization sequence.

Lemma 3.2. *Let D be a finite dimensional central division algebra over a number field F. If the index of D is square free, then the following diagram commutes,*

$$\begin{array}{ccc} K_2D & & \\ {\scriptstyle Nrd_2}\downarrow & \searrow{\scriptstyle \partial_D} & \\ K_2F & \xrightarrow[\partial_F]{} & \bigoplus_{\mathit{finite}\ \wp} (k(\wp))^* \end{array}$$

where Nrd_2 is the reduced norm of K_2 groups.

Proof. Since K_2D is generated by the Steiberg symbols $\{a,\ b\}$ with $a \in F^*$, $b \in D^*$. It suffices to prove that for any $x = \{a,\ b\}$ with $a \in F^*$, $b \in D^*$,

$$\partial_D(x) = \partial_F \cdot \mathrm{Nrd}_2(x).$$

Let L be the maximal subfield of D such that $b \in L$. Then we have the following commutative diagram([4], § 4)

$$\begin{array}{ccc} K_2L & \xrightarrow{\partial_L} & \oplus\, (l(\mathcal{P}))^* \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \psi} \\ K_2D & \xrightarrow[\partial_D]{} & \oplus\, K_1(d(\wp)) \end{array}$$

where i is induced by the ring inclusion $L \longrightarrow D$,

$$\psi = \bigoplus_{\mathcal{P},\wp} \phi(\mathcal{P}, \wp)$$

and

$$\phi(\mathcal{P}, \wp) :\ (l(\mathcal{P}))^* \longrightarrow K_1(d(\wp))$$

is given by

$$\phi(\mathcal{P}, \wp)(u) = Norm(u)$$

for $u \in l(\mathcal{P})^*$, where the $Norm$ is the natural norm mapping $l(\mathcal{P})^* \longrightarrow k(\wp)^*$, the image of ψ is $\bigoplus_{\text{finite } \wp} (k(\wp))^*$. So

$$\partial_D(x) = \partial_D \cdot i(\{a,\ b\}) = \psi \cdot \partial_L(\{a,\ b\}).$$

By the following commutative diagram

$$\begin{array}{ccc} K_2L & \xrightarrow{\partial_L} & \oplus\,(l(\mathcal{P}))^* \\ {\scriptstyle Norm}\downarrow & & \downarrow{\scriptstyle \psi} \\ K_2F & \xrightarrow[\partial_F]{} & \oplus\,(k(\wp))^*, \end{array}$$

We have

$$\psi \cdot \partial_L(\{a,\ b\}) = \partial_F \cdot Norm(\{a,\ b\}).$$

So it is sufficient to prove that

$$\partial_F \cdot \mathrm{Nrd}_2(x) = \partial_F \cdot Norm(\{a,\ b\}).$$

Note that

$$\mathrm{Nrd}_2(x) = \mathrm{Nrd}_2 \cdot i(\{a,\ b\}).$$

So we need only to prove

$$\mathrm{Nrd}_2 \cdot i(\{a,\ b\}) = Norm(\{a,\ b\}).$$

By the definition of the reduced norm Nrd_2, this equality obviously holds. □

Theorem 3.3. *Let F be a number field and R the ring of integers in F. Let D be a finite dimensional central division F-algebra with square free index and Λ a maximal R-order of D. Then*

$$K_2\Lambda \simeq K_2^+R = \ker(K_2R \longrightarrow \bigoplus_{\textit{real ramified } \wp} \{\pm 1\}\ ).$$

Proof. Due to Lemma 3.2, we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(\Lambda) & \longrightarrow & K_2(D) & \xrightarrow{\partial_D} & \bigoplus_{\text{finite } \wp} k(\wp)^* & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Nrd}_2} & & \downarrow{\scriptstyle \text{identity}} & & \\ 1 & \longrightarrow & K_2R & \longrightarrow & K_2F & \xrightarrow[\partial_F]{} & \bigoplus_{\text{finite } \wp} k(\wp)^* & \longrightarrow & 1 \end{array}$$

where f is induced by the reduced norm Nrd_2. By Theorem 1 of [1], the image of Nrd_2 is equal to

$$\ker(K_2F \longrightarrow \bigoplus_{\text{real ramified } \wp} \{\pm 1\}\ )).$$

So $f: K_2\Lambda \longrightarrow K_2^+R$ is surjective which implies $K_2\Lambda \simeq K_2^+R$. □

Proposition 3.4. *Let F be a number field and D a central division algebra over F with square free index. then $K_2D = \{\{a, b\} | a \in F^*, b \in D^*\}$, i.e., every element of K_2D is a symbol of form $\{a, b\}$ for $a \in F^*, b \in D^*$.*

Proof. By the proof of the Theorem of [5], every element of K_2F is of form $\{a, b\}$ where a is a totally negative element, i.e., a is negative at all real places. So for any element $x \in K_2D$, its image $\mathrm{Nrd}_2(x)$ in K_2F is of the form $\{a, b\}$ for some totally negative a . Since the image of K_2D under the reduced norm is K_2^+F, b must be totally real. By Eichler's Norm Theorem, b is a norm of some $h \in D^*$. So $\mathrm{Nrd}_2(x) = \mathrm{Nrd}_2(\{a, h\})$. By Theorem 3.4, Nrd_2 is injective, so $x = \{a, h\}$. □

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