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Yuzhen Peng $^a$ & Xuejun Guo $^a$

$^a$ Department of Mathematics, Nanjing University, Nanjing, China
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THE $K_1$ GROUP OF TILED ORDERS

Yuzhen Peng and Xuejun Guo
Department of Mathematics, Nanjing University, Nanjing, China

In this article, we study the $K_1$ of tiled rings over local rings. Our results generalize some existing ones obtained by Keating and Xi.

Key Words: Algebraic $K$-theory; Semiperfect ring; Tiled order.

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1. INTRODUCTION

Let $R$ be a discrete valuation ring with quotient field $F$ and maximal ideal $P$. An $R$-order $A$ in $M_n(F)$, the full $n \times n$ matrix algebra over $F$ with $n \geq 2$, is called tiled if it has the matrix form $A = (P^{\lambda_{ij}})_{n \times n}$, where the $\lambda_{ij}$s are nonnegative integers satisfying $\lambda_{ii} = 0$ for all $1 \leq i \leq n$ and $\lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}$ for all $1 \leq i, j, k \leq n$. Note further that the tiled order $A$ is semiperfect and thus Morita equivalent to a basic one. So we can always assume $\lambda_{ij} + \lambda_{ji} > 0$ for any $i \neq j$.

There have been a number of articles devoted to the study of the global dimension of tiled orders, among which we mention [2, 3, 6, 9]. On the other hand, the investigation of the algebraic $K$-theory of tiled orders remains quite limited in the literature. As far as we know, Keating [4] proved that the $G$-theory of a tiled $R$-order $A$ is related to the $K$-theories of the ground ring $R$ as well as the residue class field, and in particular if $A$ is regular then there is an isomorphism of algebraic $K$-theories

$$K_i(A) \simeq K_i(R) \oplus (n - 1)K_i(R/P), \quad i \geq 0.$$ 

For the general case, Keating [4] determined the $K_1$ of a special type of triangular orders, that is, $A = (P^{\lambda_{ij}})_{n \times n}$ with $\lambda_{ij} = 0$ for $i \geq j$ and $\lambda_{ij} = \mu > 0$ for $i < j$, in terms of the following isomorphism

$$K_1(A) \simeq K_1(R) \oplus (n - 1)K_1(R/P^\mu).$$

The above isomorphism was extended in Keating [5] to the $K$-theory of triangular matrix rings over an arbitrary ring, provided that the two-sided ideal involved is
projective. Recently, Xi [10] has generalized Keating’s result to a more general case of triangular matrix rings (see Theorem 2.1 below).

In this article, we give a description of the $K_1$ of an arbitrary tiled order (Proposition 2.9), removing the restriction of being triangular. As a consequence, we obtain a direct generalization of Keating’s result about the $K_1$ of triangular tiled orders:

**Proposition 1.1.** Let $R$ be a discrete valuation ring with maximal ideal $P$, and let $A = (P^{\ast i})_{n \times n}$ be a tiled $R$-order. If $\lambda_{ij} = k$ for any $i > j$ and $\lambda_{ij} = l$ for any $i < j$, then there is an isomorphism

$$K_1(A) \cong K_1(R) \oplus (n - 1)K_1(R/P^{k+1}).$$

Throughout, all rings are associative with identity. For a ring $R$ we denote by $U(R)$ its unit group and $\text{rad}(R)$ its Jacobson radical. For any group $G$, we mean by $G^n$ the direct product of $n$ copies of $G$.

2. MAIN RESULTS

Let $R$ be any ring and denote by $M_n(R)$ the full $n \times n$ matrix ring with entries in $R$. We consider the following subset of $M_n(R)$:

$$A = (I_{ij})_{n \times n} := \{ (a_{ij})_{n \times n} | a_{ij} \in I_{ij} \text{ for all } 1 \leq i, j \leq n \} ,$$

where the $I_{ij}$s are ideals of $R$ satisfying

(i) $I_{11} = I_{22} = \cdots = I_{nn} = R$, and
(ii) $I_{ik}I_{kj} \subseteq I_{ij}$, for every $i \neq j$ and $1 \leq k \leq n$.

The two conditions make $A$ a subring of $M_n(R)$, and we call $A$ a tiled ring over $R$. Obviously tiled orders over a discrete valuation ring are a particular example of tiled rings. The result below is due to Xi (see Theorem 1.2.(1) of [10]):

**Theorem 2.1.** Let $A$ be a tiled ring as above. If the ideals $I_{ij}$ of $R$ satisfy the following additional conditions

(1) $I_{ij} = R$ for any $i > j$,
(2) $I_{ik} \subseteq I_{ij} \cap I_{jk}$ for any $i < j < k$, then there is an isomorphism of algebraic $K$-theory

$$K_i(A) \cong K_i(R) \oplus \bigoplus_{i=1}^{n-1} K_i(R/I_{i+1}), \quad i \geq 0.$$

We remark that there has not yet been a precise formula about the $K$-theory of $A$ in the general case (without the restrictions (1) and (2) in the statement of Theorem 2.1).

Recall that a ring $R$ is called semiperfect if $R/\text{rad}(R)$ is a semisimple artinian ring with its idempotents lifted to $R$. For a semiperfect ring $R$ there is a complete set
of pairwise orthogonal primitive idempotents $e_1, e_2, \ldots, e_t$ in $R$ such that $R$ can be represented as the formal matrix ring $(R_{ij})_{i \times t}$, where $R_{ij} = e_i^* R e_j$ for $1 \leq i, j \leq t$. In particular, if $R$ satisfies the condition $R_{ij}^* R_{ji} = 0$ for any $1 \leq i \neq j \leq t$, then $K_1(R)$ is isomorphic to the direct sum of the $K_1$ of the diagonal components $R_{11}, \ldots, R_{nn}$ (see [8]). For the general case, it is proved in [1] that.

**Theorem 2.2.** Let $R$ be a semiperfect ring represented as $(R_{ij})_{i \times t}$. Then,

$$K_1(R) \cong \left( \prod_{i=1}^t U(R_{ii}) \right) / (HC) \cong \left( \bigoplus_{i=1}^t K_1(R_{ii}) \right) / (HC/C),$$

where $C$ is the subgroup of $\prod_{i=1}^t U(R_{ii})$ generated by elements of the form

$$(1 + r_{ii}s_{ii})(1 + s_{ii}r_{ii})^{-1},$$

with $r_{ii}, s_{ii} \in R_{ii}$ satisfying $e_i + r_{ii}^* s_{ii} \in U(R_{ii})$ for $1 \leq i \leq t$, and $H$ the subgroup of $\prod_{i=1}^t U(R_{ii})$ generated by elements of the form

$$(1 + r_{ij}r_{ji})(1 + r_{ji}r_{ij})^{-1},$$

with $r_{ij} \in R_{ij}$ satisfying $1 + r_{ij}r_{ji} \in U(R)$ for $1 \leq i \neq j \leq t$.

Now let $I_{ij}$ ($1 \leq i, j \leq n$) be ideals of $R$ such that $A = (I_{ij})_{n \times n}$ forms a tiled ring. As a subring of $\mathbb{M}_n(R)$, $A$ has an important feature, that is, $A$ contains all the diagonal matrix units $e_{11}, \ldots, e_{nn}$ in $\mathbb{M}_n(R)$. This observation leads to

**Lemma 2.3.** Let $R$ be a local ring. Then the tiled ring $A = (I_{ij})_{n \times n}$ is a semiperfect ring.

**Proof.** A direct consequence of Theorem 23.6 in [7].

Therefore, if $R$ is a local ring then $K_0(A)$ is isomorphic to the free abelian group of rank $n$. For $K_1$, evidently the unit group of $A$ has a diagonal subgroup as an internal direct product of $n$ copies of $U(R)$; moreover, if we denote by $V(R)$ the subgroup of $U(R)$ generated by all elements of the form $(1 + ab)(1 + ba)^{-1}$ with $a, b \in R$ satisfying $1 + ab \in U(R)$, and by $H$ the subgroup of $U(R)^n$ generated by all elements of the form

$$(1, \ldots, 1 + r_{ij}r_{ji}, 1, \ldots, 1, (1 + r_{ji}r_{ij})^{-1}, 1, \ldots, 1)$$

with $r_{ij} \in I_{ij}$ satisfying $1 + r_{ij}r_{ji} \in U(R)$ for $1 \leq i \neq j \leq n$, where $1 + r_{ij}r_{ji}$ occurs in the $i$-th spot and $(1 + r_{ji}r_{ij})^{-1}$ in the $j$-th spot, then we have by Theorem 2.2:

**Proposition 2.4.** Let $R$ be a local ring and $A = (I_{ij})_{n \times n}$ be a tiled ring over $R$. Then $K_1(A)$ is isomorphic to the quotient of $U(R)^n$ modulo the product of the subgroups $H$ and $V(R)^n$. 

Note that if $R$ is also commutative then the group $V(R)$ vanishes. Hence, we can expect a simpler description of $K_1(A)$ by calculating the group $H$. For convenience, we introduce the following notations:

Let $G$ be an abelian group and $N$ a subgroups of $G$. For $1 \leq i \neq j \leq n$, put

$$D_{ij}(N) := \{ (1, \ldots, 1, a, 1, \ldots, 1, a^{-1}, 1, \ldots, 1) \in G^n \mid a \in N \},$$

where $a$ occurs on the $i$-th spot and $a^{-1}$ on the $j$-th spot. It is clear that $D_{ij}(N)$ is a subgroup of $G^n$ and $D_{ij}(N) = D_{ji}(N)$ for every pair $i \neq j$. We call $D_{ij}(N)$ the $(i, j)$-diagonal subgroup of $G^n$ with respect to $N$.

Let $R$ now be a commutative local ring and $A = (I_{ij})_{n \times n}$ a tiled ring over $R$. Set $G = U(R)$ and $N_{ij} = 1 + I_{ij}I_{ji}$ for any $i \neq j$. If $A$ is basic, then for each pair $i \neq j$, either $I_{ij}$ or $I_{ji}$ is proper, thus $I_{ij}I_{ji}$ is contained in $\text{rad}(R)$, and hence all these $N_{ij}$ are subgroups of $G$. Furthermore, it is clear that

$$\prod_{1 \leq i < j \leq n} D_{ij}(N_{ij}) = \prod_{1 \leq i < j \leq n} D_{ij}(N_{ij}),$$

since $N_{ij} = N_{ji}$ for any $i \neq j$. We claim that

**Proposition 2.5.** Let $R$ be a commutative local ring and $A = (I_{ij})_{n \times n}$ a basic tiled ring over $R$. Then $K_1(A)$ is isomorphic to the quotient of $U(R)^n$ modulo the product of its diagonal subgroups $\prod_{1 \leq i < j \leq n} D_{ij}(N_{ij})$, where $N_{ij} = 1 + I_{ij}I_{ji}$.

**Proof.** It suffices to show that the subgroup $H$ defined in Theorem 2.2 is precisely the subgroup $\prod_{1 \leq i < j \leq n} D_{ij}(N_{ij})$ when $R$ is a commutative local ring. Obviously $H$ is contained in the product $\prod_{1 \leq i < j \leq n} D_{ij}(N_{ij})$, since every generator of $H$ falls in some $D_{ij}(N_{ij})$. The reverse containment follows from the fact that $N_{ij}$ can be generated by elements of the form $1 + r_{ij}r_{ji}$ with $r_{ij} \in I_{ij}$ and $r_{ji} \in I_{ji}$. □

**Lemma 2.6.** Let $G$ be an abelian group and $N_1, \ldots, N_{n-1}$ be subgroups of $G$, where $n \geq 2$. Then there is a group isomorphism

$$\frac{G^n}{D_{12}(N_1) \cdots D_{n-1,n}(N_{n-1})} \cong \frac{G}{N_1} \times \cdots \times \frac{G}{N_{n-1}} \times G.$$

**Proof.** Define a map from $G^n$ to $(G/N_1) \times \cdots \times (G/N_{n-1}) \times G$ by

$$(g_1, \ldots, g_n) \mapsto (g_1N_1, g_1g_2N_2, \ldots, g_1 \cdots g_{n-1}N_{n-1}, g_1 \cdots g_n),$$

then one checks that the map is a homomorphism of groups with its kernel exactly $D_{12}(N_1) \cdots D_{n-1,n}(N_{n-1})$. Moreover, the map is surjective, since for any $h_1, \ldots, h_n \in G$ we can set

$$g_1 = h_1, \quad g_2 = h_1^{-1}h_2, \quad g_3 = h_2^{-1}h_3, \ldots, g_n = h_{n-1}^{-1}h_n$$

and then $(g_1, \ldots, g_n)$ is mapped to $(h_1N_1, \ldots, h_{n-1}N_{n-1}, h_n)$. □
We are now at the position to give a partial generalization of Theorem 2.1 in the case of $R$ being a commutative local ring:

**Theorem 2.7.** Let $R$ be a commutative local ring and $A = (I_{ij})_{n \times n}$ a basic tiled ring over $R$. If $I_{ik}I_{ki} \subseteq I_{ij}I_{ji} \cap I_{jk}I_{kj}$ for any $i < j < k$, then

$$K_1(A) \simeq U(R) \oplus \left( \bigoplus_{i=1}^{n-1} U(R/I_{i+1,i+1}) \right) \simeq K_1(R) \oplus \left( \bigoplus_{i=1}^{n-1} K_1(R/I_{i+1,i+1}) \right).$$

**Proof.** Let $N_{ij} = 1 + I_{ij}I_{ji}$, $1 \leq i \neq j \leq n$. The hypotheses yields that $N_{ik} \subseteq N_{ij}$ and $N_{ik} \subseteq N_{jk}$ for any $i < j < k$, thus for each $a \in N_{ik}$,

$$(\cdots, a, \cdots, 1, \ldots, a^{-1}, \cdots) = (\cdots, a, \cdots, a^{-1}, \cdots, 1) (\cdots, 1, \ldots, a, \ldots, a^{-1}, \cdots),$$

which implies that $D_{ik}(N_{ik}) \subseteq D_{ij}(N_{ij})D_{jk}(N_{jk})$. Consequently, the product $\prod_{1 \leq i < j \leq n} D_{ij}(N_{ij})$ reduces to $D_{12}(N_{12})D_{23}(N_{23}) \cdots D_{n-1,n}(N_{n-1,n})$. Then the conclusion follows by Proposition 2.5 and Lemma 2.6. □

We note that the formula stated in Theorem 2.7 remains valid when $A$ is not necessarily basic, since every semiperfect ring is Morita equivalent to a basic one.

**Corollary 2.8.** Let $R$ be a commutative local ring and $I, J$ be two arbitrary ideals of $R$. Then for the following tiled subring in $M_n(R)$

$$A = \begin{pmatrix} R & I & \cdots & I \\ J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & R \end{pmatrix},$$

we have

$$K_1(A) \simeq U(R) \oplus (n-1)U(R/ij) \simeq K_1(R) \oplus (n-1)K_1(R/IJ).$$

Specializing to tiled orders over a discrete valuation ring, we readily get

**Proposition 2.9.** Let $R$ be a discrete valuation ring with maximal ideal $P$, and let $A = (P^{\lambda_{ij}})_{n \times n}$ be a basic tiled $R$-order. Then $K_1(A)$ is isomorphic to the quotient of $U(R)^n$ modulo the product of its diagonal subgroups

$$\prod_{1 \leq i < j \leq n} D_{ij}(1 + P^{\lambda_{ij} + \lambda_{ji}}).$$
In particular, if \( \lambda_{ij} = k \) for any \( i > j \) and \( \lambda_{ij} = l \) for any \( i < j \), then

\[
K_1(A) \cong K_1(R) \oplus (n - 1)K_1(R/P^{k+l}).
\]

**Remark 2.10.** Corollary 2.8 gives an affirmative answer in the \( K_1 \) part to the question posed at the end of [10] in case \( R \) being a commutative local ring, and Proposition 2.9 recaptures Keating’s result in [4] about the \( K_1 \) of triangular tiled orders over a discrete valuation ring.

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