ON $K_2$ OF DIVISION ALGEBRAS

XUEJUN GUO\textsuperscript{1} \hspace{1cm} ADEREMI KUKU\textsuperscript{2} \hspace{1cm} HOURONG QIN\textsuperscript{3}

\textsuperscript{1}Department of Mathematics, Nanjing University
\textsuperscript{2}The Abdus Salam International Center for Theoretical Physics, Trieste, Italy
\textsuperscript{3}Department of Mathematics, Nanjing University

Abstract: In this paper, it is proved that if $F$ is a global field, then for any integer $n > 3$, there is an extension field $E$ over $F$ of degree $n$ such that $K_2E$ is not generated by the Steinberg symbols \{a, b\} with $a \in F^*$, $b \in E^*$. However if $F$ is a number field and $D$ is a finite dimensional central division $F$-algebra with square free index, then $K_2D$ is always generated by the Steinberg symbols \{a, b\} with $a \in F^*$, $b \in D^*$. Finally the tame kernels of central division algebras over $F$ are expressed explicitly.

Keywords: $K_2$ group, central division algebra, reduced norm.


1. Introduction

Let $E/F$ be a field extension of degree $n$. If $n \leq 3$, it is well known that $K_2E$ is generated by symbols \{a, b\} with $a \in F^*$, $b \in E^*$. The proof is elementary and can be found in [7] and [13]. A very explicit version of this proposition can be found in [20] (Lemma 12.1). If $n > 3$, then in general $K_2E$ is not generated by symbols \{a, b\} with $a \in F^*$, $b \in E^*$. A counterexample can be found in Proposition 3.1 of [2].

Let $F$ be a global field. In section 2, we give a very simple counterexample. In fact, we prove (Proposition 2.3) that for any integer $n > 3$ and any global field $F$, there is an extension field $E$ over $F$ of degree $n$ such that $K_2E$ is not generated by the Steinberg symbols \{a, b\} with $a \in F^*$, $b \in E^*$.

Although $K_2E$ is not generated by the Steinberg symbols \{a, b\} with $a \in F^*$, $b \in E^*$ in general, these symbols generate a subgroup of finite index (Proposition 2.4).
In section 3, we consider the noncommutative version of this problem in case $F$ is a number field or a local field. Let $D$ be a finite dimensional central division $F$-algebra. If $F$ is a number field and the index of $D$ is square free, then $K_2D$ is generated by symbols of the form $\{a, b\}$ with $a \in F^*$, $b \in D^*$ (Theorem 3.1). If $F$ is a local field and the characteristic of $F$ is prime to the index of $D$ or the index of $D$ is square free, then $K_2D$ is generated by symbols of the form $\{a, b\}$ with $a \in F^*$, $b \in D^*$ (Theorem 3.1). The proofs depend on Merkurjev and Suslin’s results on injectivity of reduced norm for $K_2$.

Next, we study the tame kernels of central division algebras over number fields. Let $D$ be a finite dimensional central division $F$-algebra over a number field $F$ and $\Lambda$ a maximal $R$-order in $D$, where $R$ is the ring of integers of $F$. It is proved (Theorem 3.4) that

$$K_2\Lambda \simeq K_2^+R = \ker(K_2R \longrightarrow \bigoplus_{\text{real ramified } \wp} \{\pm 1\}).$$

Lenstra proved in [5] that every element of $K_2F$ is simply a symbol if $F$ is a global field. At the end of this paper, we prove that every element of $K_2D$ is a symbol of form $\{a, b\}$ with $a \in F^*$, $b \in D^*$ if $F$ is a number field and the index of $D$ is square free.

2. Examples of $K_2E$ not generated by $\{F^*, E^*\}$.

**Proposition 2.1.** Let $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{-1}, \sqrt{-3})$. Then $K_2E$ can not be generated by the Steinberg symbols $\{a, b\}$ with $a \in F^*$, $b \in E^*$.

**Proof.** Let $O_E$ be the ring of integers of $E$. Since 3 is ramified in $\mathbb{Q}(\sqrt{-3})$ and inert in $\mathbb{Q}(\sqrt{-1})$, we know that $3O_E = \mathcal{P}^2$, where $\mathcal{P}$ is a prime ideal of $O_E$ satisfying $|O_E/\mathcal{P} : \mathbb{Z}/3\mathbb{Z}| = 2$. Considering the tame mapping at $\mathcal{P}$

$$\partial_{\mathcal{P}} : K_2E \longrightarrow (O_E/\mathcal{P})^*, \quad \partial_{\mathcal{P}}(\{a, b\}) = (-1)^{\nu(a)\nu(b)} \frac{a^{\nu(b)}}{b^{\nu(a)}} \pmod{\mathcal{P}},$$

obviously $\partial_{\mathcal{P}}$ is surjective. Hence the images of the elements $\{a, b\}$ with $a \in F^*$, $b \in E^*$ would generate $(O_E/\mathcal{P})^*$ if $K_2E$ can be generated by symbols $\{a, b\}$ with $a \in F^*$, $b \in E^*$. Suppose $(O_E/\mathcal{P})^*$ is generated by $g$. Let $E_\mathcal{P}$ be the $\mathcal{P}$-adic completion of $E$. For any $a \in F^*$, assume that $a = u_\pi^{\nu_\mathcal{P}(a)}$, where $\pi$ is the uniformizer of $E_\mathcal{P}$. Then the image $\pi a$ in $(O_E/\mathcal{P})^*$ belongs to $(\mathbb{Z}/3\mathbb{Z})^* \subset (O_E/\mathcal{P})^*$. But $(\mathbb{Z}/3\mathbb{Z})^*$ is generated by $g^4$, so $\pi$ is a square in $(O_E/\mathcal{P})^*$. Since $e(\mathcal{P}/3) = f(\mathcal{P}/3) = 2$, we have $2|\nu_\mathcal{P}(a)$ for all $a \in F^*$. So $(-1)^{\nu(a)\nu(b)} a^{\nu(b)}/b^{\nu(a)}$ is always a square in $(O_E/\mathcal{P})^*$. So $K_2E$ can not be generated by the Steinberg symbols $\{a, b\}$ with $a \in F^*$, $b \in E^*$.

**Proposition 2.2.** Let $F$ be a local field with odd residue characteristic, $E$ be a field extension of $F$ of finite degree. Let $\mathcal{P}$ be the maximal ideal of the integers ring of $E$. Suppose that the residue class field index and the ramification index are all even. Then
the images of the Steinberg symbols \{a, b\} with \(a \in F^*, \ b \in E^*\) can not generate \(k(E)^*\) under the tame mapping \(\partial_P\), where \(k(E)\) is the residue class field of \(E\). So \(K_2E\) can not be generated by the Steinberg symbols \{a, b\} with \(a \in F^*, \ b \in E^*\).

**Proof.** By the same arguments as in the above proposition, it can be proved that \(\partial_P(\{a, b\})\) is always a square in \(k(E)^*\). Since \(\partial_P: K_2E \rightarrow k(E)^*\) is surjective and \(k(E)^2 \neq k(E)^*\), we know \(K_2E\) is not generated by the Steinberg symbols \{a, b\} with \(a \in F^*, \ b \in E^*\). \(\square\)

**Proposition 2.3.** Let \(F\) be a global field. Then for any integer \(n > 3\), there is an extension field \(E\) over \(F\) of degree \(n\) such that \(K_2E\) is not generated by the Steinberg symbols \{a, b\} with \(a \in F^*, \ b \in E^*\).

**Proof.** By the above proposition and the fact that the tame mapping in Quillen's localization sequence is surjective, it suffices to find a field \(E\) with \([E : F] = n\) such that the residue class field index and the ramification index are all even at some prime ideal \(\mathcal{P}\) of \(\mathcal{O}_E\). Let \(\varphi = \mathcal{P} \cap \mathcal{O}_F\). Finding such a field \(E\) is equivalent to finding an irreducible polynomial \(h(x) \in F[x]\) of degree \(n\) such that \(h(x) = h_1(x) \cdots h_r(x)\), where irreducible polynomial \(h_i(x) \in F_\varphi[x]\) and the residue class field index and the ramification index of \(F_\varphi[x]/h_1(x)\) over \(F_\varphi\) are both even, where we have used \(h_1(x)\) without loss of generality.

First, we can find a polynomial \(f(x) = f_1(x) \cdots f_r(x)\) of degree \(n\) without multiple roots, where the irreducible polynomial \(f_i(x) \in F_\varphi[x]\) for all \(1 \leq i \leq r\), such that the residue class field index and the ramification index of \(F_\varphi[x]/f_1(x)\) over \(F_\varphi\) are all even.

Second, we can find a global polynomial \(g(x) \in F[x]\) of degree \(n\) which is very close to \(f(x) \in F_\varphi[x]\) in the \(\varphi\)-adic topology. This is due to the Weak Approximation Theorem. Since \(g(x)\) is very close to \(f(x)\), we know that every root of \(g(x)\) is very close to some root of \(f(x)\). By Krasner's Lemma, \(f(x)\) and \(g(x)\) must have the same decomposition form in \(F_\varphi[x]\). So we get a global polynomial \(g(x)\) such that

\[ F_\varphi[x]/f(x) \simeq F_\varphi[x]/g(x). \]

Note that \(f(x)\) may not be irreducible. So we need to find another global polynomial \(h(x)\) such that \(h(x)\) is irreducible and \(h(x)\) is very close to \(f(x)\) in \(F_\varphi[x]\). By the Density Theorem, we can find a global prime element \(q\) such that the norm of \(q\) is sufficiently large and \(q\) is close enough to 1 in \(F_\varphi\). Assume

\[ g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0. \]

Let

\[ h(x) = x^n + qa_{n-1}x^{n-1} + \cdots + qa_1x + qa_0. \]

Then \(h(x)\) is an Eisenstein polynomial. So \(h(x)\) is irreducible and \(h(x)\) is very close to \(g(x)\) in \(F_\varphi[x]\). In \(F_\varphi[x]\), \(h(x) = h_1(x) \cdots h_r(x), h_1(x) \in F_\varphi[x]\) and the residue
class field index and the ramification index of \( F_P[x]/h_1(x) \) over \( F_P \) are all even. Let \( E = F[x]/h(x) \), then \( K_2E \) is not generated by the Steinberg symbols \( \{ a, b \} \) with \( a \in F^*, b \in E^* \).

**Proposition 2.4.** Let \( F \) be a global field. Then for any extension field \( E \) over \( F \) of degree \( n \), the subgroup \( H \) of \( K_2E \) generated by the Steinberg symbols \( \{ a, b \} \) with \( a \in F^*, b \in E^* \) is of finite index.

**Proof.** Let \( S_\infty \) denote the set of archimedean places of \( E \). If \( S \) is a non empty set of places containing \( S_\infty \), we put \( \mathcal{O}_S = \{ a \in E \mid v(a) \geq 0, \text{ for all } v \notin S \} \) be the ring of \( S \)-integers. We shall put \( K_2^SE \) the subgroup of \( K_2E \) generated by \( \{ x, y \} \), where \( x, y \in \mathcal{O}_S^* \). We can list the the finite places of \( E \), \( v_1, v_2, \ldots, v_n, \ldots \) so that \( N(v_i) \leq N(v_{i+1}) \) for all \( i \). Let \( S_m = S_\infty \cup \{ v_1, \ldots, v_m \} \). Let \( S = S_m, v = v_{m+1} \notin S, S' = S_m+1 = S \cup \{ v \} \), \( U = \mathcal{O}_S^* \). We can find an integer \( N_0 \) such that for all \( v \) satisfying \( N(v) > N_0 \), the following conditions hold(cf [19] Proposition 1)

1. \( v \) is unramified,
2. the natural quotient homomorphism \( U \longrightarrow k_v^* \) is surjective,
3. \( \partial_v : K_2^S/E/K_2^S \longrightarrow k_v^* \) is an isomorphism.

For such places, we have

\[
\partial_v : H \cap K_2^S/E/H \cap K_2^S \longrightarrow k_v^*, \quad \{ a, b \} \mapsto (-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}} \mod v
\]

is an isomorphism. So

\[
(\partial_v) : H/H \cap K_2^S \longrightarrow \bigoplus_{N(v) > N_0} k_v^*
\]

is an isomorphism which implies \( HK_2^S = K_2E \). Since \( K_2^S \) is a finite group, we know \( H \) is a subgroup of \( K_2E \) of finite index.

**3. \( K_2D \) is generated by symbols of form \( \{ a, b \} \) with \( a \in F^*, b \in D^* \)**

Let \( D \) be a finite dimensional central division algebra over \( F \). We have proved in this section that \( K_2D \) is generated by the Steinberg symbols \( \{ a, b \} \) with \( a \in F^* \) and \( b \in D^* \).

At first, let us recall the definition of reduced norm. A reduced norm homomorphism

\[
\text{Nrd}_2 : K_2D \longrightarrow K_2F
\]
is a homomorphism such that the following diagram commutes for any subfield \( L \) in \( D \) containing \( F \) such that \( [L : F] \) is equal to the index of \( D \).

\[
\begin{array}{ccc}
K_2L & \xrightarrow{i} & K_2D \\
\downarrow{N_{L/F}} & & \downarrow{\text{Nrd}_2} \\
K_2F & & 
\end{array}
\]

The existence of \( \text{Nrd}_2 \) can be deduced from the Brown-Gersten-Quillen Spectral Sequence (cf [6] for details).

We now recall some basic properties of the \( K_2 \) group of central division algebras over global field \( F \). Let \( R \) be a Dedekind domain whose field of quotients is \( F \). For each prime \( \mathfrak{p} \) of \( R \), let \( k(\mathfrak{p}) \) be the residue class field. Let \( D \) be a central division algebra over \( F \) and let \( O \) be a maximal \( R \)-order in \( D \). Each prime of \( R \) extends to a unique prime ideal of \( O \) and the corresponding residue ring is a full matrix ring over a finite extension \( d(\mathfrak{p}) \) of \( k(\mathfrak{p})([12], \text{IV, Theorem 5.9}) \). By Quillen’s localization sequence, we have the following exact sequence:

\[
1 \rightarrow K_2(\Lambda) \rightarrow K_2(D) \xrightarrow{\partial_D} \bigoplus_{\text{finite } \mathfrak{p}} K_1(d(\mathfrak{p})) \rightarrow K_1(\Lambda) \rightarrow K_1(D)
\]

It is well known that \( K_2D \) is generated by the Steinberg symbols \( \{\alpha, \beta\} \) with \( \alpha, \beta \in D^* \) (cf Theorem 9.11 and Theorem 9.12 of [10] or Theorem 138 of [15]).

**Theorem 3.1.** Let \( D \) be a division algebra over a field \( F \). Then \( K_2D \) is generated by the Steinberg symbols \( \{a, d\} \) with \( a \in F^* \) and \( d \in D^* \) in the following cases:

1. \( F \) is a number field and the index of \( D \) is square free;
2. \( F \) is a non-archimedean local field and the index of \( D \) is square free;
3. \( F \) is a non-archimedean local field and the character of the residue field is prime to the index of \( D \);

**Proof.** In these four cases, the reduced norm \( \text{Nrd}_2 : K_2D \rightarrow K_2F \) is injective (cf Proposition 26.6, Theorem 26.7 of [16]; [9] and [17]; Theorem 3 of [18]; [8] and [14]).

First let \( F \) be a global field. Let \( K_2^+F \) be the subgroup of \( K_2F \) generated by the Steinberg symbols \( \{a, b\} \) with \( a \in F^* \) and \( b \in F^+ = \{b \in F | v(b) > 0\} \) for all real places \( v \) such that \( D \) is ramified at \( v \). Since every element of \( F^+ \) is a norm of some element of \( D^* \), \( K_2^+F \) is generated by the Steinberg symbols \( \{a, n(d)\} \) with \( a \in F^* \) and \( d \in D^* \).

By Theorem 1 of [1] and the Theorem 2.2 of [3], the image of the reduced norm \( \text{Nrd}_2 \) is \( K_2^+F \) which is generated by \( \{a, b\} \) with \( a \in F^* \) and \( b \in F^+ \).

Since the reduced norm \( \text{Nrd}_2 \) is injective, we have \( K_2D \) is generated by the Steinberg symbols \( \{a, d\} \) with \( a \in F^* \) and \( d \in D^* \).
If $F$ is a non archimedean local field, then $D$ contains a maximal subfield which is unramified over $F$ (Proposition 17.7 of [11]). So the norm homomorphism from $D$ to $F$ is surjective. Obviously $K_2D$ is generated by the Steinberg symbols \{a, d\} with $a \in F^*$ and $d \in D^*$ by the injectivity of reduced norm.

\[\square\]

Keating had proved in [4] that

$$\text{Image}(K_2(D)) \xrightarrow{\partial_D} \bigoplus_{\text{finite } \wp} d(\wp)^* = \bigoplus_{\text{finite } \wp} k(\wp)^*,\$$

where $\partial_D$ is the tame mapping in Quillen’s localization sequence.

**Lemma 3.2.** Let $D$ be a finite dimensional central division algebra over a number field $F$. If the index of $D$ is square free, then the following diagram commutes,

\[
\begin{array}{ccc}
K_2D & \xrightarrow{\partial_D} & K_2F \\
\downarrow \text{Nr}d_2 & & \downarrow \text{Nr}d_2 \\
K_2F & \xrightarrow{\partial_F} & \bigoplus_{\text{finite } \wp} (k(\wp))^* \\
\end{array}
\]

where $\text{Nr}d_2$ is the reduced norm of $K_2$ groups.

**Proof.** Since $K_2D$ is generated by the Steinberg symbols \{a, b\} with $a \in F^*$, $b \in D^*$. It suffices to prove that for any $x = \{a, b\}$ with $a \in F^*$, $b \in D^*$,

$$\partial_D(x) = \partial_F \cdot \text{Nr}d_2(x).$$

Let $L$ be the maximal subfield of $D$ such that $b \in L$. Then we have the following commutative diagram([4], § 4)

\[
\begin{array}{ccc}
K_2L & \xrightarrow{\partial_L} & \bigoplus (l(\mathcal{P}))^* \\
\downarrow i & & \downarrow \psi \\
K_2D & \xrightarrow{\partial_D} & \bigoplus K_1(d(\wp)) \\
\end{array}
\]

where $i$ is induced by the ring inclusion $L \longrightarrow D$,

$$\psi = \bigoplus_{\wp} \phi(\mathcal{P}, \wp)$$

and

$$\phi(\mathcal{P}, \wp) : (l(\mathcal{P}))^* \longrightarrow K_1(d(\wp))$$

is given by

$$\phi(\mathcal{P}, \wp)(u) = \text{Norm}(u)$$
for $u \in l(\mathcal{P})^*$, where the $Norm$ is the natural norm mapping $l(\mathcal{P})^* \rightarrow k(\wp)^*$, the image of $\psi$ is $\bigoplus \ (k(\wp))^*$. So

$$\partial_D(x) = \partial_D \cdot i({a, b}) = \psi \cdot \partial_L({a, b}).$$

By the following commutative diagram

$$
\begin{array}{ccc}
K_2 L & \xrightarrow{\partial_L} & (l(\mathcal{P}))^* \\
\downarrow \text{Norm} & & \downarrow \psi \\
K_2 F & \xrightarrow{\partial_F} & (k(\wp))^*
\end{array}
$$

We have

$$\psi \cdot \partial_L({a, b}) = \partial_F \cdot \text{Norm}({a, b}).$$

So it is sufficient to prove that

$$\partial_F \cdot \text{Nrd}_2(x) = \partial_F \cdot \text{Norm}({a, b}).$$

Note that

$$\text{Nrd}_2(x) = \text{Nrd}_2 \cdot i({a, b}).$$

So we need only to prove

$$\text{Nrd}_2 \cdot i({a, b}) = \text{Norm}({a, b}).$$

By the definition of the reduced norm $\text{Nrd}_2$, this equality obviously holds. \hfill \Box

**Theorem 3.3.** Let $F$ be a number field and $R$ the ring of integers in $F$. Let $D$ be a finite dimensional central division $F$-algebra with square free index and $\Lambda$ a maximal $R$-order of $D$. Then

$$K_2 \Lambda \cong K_2^+ R = \ker(K_2 R \twoheadrightarrow \bigoplus_{\text{real ramified } \wp} \{\pm 1\} ).$$

**Proof.** Due to Lemma 3.2, we have the following commutative diagram

$$
\begin{array}{ccccccc}
1 & \rightarrow & K_2(\Lambda) & \xrightarrow{f} & K_2(D) & \xrightarrow{\partial_D} & \bigoplus_{\text{finite } \wp} k(\wp)^* & \rightarrow & 1 \\
& & \downarrow \text{Nrd}_2 & & \downarrow \text{id} & & \downarrow \text{id} & & \\
1 & \rightarrow & K_2 R & \xrightarrow{\text{Nrd}_2} & K_2 F & \xrightarrow{\partial_F} & \bigoplus_{\text{finite } \wp} k(\wp)^* & \rightarrow & 1
\end{array}
$$

where $f$ is induced by the reduced norm $\text{Nrd}_2$. By Theorem 1 of [1], the image of $\text{Nrd}_2$ is equal to

$$\ker(K_2 F \twoheadrightarrow \bigoplus_{\text{real ramified } \wp} \{\pm 1\} ).$$
So \( f : K_2 \Lambda \longrightarrow K_2^+R \) is surjective which implies \( K_2 \Lambda \simeq K_2^+R. \)

**Proposition 3.4.** Let \( F \) be a number field and \( D \) a central division algebra over \( F \) with square free index. then \( K_2D = \{ \{ a, b \} | a \in F^*, \ b \in D^* \} \), i.e., every element of \( K_2D \) is a symbol of form \( \{ a, b \} \) for \( a \in F^*, \ b \in D^* \).

*Proof.* By the proof of the Theorem of [5], every element of \( K_2F \) is of form \( \{ a, b \} \) where \( a \) is a totally negative element, i.e., \( a \) is negative at all real places. So for any element \( x \in K_2D \), its image \( \text{Nrd}_2(x) \) in \( K_2F \) is of the form \( \{ a, b \} \) for some totally negative \( a \). Since the image of \( K_2D \) under the reduced norm is \( K_2^+F \), \( b \) must be totally real. By Eichler’s Norm Theorem, \( b \) is a norm of some \( h \in D^* \). So \( \text{Nrd}_2(x) = \text{Nrd}_2(\{ a, h \}) \). By Theorem 3.4, \( \text{Nrd}_2 \) is injective, so \( x = \{ a, h \} \). \( \Box \)

**Acknowledgement** The authors are deeply grateful to Professor Fei Xu for valuable discussion. Proposition 2.2 is suggested by Professor Fei Xu. The first author would like to thank the Morningside Center of Mathematics for hospitality.

**References**


