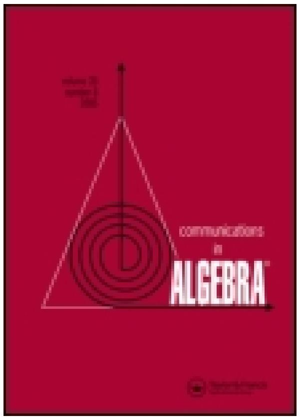

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The K_1 Group of Tiled Orders

Yuzhen Peng [a] & Xuejun Guo [a]
[a] Department of Mathematics , Nanjing University , Nanjing , China
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THE K_1 GROUP OF TILED ORDERS

Yuzhen Peng and Xuejun Guo
Department of Mathematics, Nanjing University, Nanjing, China

In this article, we study the K_1 of tiled rings over local rings. Our results generalize some existing ones obtained by Keating and Xi.

Key Words: Algebraic K-theory; Semiperfect ring; Tiled order.

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1. INTRODUCTION

Let R be a discrete valuation ring with quotient field F and maximal ideal P. An R-order A in $\mathbb{M}_n(F)$, the full $n \times n$ matrix algebra over F with $n \geqslant 2$, is called *tiled* if it has the matrix form $A = (P^{\lambda_{ij}})_{n\times n}$, where the λ_{ij}s are nonnegative integers satisfying $\lambda_{ii} = 0$ for all $1 \leqslant i \leqslant n$ and $\lambda_{ik} + \lambda_{kj} \geqslant \lambda_{ij}$ for all $1 \leqslant i, j, k \leqslant n$. Note further that the tiled order A is semiperfect and thus Morita equivalent to a basic one. So we can always assume $\lambda_{ij} + \lambda_{ji} > 0$ for any $i \neq j$.

There have been a number of articles devoted to the study of the global dimension of tiled orders, among which we mention [2, 3, 6, 9]. On the other hand, the investigation of the algebraic K-theory of tiled orders remains quite limited in the literature. As far as we know, Keating [4] proved that the G-theory of a tiled R-order A is related to the K-theories of the ground ring R as well as the residue class field, and in particular if A is regular then there is an isomorphism of algebraic K-theories

$$K_i(A) \simeq K_i(R) \oplus (n-1)K_i(R/P), \qquad i \geqslant 0.$$

For the general case, Keating [4] determined the K_1 of a special type of triangular orders, that is, $A = (P^{\lambda_{ij}})_{n\times n}$ with $\lambda_{ij} = 0$ for $i \geqslant j$ and $\lambda_{ij} = \mu > 0$ for $i < j$, in terms of the following isomorphism

$$K_1(A) \simeq K_1(R) \oplus (n-1)K_1(R/P^{\mu}).$$

The above isomorphism was extended in Keating [5] to the K-theory of triangular matrix rings over an arbitrary ring, provided that the two-sided ideal involved is

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Address correspondence to Dr. Yuzhen Peng, Department of Mathematics, Nanjing University, Nanjing 210093, China; E-mail: yzpengmath@gmail.com

projective. Recently, Xi [10] has generalized Keating's result to a more general case of triangular matrix rings (see Theorem 2.1 below).

In this article, we give a description of the K_1 of an arbitrary tiled order (Proposition 2.9), removing the restriction of being triangular. As a consequence, we obtain a direct generalization of Keating's result about the K_1 of triangular tiled orders:

Proposition 1.1. *Let R be a discrete valuation ring with maximal ideal P, and let $A = (P^{\lambda_{ij}})_{n\times n}$ be a tiled R-order. If $\lambda_{ij} = k$ for any $i > j$ and $\lambda_{ij} = l$ for any $i < j$, then there is an isomorphism*

$$K_1(A) \simeq K_1(R) \oplus (n-1)K_1(R/P^{k+l}).$$

Throughout, all rings are associative with identity. For a ring R we denote by $\mathrm{U}(R)$ its unit group and $\mathrm{rad}(R)$ its Jacobson radical. For any group G, we mean by G^n the direct product of n copies of G.

2. MAIN RESULTS

Let R be any ring and denote by $\mathbb{M}_n(R)$ the full $n \times n$ matrix ring with entries in R. We consider the following subset of $\mathbb{M}_n(R)$:

$$A = (I_{ij})_{n\times n} := \{ (a_{ij})_{n\times n} \mid a_{ij} \in I_{ij} \text{ for all } 1 \leqslant i, j \leqslant n \},$$

where the I_{ij}s are ideals of R satisfying

(i) $I_{11} = I_{22} = \cdots = I_{nn} = R$, and
(ii) $I_{ik}I_{kj} \subseteq I_{ij}$, for every $i \neq j$ and $1 \leqslant k \leqslant n$.

The two conditions make A a subring of $\mathbb{M}_n(R)$, and we call A a *tiled ring* over R. Obviously tiled orders over a discrete valuation ring are a particular example of tiled rings. The result below is due to Xi (see Theorem 1.2.(1) of [10]):

Theorem 2.1. *Let A be a tiled ring as above. If the ideals I_{ij} of R satisfy the following additional conditions*

(1) *$I_{ij} = R$ for any $i > j$,*
(2) *$I_{ik} \subseteq I_{ij} \cap I_{jk}$ for any $i < j < k$, then there is an isomorphism of algebraic K-theory*

$$K_i(A) \simeq K_i(R) \oplus \left(\bigoplus_{i=1}^{n-1} K_i(R/I_{i,i+1}) \right), \qquad i \geqslant 0.$$

We remark that there has not yet been a precise formula about the K-theory of A in the general case (without the restrictions (1) and (2) in the statement of Theorem 2.1).

Recall that a ring R is called *semiperfect* if $R/\mathrm{rad}(R)$ is a semisimple artinian ring with its idempotents lifted to R. For a semiperfect ring R there is a complete set of pairwise orthogonal primitive idempotents $e_1, e_2, \ldots, e_t$ in R such that R can be represented as the formal matrix ring $(R_{ij})_{t\times t}$, where $R_{ij} = e_iRe_j$ for $1 \leqslant i, j \leqslant t$. In particular, if R satisfies the condition $R_{ij}R_{ji} = 0$ for any $1 \leqslant i \neq j \leqslant t$, then $K_1(R)$ is isomorphic to the direct sum of the K_1 of the diagonal components $R_{11}, \ldots, R_{tt}$ (see [8]). For the general case, it is proved in [1] that.

Theorem 2.2. *Let R be a semiperfect ring represented as $(R_{ij})_{t\times t}$. Then,*

$$K_1(R) \simeq \left(\prod_{i=1}^{t} U(R_{ii})\right) \Big/ (HC) \simeq \left(\bigoplus_{i=1}^{t} K_1(R_{ii})\right) \Big/ (HC/C),$$

where C is the subgroup of $\prod_{i=1}^{t} U(R_{ii})$ generated by elements of the form

$$(1 + r_{ii}s_{ii})(1 + s_{ii}r_{ii})^{-1},$$

with $r_{ii}, s_{ii} \in R_{ii}$ satisfying $e_i + r_{ii}s_{ii} \in U(R_{ii})$ for $1 \leqslant i \leqslant t$, and H the subgroup of $\prod_{i=1}^{t} U(R_{ii})$ generated by elements of the form

$$(1 + r_{ij}r_{ji})(1 + r_{ji}r_{ij})^{-1},$$

with $r_{ij} \in R_{ij}$ satisfying $1 + r_{ij}r_{ji} \in U(R)$ for $1 \leqslant i \neq j \leqslant t$.

Now let $I_{ij}\,(1 \leqslant i, j \leqslant n)$ be ideals of R such that $A = (I_{ij})_{n\times n}$ forms a tiled ring. As a subring of $\mathbb{M}_n(R)$, A has an important feature, that is, A contains all the diagonal matrix units $\epsilon_{11}, \ldots, \epsilon_{nn}$ in $\mathbb{M}_n(R)$. This observation leads to

Lemma 2.3. *Let R be a local ring. Then the tiled ring $A = (I_{ij})_{n\times n}$ is a semiperfect ring.*

Proof. A direct consequence of Theorem 23.6 in [7]. □

Therefore, if R is a local ring then $K_0(A)$ is isomorphic to the free abelian group of rank n. For K_1, evidently the unit group of A has a diagonal subgroup as an internal direct product of n copies of $\mathrm{U}(R)$; moreover, if we denote by $\mathrm{V}(R)$ the subgroup of $\mathrm{U}(R)$ generated by all elements of the form $(1 + ab)(1 + ba)^{-1}$ with $a, b \in R$ satisfying $1 + ab \in \mathrm{U}(R)$, and by H the subgroup of $\mathrm{U}(R)^n$ generated by all elements of the form

$$(1, \ldots, 1, 1 + r_{ij}r_{ji}, 1, \ldots, 1, (1 + r_{ji}r_{ij})^{-1}, 1, \ldots, 1)$$

with $r_{ij} \in I_{ij}$ satisfying $1 + r_{ij}r_{ji} \in \mathrm{U}(R)$ for $1 \leqslant i \neq j \leqslant n$, where $1 + r_{ij}r_{ji}$ occurs in the i-th spot and $(1 + r_{ji}r_{ij})^{-1}$ in the j-th spot, then we have by Theorem 2.2:

Proposition 2.4. *Let R be a local ring and $A = (I_{ij})_{n\times n}$ be a tiled ring over R. Then $K_1(A)$ is isomorphic to the quotient of $U(R)^n$ modulo the product of the subgroups H and $V(R)^n$.*

Note that if R is also commutative then the group $\mathrm{V}(R)$ vanishes. Hence, we can expect a simpler description of $K_1(A)$ by calculating the group H. For convenience, we introduce the following notations:

Let G be an abelian group and N a subgroups of G. For $1 \leqslant i \neq j \leqslant n$, put

$$D_{ij}(N) := \{\ (1, \ldots, 1, a, 1, \ldots, 1, a^{-1}, 1, \ldots, 1) \in G^n \mid a \in N\ \},$$

where a occurs on the i-th spot and a^{-1} on the j-th spot. It is clear that $D_{ij}(N)$ is a subgroup of G^n and $D_{ij}(N) = D_{ji}(N)$ for every pair $i \neq j$. We call $D_{ij}(N)$ the (i, j)-*diagonal subgroup* of G^n with respect to N.

Let R now be a commutative local ring and $A = (I_{ij})_{n\times n}$ a tiled ring over R. Set $G = \mathrm{U}(R)$ and $N_{ij} = 1 + I_{ij}I_{ji}$ for any $i \neq j$. If A is basic, then for each pair $i \neq j$, either I_{ij} or I_{ji} is proper, thus $I_{ij}I_{ji}$ is contained in $\mathrm{rad}(R)$, and hence all these N_{ij}s are subgroups of G. Furthermore, it is clear that

$$\prod_{1\leqslant i\neq j\leqslant n} D_{ij}(N_{ij}) = \prod_{1\leqslant i<j\leqslant n} D_{ij}(N_{ij}),$$

since $N_{ij} = N_{ji}$ for any $i \neq j$. We claim that

Proposition 2.5. *Let R be a commutative local ring and $A = (I_{ij})_{n\times n}$ a basic tiled ring over R. Then $K_1(A)$ is isomorphic to the quotient of $U(R)^n$ modulo the product of its diagonal subgroups $\prod_{1\leqslant i<j\leqslant n} D_{ij}(N_{ij})$, where $N_{ij} = 1 + I_{ij}I_{ji}$.*

Proof. It suffices to show that the subgroup H defined in Theorem 2.2 is precisely the subgroup $\prod_{1\leqslant i<j\leqslant n} D_{ij}(N_{ij})$ when R is a commutative local ring. Obviously H is contained in the product $\prod_{1\leqslant i\neq j\leqslant n} D_{ij}(N_{ij})$, since every generator of H falls in some $D_{ij}(N_{ij})$. The reverse containment follows from the fact that N_{ij} can be generated by elements of the form $1 + r_{ij}r_{ji}$ with $r_{ij} \in I_{ij}$ and $r_{ji} \in I_{ji}$. □

Lemma 2.6. *Let G be an abelian group and $N_1, \ldots, N_{n-1}$ be subgroups of G, where $n \geqslant 2$. Then there is a group isomorphism*

$$\frac{G^n}{D_{12}(N_1)\cdots D_{n-1,n}(N_{n-1})} \simeq \frac{G}{N_1} \times \cdots \times \frac{G}{N_{n-1}} \times G.$$

Proof. Define a map from G^n to $(G/N_1) \times \cdots \times (G/N_{n-1}) \times G$ by

$$(g_1, \ldots, g_n) \mapsto (g_1N_1,\ g_1g_2N_2,\ \cdots,\ g_1\cdots g_{n-1}N_{n-1},\ g_1\cdots g_n),$$

then one checks that the map is a homomorphism of groups with its kernel exactly $D_{12}(N_1)\cdots D_{n-1,n}(N_{n-1})$. Moreover, the map is surjective, since for any $h_1, \ldots, h_n \in G$ we can set

$$g_1 = h_1, \qquad g_2 = h_1^{-1}h_2, \qquad g_3 = h_2^{-1}h_3, \ldots, g_n = h_{n-1}^{-1}h_n$$

and then $(g_1, \ldots, g_n)$ is mapped to $(h_1N_1,\ \cdots,\ h_{n-1}N_{n-1},\ h_n)$. □

We are now at the position to give a partial generalization of Theorem 2.1 in the case of R being a commutative local ring:

Theorem 2.7. *Let R be a commutative local ring and $A = (I_{ij})_{n\times n}$ a basic tiled ring over R. If $I_{ik}I_{ki} \subseteq I_{ij}I_{ji} \cap I_{jk}I_{kj}$ for any $i < j < k$, then*

$$K_1(A) \simeq U(R) \oplus \left(\bigoplus_{i=1}^{n-1} U(R/I_{i,i+1}I_{i+1,i})\right)$$
$$\simeq K_1(R) \oplus \left(\bigoplus_{i=1}^{n-1} K_1(R/I_{i,i+1}I_{i+1,i})\right).$$

Proof. Let $N_{ij} = 1 + I_{ij}I_{ji}$, $1 \leqslant i \neq j \leqslant n$. The hypotheses yields that $N_{ik} \subseteq N_{ij}$ and $N_{ik} \subseteq N_{jk}$ for any $i < j < k$, thus for each $a \in N_{ik}$,

$$\begin{aligned}&(\cdots, a, \ldots, 1, \ldots, a^{-1}, \cdots)\\ &\quad = (\cdots, a, \ldots, a^{-1}, \ldots, 1)(\cdots, 1, \ldots, a, \ldots, a^{-1}, \cdots),\end{aligned}$$

which implies that $D_{ik}(N_{ik}) \subseteq D_{ij}(N_{ij})D_{jk}(N_{jk})$. Consequently, the product $\prod_{1\leqslant i<j\leqslant n} D_{ij}(N_{ij})$ reduces to $D_{12}(N_{12})D_{23}(N_{23})\cdots D_{n-1,n}(N_{n-1,n})$. Then the conclusion follows by Proposition 2.5 and Lemma 2.6. □

We note that the formula stated in Theorem 2.7 remains valid when A is not necessarily basic, since every semiperfect ring is Morita equivalent to a basic one.

Corollary 2.8. *Let R be a commutative local ring and I, J be two arbitrary ideals of R. Then for the following tiled subring in $\mathbb{M}_n(R)$*

$$A = \begin{pmatrix} R & I & \cdots & I \\ J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & R \end{pmatrix},$$

we have

$$K_1(A) \simeq U(R) \oplus (n-1)U(R/IJ)$$
$$\simeq K_1(R) \oplus (n-1)K_1(R/IJ).$$

Specializing to tiled orders over a discrete valuation ring, we readily get

Proposition 2.9. *Let R be a discrete valuation ring with maximal ideal P, and let $A = (P^{\lambda_{ij}})_{n\times n}$ be a basic tiled R-order. Then $K_1(A)$ is isomorphic to the quotient of $U(R)^n$ modulo the product of its diagonal subgroups*

$$\prod_{1\leqslant i<j\leqslant n} D_{ij}(1 + P^{\lambda_{ij}+\lambda_{ji}}).$$

In particular, if $\lambda_{ij} = k$ *for any* $i > j$ *and* $\lambda_{ij} = l$ *for any* $i < j$, *then*

$$K_1(A) \simeq K_1(R) \oplus (n-1)K_1(R/P^{k+l}).$$

Remark 2.10. Corollary *2.8* gives an affirmative answer in the K_1 part to the question posed at the end of *[10]* in case R being a commutative local ring, and Proposition *2.9* recaptures Keating's result in *[4]* about the K_1 of triangular tiled orders over a discrete valuation ring.

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