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On the representation numbers of ternary quadratic forms and modular forms of weight 3/2 ☆

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ABSTRACT

In this paper, we give an alternative proof of Berkovich and Jagy's genus identity and verify a series of conjectures raised by Cooper and Lam on the number of solutions of $n^2 = x^2 + by^2 + cz^2$ and two conjectures raised by Sun on the number of solutions of $p = x^2 + y^2 + 3z^2$, $3p = x^2 + y^2 + 3z^2$.

1. Introduction

Let $Q(x, y, z)$ be a positive definite ternary quadratic form with integer coefficients. We denote by $R_Q(n)$ the representation number of the integer n by Q, that is, the number of integral solutions (x, y, z) of the equation $Q(x, y, z) = n$. In particular, if

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$Q(x,y,z) = ax^2 + by^2 + cz^2 + ryz + szx + txy$, we write $R_{(a,b,c,r,s,t)}(n)$ for $R_Q(n)$. If $r = s = t = 0$, we will also write $R_{(a,b,c,r,s,t)}(n)$ simply as $R_{(a,b,c)}(n)$. In particular, we write $S(n) = R_{(1,1,1,0,0,0)}(n)$ for short. Recall that the matrix associated to Q is

$$A = \begin{pmatrix} 2a & t & s \\ xt & 2b & r \\ s & r & 2c \end{pmatrix}.$$

The discriminant of Q is defined to be $\det(A)/2$, the level of Q is the minimal positive integer N such that NA^{-1} is an integral matrix with even diagonal entries, and the class number of Q is the number of equivalent classes in the genus of Q.

By Theorem 10.1 of [21],

$$\theta_Q(z) = \sum_{n \geqslant 0} R_Q(n) e^{2\pi i n z}$$

is a holomorphic function in the complex upper half-plane. Furthermore $\theta_Q(z)$ is a modular form of weight 3/2 with level N and character $\chi = (\frac{2\det(A)}{\bullet})$. Next we will give a brief review of the definition and relevant results of a modular form of weight 3/2.

Let N be a positive integer divisible by 4 and ω a Dirichlet character modulo N. Let $\Gamma_0(N)$ be the subgroup of $SL_2(\mathbb{Z})$ consisting of all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $N \mid c$. For a complete definition of modular forms with half-integral weights, one can see Chapter IV of [8]. A modular form f is said to have weight 3/2, level N and character ω if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and all $z \in \mathbb{C}$ with positive imaginary part,

$$f\left(\frac{az+b}{cz+d}\right) = \omega(d) \cdot \varepsilon_d \cdot \left(\frac{c}{d}\right) \cdot (cz+d)^{\frac{3}{2}} f(z),$$

where ω is a Dirichlet character and

$$\varepsilon_d = \begin{cases} 1, & d \equiv 1 \pmod 4, \\ i, & d \equiv 3 \pmod 4. \end{cases}$$

Denote by $\mathscr{M}(N,\omega)$ the complex linear space of modular forms of weight 3/2, level N and character ω. Let $\mathscr{S}(N,\omega)$ be the subspace of cusp forms in $\mathscr{M}(N,\omega)$ and $\mathscr{E}(N,\omega)$ the orthogonal complement of $\mathscr{S}(N,\omega)$ in $\mathscr{M}(N,\omega)$ with respect to Petersson inner product. One can see p. 1164 of [14] for the details of discussion about $\mathscr{E}(N,\omega)$.

It is clear that, if we can get an explicit expression of the function $\theta_Q(z)$, then we immediately derive an explicit formula for the representation number $R_Q(n)$. We note that, Pei constructed in [13–15], for some particular N and ω, an explicit basis for the space $\mathscr{E}(N,\omega)$. By virtue of these results we are able to reprove Berkovich and Jagy's beautiful identity in [1,3] and solve several conjectures proposed in Cooper and Lam [4] and Sun [20].

Now we give an outline of this paper. In Section 2, we will give an alternative proof of Berkovich and Jagy's identity (see below). Berkovich [1] established the following two interesting equalities connecting the representation numbers of three ternary forms:

Theorem 1.1. *(See Theorems 5.2 and 5.3 of [1].) For any natural number n,*

$$S(9n) - 3S(n) = 2R_{(1,1,3,0,0,1)}(n) - 4R_{(4,3,4,0,4,0)}(n), \tag{1.1}$$

$$S(25n) - 5S(n) = 4R_{(2,2,2,-1,1,1)}(n) - 8R_{(7,8,8,-4,8,8)}(n). \tag{1.2}$$

Berkovich and Jagy continued to investigate the value of $S(p^2n) - pS(n)$ for arbitrary odd prime p. They constructed two genera $TG_{1,p}$ and $TG_{2,p}$, where $TG_{1,p}$ consists of all the ternary quadratic forms with discriminant p^2, while $TG_{2,p}$ is the set of ternary quadratic forms $ax^2 + by^2 + cz^2 + ryz + szx + txy$ with discriminant $16p^2$ satisfying two conditions, namely, r, s, t are even and

$$R_{(a,b,c,r,s,t)}(n) = 0, \quad n \equiv 1, 2 \pmod 4. \tag{$*$}$$

Then the generalization of Theorem 1.1 reads as follows:

Theorem 1.2. *(See Theorem 1.3 of [3].) Let p be an odd prime. Then for any natural number n,*

$$S\left(p^2n\right) - pS(n) = 48 \sum_{Q\in TG_{1,p}} \frac{R_Q(n)}{|\mathrm{Aut}(Q)|} - 96 \sum_{Q\in TG_{2,p}} \frac{R_Q(n)}{|\mathrm{Aut}(Q)|}, \tag{1.3}$$

where $\mathrm{Aut}(Q)$ *is the finite group of integral automorphs of Q, and a sum over forms in a genus is understood to be the finite sum resulting from taking a single representative from each equivalence class of forms.*

Berkovich, Hanke and Jagy generalize the above theorem to general S-genus identity in [2]. Berkovich and Jagy proved the above theorems by computing the local factors in the Siegel–Weil formula. In this paper, we will give an alternative proof of the above result from the perspective of Eisenstein spaces. By our method, one can see that in Theorem 1.2, Berkovich and Jagy are studying four vectors in a 3-dimensional vector space $\mathscr{E}(4p, \mathrm{id})$. Furthermore, the beautiful identity of Berkovich and Jagy in fact gave an explicit basis of $\mathscr{E}(4p, \mathrm{id})$. Usually, it is difficult to give the explicit basis of the Eisenstein space. Since any four vectors in a 3-dimensional vector space are necessarily linearly dependent, one can find exactly the coefficients by the comparison of several suitable terms in the Fourier expansions of the corresponding modular forms.

We note that the Kudla matching in [9] gives similar identities to Theorem 1.2. By the virtue of Kudla matching, Tuoping Du found some explicit identities for different quaternary quadratic forms which are the reduced norm of certain quaternion orders in his PhD thesis [5].

Table 1
Data for Conjecture 1.4.

b	c
1	1, 2, 3, 4, 5, 6, 8, 9, 12, 21, 24
2	2, 3, 4, 5, 6, 8, 10, 13, 16, 22, 40, 70
3	3, 4, 5, 6, 9, 10, 12, 18, 21, 30, 45
4	4, 6, 8, 12, 24
5	5, 8, 10, 13, 25, 40
6	6, 9, 16, 18, 24
8	8, 10, 13, 16, 40
9	9, 12, 21, 24
10	30
12	12
16	24
21	21
24	24

In Sections 3–5, we verify a series of conjectures of Cooper and Lam. The following classical result, which is about the representation number of a square integer by $x^2 + y^2 + z^2$, is well known:

Theorem 1.3. *(See Hurwitz [7].) Let n be a positive integer with prime factorization*

$$n = 2^{e_2} \prod_{p\,\mathrm{odd}} p^{e_p}. \tag{1.4}$$

Then

$$S(n^2) = 6 \prod_{p\,\mathrm{odd}} \left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-1}{p}\right) \frac{p^{e_p}-1}{p-1} \right),$$

where $(\frac{a}{p})$ is the Legendre symbol.

Recently Cooper and Lam [4] established analogues of Hurwitz's formula for the cases

$$(a,b,c) = (1,1,2), (1,1,3), (1,2,2) \text{ and } (1,3,3),$$

and they were inspired, combining with computer investigations, to propose the following conjecture:

Conjecture 1.4. *Let n be a positive integer with prime factorization given by (1.4). Then when (b,c) takes the values in Table 1, the representation number $S_{(1,b,c)}(n^2)$ is a product of two integers:*

$$S_{(1,b,c)}(n^2) = G(b,c,n) \cdot H(b,c,n),$$

where

Table 2
Data for Conjecture 1.5.

b	c
1	1, 2, 3, 4, 5, 6, 8, 9, 12, 21, 24
2	2, 3, 4, 5, 6, 8, 10, 16
3	3, 4, 6, 9, 10, 12, 18, 30
4	4, 6, 8, 12, 24
5	5, 8, 10, 25, 40
6	6, 9, 16, 18, 24
8	8, 16, 40
9	9, 12, 21, 24
10	30
12	12
16	24
21	21
24	24

$$H(b,c,n)=\prod_{p\nmid 2bc}\left(\frac{p^{e_p+1}-1}{p-1}-\left(\frac{-bc}{p}\right)\frac{p^{e_p}-1}{p-1}\right),$$

and

$$G(b,c,n)=\prod_{p\mid 2bc} g(b,c,p,e_p),$$

in which $g(b,c,p,e_p)$ has to be determined on an individual and case-by-case basis.

Recall that Ramanujan's theta function, denoted by $\varphi(q)$, is defined to be

$$\varphi(q)=\sum_{j=-\infty}^{\infty} q^{j^2},\quad \text{where } q=e^{2\pi iz}.$$

Let b and c be fixed integers, the *Hecke operator* $T_{p^2}(b,c)$ is defined for any prime $p\nmid 2bc$ by

$$T_{p^2}(b,c)\left(\sum_{j=0}^{\infty}c_jq^j\right)=\sum_{j=0}^{\infty}c_{p^2j}q^j+\sum_{j=0}^{\infty}\left(\left(\frac{-bcj}{p}\right)\right)c_jq^j+\sum_{j=0}^{\infty}c_jq^{p^2j},$$

where $(\frac{-bcj}{p})$ is the Legendre symbol. One can see Theorem 1.7 of [19] for details on the Hecke operator.

Also based on computer investigations, Cooper and Lam [4] made the following conjecture:

Conjecture 1.5. *Let b and c take any of the values given in Table 2. Then for any prime p with $p\nmid 2bc$, we have*

$$T_{p^2}(b,c)\big(\varphi(q)\varphi\big(q^b\big)\varphi\big(q^c\big)\big)=(p+1)\varphi(q)\varphi\big(q^b\big)\varphi\big(q^c\big).$$

Table 3
Conjecture 1.4 holds for the following cases.

b	c
1	1, 2, 3, 4, 5, 6, 8
2	2, 3, 4, 6
3	3, 6
4	4, 8
5	5
6	6

Cooper and Lam verified Conjecture 1.4 in the cases

$$(b, c) = (1, 1), (1, 2), (1, 3), (2, 2), (3, 3).$$

In this paper, we will show that Conjecture 1.4 still holds for 12 other cases and Conjecture 1.5 holds for all the cases listed in Table 2 and no more. We put 17 cases which hold for Conjecture 1.4 in Table 3.

Indeed, Pei [13–15] has shown that, for some particular N and ω, the space $\mathscr{S}(N, \omega)$ is trivial and thus $\mathscr{M}(N, \omega)$ coincides with $\mathscr{E}(N, \omega)$; moreover, $\mathscr{E}(N, \omega)$ is generated by Eisenstein series and an explicit basis for it could be constructed. Based on these results, Pei derived analytic formulas for the representation number $R_Q(m)$ when Q is a "diagonal form" $ax^2 + by^2 + cz^2$ for several small a, b, c, involving special values of some particular L-functions.

We shall show in Sections 3 and 4 that Pei's analytic formulas reduce to explicit formulas of Hurwitz type when m is a square, which cover several cases of Cooper and Lam's Conjecture 1.4.

For those cases in Table 1 that are not covered in Table 3, it is also potential to derive the corresponding formulas for representation numbers through similar arguments. The idea goes as follows. Let Q be a positive definite ternary form with integer coefficients. Then the associated modular form $\theta_Q(z)$ has weight 3/2 and level N. If Q has class number 1, then $\theta_Q(z)$ lies in the space $\mathscr{E}(N, \omega)$. Note that Pei has constructed in [13,15] an explicit basis for $\mathscr{E}(N, \omega)$, so it is possible to write $\theta_Q(z)$ explicitly as a linear combination of this basis—generally speaking this can be done by computing the values of $\theta_Q(z)$ at all its cusps. Here we simply benefit from Pei's computations for cases in Table 3, which was carried out in [14]. For other cases in Table 1, we believe Cooper and Lam's conjecture holds too, although we have not been able to provide a rigorous proof in this paper.

Conjecture 1.5 concerns Hecke operators on the space $\mathscr{E}(N, \omega)$. If the class number of a ternary quadratic form Q is 1, then the action of T_{p^2} on $\theta_Q(z)$ is given as Theorem 10.2 of [21]. The details of the proof of Theorem 10.2 of [21] can be found in [17] and [18]. In [16], Pei credited this result to Eichler. For all the cases of Table 2, the class number of Q is 1 by the list of all diagonal (primitive) quadratic forms of class number one at http://www.kobepharma-u.ac.jp/~math/notes/note03.html. Hence Conjecture 1.5 follows directly from Theorem 10.2 of [21].

Pei's results on the Eisenstein series of weight 3/2 can be used to compute not only the representation numbers of squares by ternary quadratic forms, but also the representation numbers of general integers by the forms. In Section 5 we use this method to prove Conjectures 18 and 19 of [20]. More precisely, we prove that

$$S_{(1,1,3)}(p) = \begin{cases} 12h(-3p), & \text{if } p \equiv 1 \pmod 8, \\ 8h(-3p), & \text{if } p \equiv 5 \pmod 8, \\ 2h(-3p), & \text{if } p \equiv 3 \pmod 4; \end{cases}$$

and

$$S_{(1,1,3)}(3p) = \begin{cases} 4h(-p), & \text{if } p \equiv 1 \pmod 4, \\ 24h(-p), & \text{if } p \equiv 3 \pmod 8, \\ 16h(-p), & \text{if } p \equiv 7 \pmod 8, \end{cases}$$

where $h(d)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{d}\,)$.

2. Berkovich and Jagy's genus identity

We will use the same notations as those in the Introduction. Let p be an odd prime. Let $\theta(z) = \sum_{n=0}^{\infty} S(n)q^n$, where $q = e^{2\pi iz}$. Then $\theta(z)$ is in the space $\mathscr{E}(4, \mathrm{id}) \subset \mathscr{E}(4p, \mathrm{id})$, where we use "id" for the trivial character. By [12, Proposition 2.22], we see that the function $\psi_p(z) = \sum_{n=0}^{\infty} S(p^2 n)q^n$ is in $\mathscr{E}(4p, \mathrm{id})$.

By Section 6 of [3], $TG_{1,p}$ consists of all the ternary quadratic forms with discriminant p^2 and $TG_{2,p}$ consists of the set of ternary quadratic forms with discriminant $16p^2$ and particular genus symbols. Then by Proposition 5 of [10], both of the levels of ternary forms in $TG_{1,p}$ and $TG_{2,p}$ are $4p$ which implies that the theta series of $TG_{1,p}$ and $TG_{2,p}$ are all in $\mathscr{E}(4p, \mathrm{id})$. Let

$$\theta_i(z) = \sum_{n=0}^{\infty} \left(\sum_{Q \in TG_{i,p}} \frac{R_Q(n)}{|\mathrm{Aut}(Q)|} \right) q^n, \quad i = 1, 2.$$

Then $\theta(z)$, $\theta_1(z)$, $\theta_2(z)$ are all in $\mathscr{E}(4p, \mathrm{id})$.

Theorem 2.1. *The theta series $\theta(z)$, $\theta_1(z)$, $\theta_2(z)$ are linearly independent in $\mathscr{E}(4p, \mathrm{id})$ and*

$$\psi_p(z) = p\theta(z) + 48\theta_1(z) - 96\theta_2(z).$$

Note that $\dim \mathscr{E}(4p, \mathrm{id}) = 3$, so $\{\theta(z), \theta_1(z), \theta_2(z)\}$ is a basis of $\mathscr{E}(4p, \mathrm{id})$. One can also see Corollary 10.1 of [21] for another explicit basis.

Before the proof of Theorem 2.1, we will cite the following lemma which is a standard result derived from the analytic class number formula for imaginary quadratic number fields.

Lemma 2.2 *(Analytic class number formula). (See Theorem 4.9 of [22].) Let p be an odd prime, and $h(-p)$ the class number of $\mathbb{Q}(\sqrt{-p})$. Then*

$$\prod_{p' \text{ odd}} \left(1 + \frac{1}{p'}\left(\frac{-1}{p'}\right)\right) = \frac{8L(1,\chi_{-1})}{\pi^2} = \frac{2}{\pi};$$

$$\prod_{p' \text{ odd}} \left(1 + \frac{1}{p'}\left(\frac{-2}{p'}\right)\right) = \frac{8L(1,\chi_{-2})}{\pi^2} = \frac{2\sqrt{2}}{\pi};$$

$$\prod_{p' \neq 2,p} \left(1 + \frac{1}{p'}\left(\frac{-p}{p'}\right)\right) = \frac{8p^2L(1,\chi_{-p})}{\pi^2(p^2-1)},$$

where

$$L(1,\chi_{-p}) = \begin{cases} \frac{\pi}{3\sqrt{3}}, & \text{if } p = 3, \\ \frac{\pi h(-p)}{\sqrt{p}}, & \text{if } p \equiv 3 \pmod 4 \text{ and } p \neq 3, \\ \frac{\pi h(-p)}{2\sqrt{p}}, & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

Lemma 2.3 *(Siegel–Weil formula). Let*

$$\theta_i(z) = \sum_{n=0}^{\infty} r_{in} q^n, \quad i = 1, 2.$$

Then

$$r_{1n} = \frac{\pi(p-1)\sqrt{n}}{12p} \cdot \prod_{p'} d_{1,p'}(n),$$

$$r_{2n} = \frac{\pi(p-1)\sqrt{n}}{48p} \cdot \prod_{p'} d_{2,p'}(n),$$

where the local factors in the product are

(1) $p' = 2$. *Assume $n = 4^a k$, $4 \nmid k$. Then*

$$d_{1,2}(n) = \begin{cases} \frac{3}{2}, & \text{if } k \equiv 7 \pmod 8, \\ \frac{3}{2} - \frac{1}{2^{a+1}}, & \text{if } k \equiv 3 \pmod 8, \\ \frac{3}{2} - \frac{3}{2^{a+2}}, & \text{if } k \equiv 1, 2 \pmod 4; \end{cases}$$

and

$$d_{2,2}(n) = \begin{cases} 3, & \text{if } k \equiv 7 \pmod 8, \\ 3 - \frac{1}{2^{a-1}}, & \text{if } k \equiv 3 \pmod 8, \\ 3 - \frac{3}{2^a}, & \text{if } k \equiv 1, 2 \pmod 4. \end{cases}$$

(2) $p' = p$.

$$d_{1,p}(n) = d_{2,p}(n) = \begin{cases} \frac{1}{p^k}(1 - (\frac{-\ell}{p})), & \text{if } n = \ell p^{2k},\ p \nmid \ell, \\ \frac{1}{p^k}(1 + \frac{1}{p}), & \text{if } n = \ell p^{2k+1},\ p \nmid \ell. \end{cases}$$

(3) $p' \neq 2, p$.

$$d_{1,p'}(n) = d_{2,p'}(n) = \begin{cases} (\frac{1}{p'} + 1) + \frac{1}{p'^{k+1}}((\frac{-\ell}{p'}) - 1), & \text{if } n = \ell p'^{2k},\ p' \nmid \ell, \\ (\frac{1}{p'} + 1)(1 - \frac{1}{p'^{k+1}}), & \text{if } n = \ell p'^{2k+1},\ p' \nmid \ell. \end{cases}$$

We will not prove the above lemma. One can find every detail of the proof in the well-written paper [3]. In fact, we can also prove Berkovich and Jagy's identity by carefully computing the values of theta series at all the cusps of $\Gamma_0(4p)$. However since everything we need is contained in [3], it is hard to resist using their computation directly.

Lemma 2.4. *If* $p \equiv 1 \pmod 4$*, then*

$$S(p^3) - pS(p) = 12h(-p).$$

Proof. This lemma follows from the explicit representation formula for the sum of 3 squares. For the details of the proof, one can see Eq. (3) of [6] and Lemma 2.2 (the analytic class number formula) above. □

Proof of Theorem 2.1.

(1) If $p \equiv 3 \pmod 4$, then considering the coefficients of the constant term, q-term and the q^4-term by Lemma 2.3, we have

$$\theta(z) = 1 + 6 \cdot q + * \cdot q^2 + * \cdot q^3 + 6 \cdot q^4 + \cdots,$$
$$\theta_1(z) = \frac{p-1}{48} + \frac{1}{4} \cdot q + * \cdot q^2 + * \cdot q^3 + \frac{3}{4} \cdot q^4 + \cdots,$$
$$\theta_2(z) = \frac{p-1}{48} + 0 \cdot q + * \cdot q^2 + * \cdot q^3 + \frac{1}{4} \cdot q^4 + \cdots,$$

where the symbol "$*$" means that we don't need to consider this coefficient. Since the matrix

$$A = \begin{pmatrix} 1 & (p-1)/48 & (p-1)/48 \\ 6 & 1/4 & 0 \\ 6 & 3/4 & 1/4 \end{pmatrix}$$

is invertible, $\theta(z)$, $\theta_1(z)$, $\theta_2(z)$ are linearly independent. And for $p \equiv 3 \pmod 4$, we have

$$\begin{aligned}\psi_p(z) &= 1 + 6(p+2)\cdot q + * \cdot q^2 + * \cdot q^3 + 6(p+2)\cdot q^4 + \cdots \\ &= p\theta(z) + 48\theta_1(z) - 96\theta_2(z).\end{aligned}$$

(2) If $p \equiv 1 \pmod 4$, then by Lemma 2.3, we have

$$\begin{aligned}\theta(z) &= 1 + 6\cdot q + \cdots + S(p)\cdot q^p + \cdots, \\ \theta_1(z) &= \frac{p-1}{48} + \frac{1}{4}\cdot q + \cdots + \frac{1}{4}h(-p)\cdot q^p + \cdots, \\ \theta_2(z) &= \frac{p-1}{48} + 0\cdot q + \cdots + 0\cdot q^p + \cdots.\end{aligned}$$

Since the matrix

$$A = \begin{pmatrix} 1 & \frac{p-1}{48} & \frac{p-1}{48} \\ 6 & 0 & 0 \\ S(p) & \frac{1}{4}h(-p) & 0 \end{pmatrix}$$

is invertible, $\theta(z)$, $\theta_1(z)$, $\theta_2(z)$ are linearly independent. And by Lemma 2.4, we have

$$\begin{aligned}\psi_p(z) &= 1 + 6(p+2)\cdot q + \cdots + S(p^3)\cdot q^p + \cdots \\ &= p\theta(z) + 48\theta_1(z) - 96\theta_2(z). \qquad \square\end{aligned}$$

3. Representation numbers of natural numbers by diagonal ternary forms

We start by describing Pei's results about the representation numbers of an arbitrary integer by certain positive definite ternary quadratic forms. To do this we have to introduce several notations. For details of Pei's work on this issue we refer to [13–16,21].

Throughout this section we assume m and D are fixed positive integers, and D is odd and square-free.

The function $L_N(s,\omega)$. Let ω be a Dirichlet character. Recall that the *Dirichlet L-series* associated to ω is by definition

$$L(s,\omega) := \sum_{k=1}^{\infty} \frac{\omega(k)}{k^s},$$

where s is a complex variable. The series $L(s,\omega)$ represents an analytic function on the half-plane $\mathrm{Re}(s) > 1$ and admits an analytic continuation to the whole complex plane (with a pole at $s = 1$ in the case ω being trivial).

Let N be a positive integer and χ a Dirichlet character. Define

$$L_N(s,\omega) := \sum_{\substack{k=1\\(k,N)=1}}^{\infty} \frac{\omega(k)}{k^s}.$$

Of course $L_N(s,\omega) = L(s,\omega)$ when the modulus of ω is exactly N.

The character χ_t. Recall that for any two integers a and b there is defined a *Kronecker symbol* $(\frac{a}{b})$. For any fixed nonzero integer t, we can define a function χ_t on the integers as follows: Let $t = qs^2$ with q the square-free part of t. Then define $\chi_t = (\frac{q}{\bullet})$ when $q \equiv 1 \pmod 4$ and $\chi_t = (\frac{4q}{\bullet})$ when $q \equiv 2, 3 \pmod 4$. One sees that χ_t is a quadratic Dirichlet character.

The sum $\sigma(m, 4D)$. Denote by $\sigma(m, 4D)$ the sum

$$\sum \frac{\mu(a)\chi_{-m}(a)}{ab},$$

where μ is the Möbius function, and the summation ranges over all the positive integers a, b that are both coprime to $4D$ and satisfy $(ab)^2 \mid m$.

The function $\lambda(m, 4D)$. Put

$$\lambda(m, 4D) := \frac{L_{4D}(1, \chi_{-m})}{L_{4D}(2, \mathrm{id})} \cdot \sigma(m, 4D),$$

where id denotes the trivial character mod $4D$.

The functions $\alpha(m)$ and $A(p, m)$. For any prime p, let $h_p(m)$ be the natural number such that $p^{h_p(m)} \parallel m$ and $h'_p(m) := \frac{m}{p^{h_p(m)}}$ the p'-part of m. Define

$$\alpha(m) := \begin{cases} 3 \cdot 2^{-\frac{1+h_2(m)}{2}}, & \text{if } h_2(m) \text{ is odd},\\ 3 \cdot 2^{-1-\frac{h_2(m)}{2}}, & \text{if } h_2(m) \text{ is even and } h'_2(m) \equiv 1 \pmod 4,\\ 2^{-\frac{h_2(m)}{2}}, & \text{if } h_2(m) \text{ is even and } h'_2(m) \equiv 3 \pmod 8,\\ 0, & \text{if } h_2(m) \text{ is even and } h'_2(m) \equiv 7 \pmod 8; \end{cases}$$

and

$$A(p, m) := \begin{cases} p^{-1} - (1+p)p^{-\frac{3+h_p(m)}{2}}, & \text{if } h_p(m) \text{ is odd},\\ p^{-1} - 2p^{-1-\frac{h_p(m)}{2}}, & \text{if } h_p(m) \text{ is even and } (\frac{-h'_p(m)}{p}) = -1,\\ p^{-1}, & \text{if } h_p(m) \text{ is even and } (\frac{-h'_p(m)}{p}) = 1, \end{cases}$$

where p is an odd prime.

Now we are ready to formulate Pei's results about representation numbers of m by certain positive definite ternary quadratic forms in the following list:

$$S_{(1,1,1)}(m) = 2\pi\sqrt{m}\cdot\lambda(m,4)\cdot\alpha(m);$$

$$S_{(1,2,2)}(m) = \begin{cases} 2\pi\sqrt{m}\cdot\lambda(m,4)\cdot\alpha(m), & \text{if } m\equiv 0,3 \pmod 4,\\ 2\pi\sqrt{m}\cdot\lambda(m,4)\cdot(\alpha(m)-1), & \text{if } m\equiv 1,2 \pmod 4;\end{cases}$$

$$S_{(1,3,3)}(m) = 2\pi\sqrt{m}\cdot\lambda(m,12)\cdot\left(\frac{1}{3}-A(3,m)\right)\cdot\big(2-\alpha(m)\big);$$

$$S_{(1,5,5)}(m) = 2\pi\sqrt{m}\cdot\lambda(m,20)\cdot\alpha(m)\cdot\left(A(5,m)+\frac{1}{5}\right);$$

$$S_{(1,6,6)}(m) = \begin{cases} 2\pi\sqrt{m}\cdot\lambda(m,12)\cdot(\frac{1}{3}-A(3,m))\cdot(1-\alpha(m)), & \text{if } m\equiv 0,3 \pmod 4,\\ 2\pi\sqrt{m}\cdot\lambda(m,12)\cdot(\frac{1}{3}-A(3,m))\cdot(2-\alpha(m)), & \text{if } m\equiv 1,2 \pmod 4;\end{cases}$$

$$S_{(2,3,6)}(m) = \begin{cases} 2\pi\sqrt{m}\cdot\lambda(m,12)\cdot(\frac{1}{3}+A(3,m))\cdot\alpha(m), & \text{if } m\equiv 0,3 \pmod 4,\\ 2\pi\sqrt{m}\cdot\lambda(m,12)\cdot(\frac{1}{3}+A(3,m))\cdot(\alpha(m)-1), & \text{if } m\equiv 1,2 \pmod 4;\end{cases}$$

$$S_{(1,1,4)}(m) = \begin{cases} 2\pi\sqrt{m}\cdot\lambda(m,4)\cdot\alpha(m), & \text{if } m\equiv 0 \pmod 4,\\ 2\pi\sqrt{m}\cdot\lambda(m,4), & \text{if } m\equiv 1 \pmod 4,\\ \pi\sqrt{m}\cdot\lambda(m,4), & \text{if } m\equiv 2 \pmod 4,\\ 0, & \text{if } m\equiv 3 \pmod 4;\end{cases}$$

$$S_{(1,4,4)}(m) = \begin{cases} 2\pi\sqrt{m}\cdot\lambda(m,4)\cdot\alpha(m), & \text{if } m\equiv 0 \pmod 4,\\ \pi\sqrt{m}\cdot\lambda(m,4), & \text{if } m\equiv 1 \pmod 4,\\ 0, & \text{if } m\equiv 2,3 \pmod 4;\end{cases}$$

$$S_{(1,2,4)}(m) = \begin{cases} \pi\sqrt{2m}\cdot\lambda(2m,4)\cdot(2\alpha(2m)-\frac{5}{2}), & \text{if } m \text{ is odd},\\ \pi\sqrt{2m}\cdot\lambda(2m,4)\cdot 2\alpha(2m), & \text{if } m\equiv 0,6 \pmod 8,\\ \pi\sqrt{2m}\cdot\lambda(2m,4)\cdot(2\alpha(2m)-1), & \text{if } m\equiv 2,4 \pmod 8;\end{cases}$$

$$S_{(1,1,8)}(m) = \begin{cases} \frac{1}{\sqrt{2}}\pi\sqrt{m}\cdot\lambda(2m,4)\cdot\alpha(\frac{m}{8}), & \text{if } m\equiv 0 \pmod 8,\\ \frac{1}{\sqrt{2}}\pi\sqrt{m}\cdot\lambda(2m,4), & \text{if } m\equiv 4 \pmod 8,\\ \sqrt{2}\pi\sqrt{m}\cdot\lambda(2m,4), & \text{if } m\equiv 1 \pmod 4,\\ \sqrt{2}\pi\sqrt{m}\cdot\lambda(2m,4), & \text{if } m\equiv 2 \pmod 4,\\ 0, & \text{if } m\equiv 3 \pmod 4;\end{cases}$$

$$S_{(1,4,8)}(m) = \begin{cases} \frac{1}{\sqrt{2}}\pi\sqrt{m}\cdot\lambda(2m,4)\cdot\alpha(\frac{m}{8}), & \text{if } m\equiv 0 \pmod 8,\\ \frac{1}{\sqrt{2}}\pi\sqrt{m}\cdot\lambda(2m,4), & \text{if } m\equiv 4 \pmod 8,\\ \frac{1}{\sqrt{2}}\pi\sqrt{m}\cdot\lambda(2m,4), & \text{if } m\equiv 1 \pmod 4,\\ 0, & \text{if } m\equiv 2,3 \pmod 4.\end{cases}$$

Moreover, there are several relations between certain representation numbers:

$$
\begin{aligned}
S_{(1,1,2)}(m) &= S_{(1,2,2)}(2m), \\
S_{(1,1,3)}(m) &= S_{(1,3,3)}(3m), \\
S_{(1,1,5)}(m) &= S_{(1,5,5)}(5m), \\
S_{(2,3,3)}(m) &= S_{(1,6,6)}(2m), \\
S_{(2,2,3)}(m) &= S_{(1,6,6)}(3m), \\
S_{(1,1,6)}(m) &= S_{(1,6,6)}(6m), \\
S_{(1,3,6)}(m) &= S_{(2,3,6)}(2m), \\
S_{(1,2,6)}(m) &= S_{(2,3,6)}(3m), \\
S_{(1,2,3)}(m) &= S_{(2,3,6)}(6m).
\end{aligned}
$$

Explicitly,

$$
S_{(1,1,2)}(m) = \begin{cases} 2\pi\sqrt{2m} \cdot \lambda(2m,4) \cdot (\alpha(2m) - 1), & \text{if } m \text{ is odd}, \\ 2\pi\sqrt{2m} \cdot \lambda(2m,4) \cdot \alpha(2m), & \text{if } m \text{ is even}; \end{cases}
$$

$$
S_{(1,1,3)}(m) = 2\pi\sqrt{3m} \cdot \lambda(3m,12) \cdot \left(\frac{1}{3} - A(3,3m)\right) \cdot \big(2 - \alpha(3m)\big);
$$

$$
S_{(1,1,5)}(m) = 2\pi\sqrt{5m} \cdot \lambda(5m,20) \cdot \alpha(5m) \cdot \left(A(5,5m) + \frac{1}{5}\right);
$$

$$
S_{(2,3,3)}(m) = \begin{cases} 2\pi\sqrt{2m} \cdot \lambda(2m,12) \cdot (\frac{1}{3} - A(3,2m)) \cdot (2 - \alpha(2m)), & \text{if } m \text{ is odd}, \\ 2\pi\sqrt{2m} \cdot \lambda(2m,12) \cdot (\frac{1}{3} - A(3,2m)) \cdot (1 - \alpha(2m)), & \text{if } m \text{ is even}; \end{cases}
$$

$$
S_{(2,2,3)}(m) = \begin{cases} 2\pi\sqrt{3m} \cdot \lambda(3m,12) \cdot (\frac{1}{3} - A(3,3m)) \cdot (1 - \alpha(3m)), & \text{if } m \equiv 0,1 \pmod 4, \\ 2\pi\sqrt{3m} \cdot \lambda(3m,12) \cdot (\frac{1}{3} - A(3,3m)) \cdot (2 - \alpha(3m)), & \text{if } m \equiv 2,3 \pmod 4; \end{cases}
$$

$$
S_{(1,1,6)}(m) = \begin{cases} 2\pi\sqrt{6m} \cdot \lambda(6m,12) \cdot (\frac{1}{3} - A(3,6m)) \cdot (2 - \alpha(6m)), & \text{if } m \text{ is odd}, \\ 2\pi\sqrt{6m} \cdot \lambda(6m,12) \cdot (\frac{1}{3} - A(3,6m)) \cdot (1 - \alpha(6m)), & \text{if } m \text{ is even}; \end{cases}
$$

$$
S_{(1,3,6)}(m) = \begin{cases} 2\pi\sqrt{2m} \cdot \lambda(2m,12) \cdot (\frac{1}{3} + A(3,2m)) \cdot (\alpha(2m) - 1), & \text{if } m \text{ is odd}, \\ 2\pi\sqrt{2m} \cdot \lambda(2m,12) \cdot (\frac{1}{3} + A(3,2m)) \cdot \alpha(2m), & \text{if } m \text{ is even}; \end{cases}
$$

$$
S_{(1,2,6)}(m) = \begin{cases} 2\pi\sqrt{3m} \cdot \lambda(3m,12) \cdot (\frac{1}{3} + A(3,3m)) \cdot \alpha(3m), & \text{if } m \equiv 0,1 \pmod 4, \\ 2\pi\sqrt{3m} \cdot \lambda(3m,12) \cdot (\frac{1}{3} + A(3,3m)) \cdot (\alpha(3m) - 1), & \text{if } m \equiv 2,3 \pmod 4; \end{cases}
$$

$$
S_{(1,2,3)}(m) = \begin{cases} 2\pi\sqrt{6m} \cdot \lambda(6m,12) \cdot (\frac{1}{3} + A(3,6m)) \cdot (\alpha(6m) - 1), & \text{if } m \text{ is odd}, \\ 2\pi\sqrt{6m} \cdot \lambda(6m,12) \cdot (\frac{1}{3} + A(3,6m)) \cdot \alpha(6m), & \text{if } m \text{ is even}. \end{cases}
$$

In the next section we shall restrict our attention to the situation of m being a square integer, in which we derive explicit formulas of Hurwitz type from Pei's formulas.

4. Representation numbers of perfect squares by diagonal ternary forms

Our results follow from the computation of the values of the expressions $\lambda(m, D)$, $\alpha(m)$ and $A(p, m)$, in which m is replaced by a square integer. We maintain the assumption that D is an odd positive square-free integer and n a positive integer with prime factorization given by (1.4).

Let ℓ be a positive integer all of whose prime factors divide $2D$.

4.1. The computation of $\sigma(\ell n^2, 4D)$

Recall that the expression $\sigma(\ell n^2, 4D)$ is by definition the sum

$$\sum \frac{\mu(a)\chi_{-\ell n^2}(a)}{ab},$$

where μ is the Möbius function, and the summation is over all the positive integers a, b that are both coprime to $4D$ and satisfy $ab \mid n$. Note that $\chi_{-\ell n^2} = \chi_{-\ell}$ for any n.

Lemma 4.1. *The function*

$$F(\ell, n) := \sigma\bigl(\ell n^2, 4D\bigr)$$

is multiplicative in n for n being coprime to $4D$.

Proof. Since $(n, 4D) = 1$, we can write $F(\ell, n)$ as

$$F(\ell, n) = \sum_{d|n} \frac{1}{d} \sum_{a|d} \mu(a)\chi_{-\ell}(a).$$

Notice that $\mu(a)\chi_{-\ell}(a)$ is multiplicative in a, thus $\sum_{a|d} \mu(a)\chi_{-\ell}(a)$ is multiplicative in d, then $\frac{1}{d}\sum_{a|d} \mu(a)\chi_{-\ell}(a)$ is also multiplicative in d, and finally $F(\ell, n)$ is multiplicative in n. $\square$

Lemma 4.2. *Let u be the square-free part of ℓ. Then*

$$\sigma\bigl(\ell n^2, 4D\bigr) = \frac{1}{n}\biggl(\prod_{p|4D} p^{e_p}\biggr) \cdot \biggl(\prod_{p\nmid 4D}\biggl(\frac{p^{e_p+1}-1}{p-1} - \biggl(\frac{-u}{p}\biggr) \cdot \frac{p^{e_p}-1}{p-1}\biggr)\biggr).$$

Proof. Due to Lemma 4.1, to compute $F(\ell, n)$ we need only calculate the value of F at a power of a prime, say p^e, $e \geqslant 0$, which is coprime to $4D$. Note that $\mu(p^i) = 0$ for $i \geqslant 2$. We have

$$\begin{aligned}
F\bigl(\ell, p^e\bigr) &= \sum_{k=0}^{e} \frac{1}{p^k} \sum_{i=0}^{k} \mu\bigl(p^i\bigr)\chi_{-\ell}\bigl(p^i\bigr) \\
&= \sum_{i=0}^{e} \mu\bigl(p^i\bigr)\chi_{-\ell}\bigl(p^i\bigr) \sum_{k=i}^{e} \frac{1}{p^k} \\
&= \mu(1)\chi_{-\ell}(1) \sum_{k=0}^{e} \frac{1}{p^k} + \mu(p)\chi_{-\ell}(p) \sum_{k=1}^{e} \frac{1}{p^k} \\
&= \frac{1-\frac{1}{p^{e+1}}}{1-\frac{1}{p}} - \chi_{-\ell}(p) \cdot \frac{\frac{1}{p}-\frac{1}{p^{e+1}}}{1-\frac{1}{p}} \\
&= \frac{1}{p^e}\biggl(\frac{p^{e+1}-1}{p-1} - \chi_{-\ell}(p) \cdot \frac{p^e-1}{p-1}\biggr).
\end{aligned}$$

Put $\ell' = \ell \cdot \prod_{p|4D} p^{2e_p}$. Then

$$\begin{aligned}
F(\ell, n) &= F\biggl(\ell', \prod_{p\nmid 4D} p^{e_p}\biggr) = \prod_{p\nmid 4D} F\bigl(\ell', p^{e_p}\bigr) \\
&= \frac{1}{n}\biggl(\prod_{p|4D} p^{e_p}\biggr) \cdot \biggl(\prod_{p\nmid 4D}\biggl(\frac{p^{e_p+1}-1}{p-1} - \chi_{-\ell'}(p) \cdot \frac{p^{e_p}-1}{p-1}\biggr)\biggr).
\end{aligned}$$

Finally, note that $\chi_{-\ell'}(p) = (\frac{-u}{p})$ for any prime $p \nmid 4D$. $\square$

4.2. The computation of $\lambda(\ell n^2, 4D)$

By definition,

$$\lambda\bigl(\ell n^2, 4D\bigr) = \frac{L_{4D}(1, \chi_{-\ell})}{L_{4D}(2, \mathrm{id})} \cdot \sigma\bigl(\ell n^2, 4D\bigr),$$

where id is the trivial character mod $4D$. For the calculation of $\frac{L_{4D}(1,\chi_{-\ell})}{L_{4D}(2,\mathrm{id})}$, the following lemma is critical.

Lemma 4.3. *Suppose D is an odd prime.*

(i) *If* id *is the trivial character* mod $4D$, *then*

$$L_{4D}(2, \mathrm{id}) = \frac{3(D^2-1)}{4D^2}\zeta(2),$$

where $\zeta(s)$ is the classical Riemann-zeta function.

(ii) *If ω is a nontrivial Dirichlet character, then*

$$L_{4D}(1, \omega) = \biggl(1 - \frac{\omega(2)}{2} - \frac{\omega(D)}{D} + \frac{\omega(2D)}{2D}\biggr)L(1, \omega).$$

Therefore

$$\frac{L_{4D}(1,\omega)}{L_{4D}(2,\mathrm{id})} = \frac{4D^2(1-\frac{\omega(2)}{2}-\frac{\omega(D)}{D}+\frac{\omega(2D)}{2D})}{3(D^2-1)} \cdot \frac{L(1,\omega)}{\zeta(2)}.$$

Proof. (i) Since id is induced by the trivial character mod 1, we have

$$L_{4D}(2,\mathrm{id}) = L(2,\mathrm{id}) = \zeta(2) \cdot \prod_{p|4D}\left(1-\frac{1}{p^2}\right),$$

as asserted.

(ii) We write $L(1,\omega)$ as

$$L(1,\omega) = \sum_{\substack{k=1\\(k,4D)=1}}^{\infty} \frac{\omega(k)}{k} + \sum_{\substack{k=1\\(k,4D)\neq 1}}^{\infty} \frac{\omega(k)}{k},$$

where the first additive item is exactly $L_{4D}(1,\omega)$, while the second is

$$\begin{aligned}\sum_{\substack{k=1\\(k,4D)\neq 1}}^{\infty} \frac{\omega(k)}{k} &= \sum_{\substack{k=1\\2|k}}^{\infty} \frac{\omega(k)}{k} + \sum_{\substack{k=1\\D|k}}^{\infty} \frac{\omega(k)}{k} - \sum_{\substack{k=1\\2D|k}}^{\infty} \frac{\omega(k)}{k}\\ &= \left(\frac{\omega(2)}{2}+\frac{\omega(D)}{D}-\frac{\omega(2D)}{2D}\right)L(1,\omega).\end{aligned}$$

Then the result follows. □

By the above lemma, in order to compute $\frac{L_{4D}(1,\chi_{-\ell})}{L_{4D}(2,\mathrm{id})}$, it suffices to compute the value of $L(1,\chi_{-\ell})$, which is related to the notion of *Gauss sum*. Recall that the Gauss sum of a Dirichlet character ω with its conductor d is defined to be

$$\tau(\omega) := \sum_{k=1}^{d} \omega(k) e^{\frac{2\pi i k}{d}}.$$

The following lemma can be found in [22, Theorem 4.9]:

Lemma 4.4. *If ω is an odd Dirichlet character with its conductor d, then*

$$L(1,\omega) = \frac{\pi i \cdot \tau(\omega)}{d^2} \sum_{k=1}^{d} \overline{\omega(k)} k,$$

where $^-$ is the usual conjugation of a complex number.

Now we can calculate the value of $\lambda(\ell n^2, D)$:

Proposition 4.5. *Let n be a positive integer with prime factorization given by* (1)*.*

(i) *For $\ell = 1$ and $D = 1, 3, 5$,*

$$\lambda\left(n^2, 4\right) = \frac{2}{\pi}\sigma\left(n^2, 4\right) = \frac{2^{e_2+1}}{\pi n}\prod_{p\nmid 2}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-1}{p}\right)\frac{p^{e_p}-1}{p-1}\right),$$

$$\lambda\left(n^2, 12\right) = \frac{3}{\pi}\sigma\left(n^2, 12\right) = \frac{2^{e_2}3^{e_3+1}}{\pi n}\prod_{p\nmid 6}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-1}{p}\right)\frac{p^{e_p}-1}{p-1}\right),$$

$$\lambda\left(n^2, 20\right) = \frac{5}{3\pi}\sigma\left(n^2, 20\right) = \frac{2^{e_2}5^{e_5+1}}{3\pi n}\prod_{p\nmid 10}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-1}{p}\right)\frac{p^{e_p}-1}{p-1}\right);$$

(ii) *For $\ell = 2$ and $D = 1, 3$,*

$$\lambda\left(2n^2, 4\right) = \frac{2\sqrt{2}}{\pi}\sigma\left(2n^2, 4\right) = \frac{2^{e_2+1}\sqrt{2}}{\pi n}\prod_{p\nmid 2}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-2}{p}\right)\frac{p^{e_p}-1}{p-1}\right),$$

$$\lambda\left(2n^2, 12\right) = \frac{3\sqrt{2}}{2\pi}\sigma\left(2n^2, 12\right) = \frac{2^{e_2-1}3^{e_3+1}\sqrt{2}}{\pi n}\prod_{p\nmid 6}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-2}{p}\right)\frac{p^{e_p}-1}{p-1}\right);$$

(iii) *For $\ell = 3$ and $D = 1, 3$,*

$$\lambda\left(3n^2, 4\right) = \frac{4\sqrt{3}}{3\pi}\sigma\left(3n^2, 4\right) = \frac{2^{e_2+2}\sqrt{3}}{3\pi n}\prod_{p\nmid 2}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-3}{p}\right)\frac{p^{e_p}-1}{p-1}\right),$$

$$\lambda\left(3n^2, 12\right) = \frac{3\sqrt{3}}{2\pi}\sigma\left(3n^2, 12\right) = \frac{2^{e_2-1}3^{e_3+1}\sqrt{3}}{\pi n}\prod_{p\nmid 6}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-3}{p}\right)\frac{p^{e_p}-1}{p-1}\right);$$

(iv) *For $\ell = 5$ and $D = 5$,*

$$\lambda\left(5n^2, 20\right) = \frac{5\sqrt{5}}{3\pi}\sigma\left(5n^2, 20\right) = \frac{2^{e_2}5^{e_5+1}\sqrt{5}}{3\pi n}\prod_{p\nmid 10}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-5}{p}\right)\frac{p^{e_p}-1}{p-1}\right);$$

(v) *For $\ell = 6$ and $D = 3$,*

$$\lambda\left(6n^2, 12\right) = \frac{3\sqrt{6}}{2\pi}\sigma\left(6n^2, 12\right) = \frac{2^{e_2-1}3^{e_3+1}\sqrt{6}}{\pi n}\prod_{p\nmid 6}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-6}{p}\right)\frac{p^{e_p}-1}{p-1}\right).$$

Proof. (i) The associated Gauss sum of $\chi_{-1} = (\frac{-4}{\bullet})$ is

$$\tau(\chi_{-1}) = \chi_{-1}(1)\mathrm{e}^{\frac{\pi \mathrm{i}}{2}} + \chi_{-1}(3)\mathrm{e}^{\frac{3\pi \mathrm{i}}{2}} = \mathrm{e}^{\frac{\pi \mathrm{i}}{2}} - \mathrm{e}^{\frac{3\pi \mathrm{i}}{2}} = 2i,$$

and hence

$$L(1,\chi_{-1}) = \frac{\pi \mathrm{i} \cdot \tau(\chi_{-1})}{4^2}\big[\chi_{-1}(1) + \chi_{-1}(3)3\big] = \frac{\pi}{4}.$$

It is clear that $L_4(2,\mathrm{id}) = \frac{\pi^2}{8}$ and $L_4(1,\chi_{-1}) = L(1,\chi_{-1}) = \frac{\pi}{4}$, so

$$\frac{L_4(1,\chi_{-1})}{L_4(2,\mathrm{id})} = \frac{2}{\pi}.$$

Thanks to Lemma 4.3, we get

$$\frac{L_{12}(1,\chi_{-1})}{L_{12}(2,\mathrm{id})} = \frac{3}{\pi} \quad \text{and} \quad \frac{L_{20}(1,\chi_{-1})}{L_{20}(2,\mathrm{id})} = \frac{5}{3\pi}.$$

(ii) The associated Gauss sum of $\chi_{-2} = (\frac{-8}{\bullet})$ is

$$\begin{aligned}\tau(\chi_{-2}) &= \chi_{-2}(1)e^{\frac{\pi i}{4}} + \chi_{-2}(3)e^{\frac{3\pi i}{4}} + \chi_{-2}(5)e^{\frac{5\pi i}{4}} + \chi_{-2}(7)e^{\frac{7\pi i}{4}} \\ &= e^{\frac{\pi i}{4}} + e^{\frac{3\pi i}{4}} - e^{\frac{5\pi i}{4}} - e^{\frac{7\pi i}{4}} = 2\sqrt{2}i,\end{aligned}$$

and so

$$\begin{aligned}L_4(1,\chi_{-2}) &= L(1,\chi_{-2}) \\ &= \frac{\pi i \cdot \tau(\chi_{-2})}{8^2}\big[\chi_{-2}(1) + \chi_{-2}(3)3 + \chi_{-2}(5)5 + \chi_{-2}(7)7\big] \\ &= \frac{\sqrt{2}\pi}{4}.\end{aligned}$$

Hence $\frac{L_4(1,\chi_{-2})}{L_4(2,\mathrm{id})} = \frac{2\sqrt{2}}{\pi}$. By Lemma 4.3, we have

$$\frac{L_{12}(1,\chi_{-2})}{L_{12}(2,\mathrm{id})} = \frac{3\sqrt{2}}{2\pi}.$$

(iii) The associated Gauss sum of $\chi_{-3} = (\frac{-3}{\bullet})$ is

$$\begin{aligned}\tau(\chi_{-3}) &= \chi_{-3}(1)e^{\frac{2\pi i}{3}} + \chi_{-3}(2)e^{\frac{4\pi i}{3}} \\ &= e^{\frac{2\pi i}{3}} - e^{\frac{4\pi i}{3}} = \sqrt{3}i,\end{aligned}$$

and so

$$L(1,\chi_{-3}) = \frac{\pi i \cdot \tau(\chi_{-3})}{3^2}\big[\chi_{-3}(1) + \chi_{-3}(2)2\big] = \frac{\sqrt{3}\pi}{9}.$$

Henceforth,

$$L_4(1,\chi_{-3}) = \left(1 - \frac{\chi_{-3}(2)}{2}\right) L(1,\chi_{-3}) = \frac{\sqrt{3}\pi}{6},$$

and then $\frac{L_4(1,\chi_{-3})}{L_4(2,\mathrm{id})} = \frac{4\sqrt{3}}{3\pi}$.

By Lemma 4.3,

$$\frac{L_{12}(1,\chi_{-3})}{L_{12}(2,\mathrm{id})} = \frac{3\sqrt{3}}{2\pi}.$$

(iv) The associated Gauss sum of $\chi_{-5} = (\frac{-20}{\bullet})$ is

$$\begin{aligned}
\tau(\chi_{-5}) &= \sum_{k=1}^{20} \chi_{-5}(k) e^{\frac{\pi i k}{10}} \\
&= \chi_{-5}(1) e^{\frac{\pi i}{10}} + \chi_{-5}(3) e^{\frac{3\pi i}{10}} + \chi_{-5}(7) e^{\frac{7\pi i}{10}} + \chi_{-5}(9) e^{\frac{9\pi i}{10}} \\
&\quad + \chi_{-5}(11) e^{\frac{11\pi i}{10}} + \chi_{-5}(13) e^{\frac{13\pi i}{10}} + \chi_{-5}(17) e^{\frac{17\pi i}{10}} + \chi_{-5}(19) e^{\frac{19\pi i}{10}} \\
&= e^{\frac{\pi i}{10}} + e^{\frac{3\pi i}{10}} + e^{\frac{7\pi i}{10}} + e^{\frac{9\pi i}{12}} - e^{\frac{11\pi i}{10}} - e^{\frac{13\pi i}{10}} - e^{\frac{17\pi i}{10}} - e^{\frac{19\pi i}{10}} \\
&= 2\left(e^{\frac{\pi i}{10}} + e^{\frac{3\pi i}{10}} + e^{\frac{7\pi i}{10}} + e^{\frac{9\pi i}{10}}\right) \\
&= 4i\left(\sin\frac{\pi}{10} + \sin\frac{3\pi}{10}\right) = 2\sqrt{5} i.
\end{aligned}$$

Therefore

$$L(1,\chi_{-5}) = \frac{\pi i \cdot \tau(\chi_{-5})}{20^2} \sum_{k=1}^{20} \chi_{-5}(k) k = \frac{\sqrt{5}\pi}{5}.$$

Then by Lemma 4.3,

$$\frac{L_{20}(1,\chi_{-5})}{L_{20}(2,\mathrm{id})} = \frac{5\sqrt{5}}{3\pi}.$$

(v) The associated Gauss sum of $\chi_{-6} = (\frac{-24}{\bullet})$ is

$$\begin{aligned}
\tau(\chi_{-6}) &= \sum_{k=1}^{24} \chi_{-6}(k) e^{\frac{\pi i k}{12}} \\
&= \chi_{-6}(1) e^{\frac{\pi i}{12}} + \chi_{-6}(5) e^{\frac{5\pi i}{12}} + \chi_{-6}(7) e^{\frac{7\pi i}{12}} + \chi_{-6}(11) e^{\frac{11\pi i}{12}} \\
&\quad + \chi_{-6}(13) e^{\frac{13\pi i}{12}} + \chi_{-6}(17) e^{\frac{17\pi i}{12}} + \chi_{-6}(19) e^{\frac{19\pi i}{12}} + \chi_{-6}(23) e^{\frac{23\pi i}{12}} \\
&= e^{\frac{\pi i}{12}} + e^{\frac{5\pi i}{12}} + e^{\frac{7\pi i}{12}} + e^{\frac{11\pi i}{12}} - e^{\frac{13\pi i}{12}} - e^{\frac{17\pi i}{12}} - e^{\frac{19\pi i}{12}} - e^{\frac{23\pi i}{12}} \\
&= 2\left(e^{\frac{\pi i}{12}} + e^{\frac{5\pi i}{12}} + e^{\frac{7\pi i}{12}} + e^{\frac{11\pi i}{12}}\right) \\
&= 4i\left(\sin\frac{\pi}{12} + \sin\frac{5\pi}{12}\right) = 2\sqrt{6} i,
\end{aligned}$$

and so

$$L(1,\chi_{-6}) = \frac{\pi i\cdot\tau(\chi_{-6})}{24^2}\sum_{k=1}^{24}\chi_{-6}(k)k = \frac{\sqrt{6}\pi}{6}.$$

By Lemma 4.3, we get

$$\frac{L_{12}(1,\chi_{-6})}{L_{12}(2,\mathrm{id})} = \frac{3\sqrt{6}}{2\pi}. \qquad \square$$

4.3. The computation of $\alpha(\ell n^2)$ and $A(p,\ell n^2)$

We get by definition that

$$\alpha\big(n^2\big) = \alpha\big(2n^2\big) = \alpha\big(5n^2\big) = \alpha\big(6n^2\big) = \frac{3}{2^{e_2+1}}$$

and $\alpha(3n^2) = \frac{1}{2^{e_2}}$; moreover, if $8 \mid n^2$, i.e., $e_2 \geqslant 2$, then $\alpha(\frac{n^2}{8}) = \frac{3}{2^{e_2-1}}$.

For $A(p,\ell n^2)$, we have

$$A\big(3,n^2\big) = \frac{1}{3} - \frac{2}{3^{e_3+1}}, \qquad A\big(3,2n^2\big) = \frac{1}{3}, \qquad A\big(3,3n^2\big) = A\big(3,6n^2\big) = \frac{1}{3} - \frac{4}{3^{e_3+2}}$$

and

$$A\big(5,5n^2\big) = \frac{1}{5} - \frac{6}{5^{e_5+2}}.$$

4.4. Explicit formulas

Now we have been sufficiently equipped to derive explicit formulas for the representation number of a square by quadratic forms of type $ax^2+by^2+cz^2$, where a, b, c are small positive integers. We denote by $H_{(a,b,c)}(n)$ the following product

$$H_{(a,b,c)}(n) = \prod_{p\nmid 2abc}\left(\frac{p^{e_p+1}-1}{p-1} - \left(\frac{-abc}{p}\right)\frac{p^{e_p}-1}{p-1}\right).$$

Theorem 4.6. *Let n be a positive integer with prime factorization given by* (1)*. Then the representation numbers $r_{(a,b,c)}(n^2)$ are listed in Table 4 which accounts for all of the results in Table 3 as well as the results* $(2,2,3)$*,* $(2,3,3)$ *and* $(2,3,6)$*.*

Proof. We carry out the computation case by case as follows.

Case $(a,b,c) = (1,1,1)$. Since $\alpha(n^2) = \frac{3}{2^{e_2+1}}$, we have

$$r_{(1,1,1)}\big(n^2\big) = 2\pi\sqrt{n^2}\cdot\lambda\big(n^2,4\big)\cdot\alpha\big(n^2\big)$$

Table 4
The representation numbers $r_{(a,b,c)}(n^2)$.

(a,b,c)	$S_{(a,b,c)}(n^2)$
$(1,1,1)$	$6H_{(1,1,1)}(n)$
$(1,1,2)$	$4H_{(1,1,2)}(n)$, if n is odd, $12H_{(1,1,2)}(n)$, if n is even
$(1,1,3)$	$4(2^{e_2+1}-1)H_{(1,1,3)}(n)$
$(1,1,4)$	$4H_{(1,1,4)}(n)$, if n is odd, $6H_{(1,1,4)}(n)$, if n is even
$(1,1,5)$	$2(5^{e_5+1}-3)H_{(1,1,5)}(n)$
$(1,1,6)$	$4H_{(1,1,6)}(n)$, if n is odd, $4(2^{e_2+1}-3)H_{(1,1,6)}(n)$, if n is even
$(1,1,8)$	$4H_{(1,1,8)}(n)$, if n is odd, $4H_{(1,1,8)}(n)$, if $n \equiv 2 \pmod 4$, $12H_{(1,1,8)}(n)$, if $n \equiv 0 \pmod 4$
$(1,2,2)$	$2H_{(1,2,2)}(n)$, if n is odd, $6H_{(1,2,2)}(n)$, if n is even
$(1,2,3)$	$2(3^{e_3+1}-2)H_{(1,2,3)}(n)$, if n is odd, $6(3^{e_3+1}-2)H_{(1,2,3)}(n)$, if n is even
$(1,2,4)$	$2H_{(1,2,4)}(n)$, if n is odd, $4H_{(1,2,4)}(n)$, if $n \equiv 2 \pmod 4$, $12H_{(1,2,4)}(n)$, if $n \equiv 0 \pmod 4$
$(1,2,6)$	$2(3^{e_3+1}-2)H_{(1,2,6)}(n)$
$(1,3,3)$	$2(2^{e_2+2}-3)H_{(1,3,3)}(n)$
$(1,3,6)$	$2\cdot 3^{e_3}H_{(1,3,6)}(n)$, if n is odd, $2\cdot 3^{e_3+1}H_{(1,3,6)}(n)$, if n is even
$(1,4,4)$	$2H_{(1,4,4)}(n)$, if n is odd, $6H_{(1,4,4)}(n)$, if n is even
$(1,4,8)$	$2H_{(1,4,8)}(n)$, if n is odd, $4H_{(1,4,8)}(n)$, if $n \equiv 2 \pmod 4$, $12H_{(1,4,8)}(n)$, if $n \equiv 0 \pmod 4$
$(1,5,5)$	$2\cdot 5^{e_5}H_{(1,5,5)}(n)$
$(1,6,6)$	$2H_{(1,6,6)}(n)$, if n is odd, $2(2^{e_2+1}-3)H_{(1,6,6)}(n)$, if n is even
$(2,2,3)$	$4(2^{e_2}-1)H_{(2,2,3)}(n)$
$(2,3,3)$	0
$(2,3,6)$	$2(3^{e_3}-1)H_{(2,3,6)}(n)$, if n is odd, $6(3^{e_3}-1)H_{(2,3,6)}(n)$, if n is even

$$= 2\pi n \cdot \frac{2^{e_2+1}}{\pi n} H_{(1,1,1)}(n) \cdot \frac{3}{2^{e_2+1}}$$
$$= 6H_{(1,1,1)}(n).$$

Case $(a,b,c) = (1,1,4)$. Note that

$$2\pi\sqrt{n^2} \cdot \lambda\left(n^2, 4\right) = 2\pi n \cdot \frac{2^{e_2+1}}{\pi n} H_{(1,1,4)}(n) = 2^{e_2+2} H_{(1,1,4)}(n).$$

If n is odd, then

$$r_{(1,1,4)}\bigl(n^2\bigr) = 2\pi\sqrt{n^2}\cdot\lambda\bigl(n^2,4\bigr) = 4H_{(1,1,4)}(n);$$

otherwise,

$$r_{(1,1,4)}\bigl(n^2\bigr) = 2\pi\sqrt{n^2}\cdot\lambda\bigl(n^2,4\bigr)\cdot\alpha\bigl(n^2\bigr) = 2^{e_2+2}H_{(1,1,4)}(n)\cdot\frac{3}{2^{e_2+1}} = 6H_{(1,1,4)}(n).$$

Case $(a,b,c)=(1,2,2)$. Similarly,

$$2\pi\sqrt{n^2}\cdot\lambda\bigl(n^2,4\bigr) = 2^{e_2+2}H_{(1,2,2)}(n).$$

If n is odd, then

$$r_{(1,2,2)}\bigl(n^2\bigr) = 2\pi\sqrt{n^2}\cdot\lambda\bigl(n^2,4\bigr)\cdot\bigl(\alpha\bigl(n^2\bigr)-1\bigr) = 4H_{(1,2,2)}(n)\cdot\frac{1}{2} = 2H_{(1,2,2)}(n);$$

otherwise,

$$r_{(1,2,2)}\bigl(n^2\bigr) = 2\pi\sqrt{n^2}\cdot\lambda\bigl(n^2,4\bigr)\cdot\alpha\bigl(n^2\bigr) = 2^{e_2+2}H_{(1,2,2)}(n)\cdot\frac{3}{2^{e_2+1}} = 6H_{(1,2,2)}(n).$$

Case $(a,b,c)=(1,1,2)$. Note that $\alpha(2n^2)=\frac{3}{2^{e_2+1}}$ and

$$2\pi\sqrt{2n^2}\cdot\lambda\bigl(2n^2,4\bigr) = 2\sqrt{2}\pi n\cdot\frac{2^{e_2+1}\sqrt{2}}{\pi n}H_{(1,1,2)}(n) = 2^{e_2+3}H_{(1,1,2)}(n).$$

If n is odd, then

$$r_{(1,1,2)}\bigl(n^2\bigr) = 2\pi\sqrt{2n^2}\cdot\lambda\bigl(2n^2,4\bigr)\cdot\bigl(\alpha\bigl(2n^2\bigr)-1\bigr) = 8H_{(1,1,2)}(n)\cdot\frac{1}{2} = 4H_{(1,1,2)}(n);$$

otherwise,

$$r_{(1,1,2)}\bigl(n^2\bigr) = 2\pi\sqrt{2n^2}\cdot\lambda\bigl(2n^2,4\bigr)\cdot\alpha\bigl(2n^2\bigr) = 2^{e_2+3}H_{(1,1,2)}(n)\cdot\frac{3}{2^{e_2+1}} = 12H_{(1,2,2)}(n).$$

Case $(a,b,c)=(1,2,4)$. Note that $2\alpha(2n^2)=\frac{3}{2^{e_2}}$ and

$$\pi\sqrt{2n^2}\cdot\lambda\bigl(2n^2,4\bigr) = \sqrt{2}\pi n\cdot\frac{2^{e_2+1}\sqrt{2}}{\pi n}H_{(1,2,4)}(n) = 2^{e_2+2}H_{(1,2,4)}(n).$$

If n is odd, then

$$r_{(1,2,4)}\bigl(n^2\bigr) = \pi\sqrt{2n^2}\cdot\lambda\bigl(2n^2,4\bigr)\cdot\left(2\alpha\bigl(2n^2\bigr)-\frac{5}{2}\right) = 4H_{(1,2,4)}(n)\cdot\frac{1}{2} = 2H_{(1,2,4)}(n);$$

if $n\equiv 2 \pmod 4$, then

$$r_{(1,2,4)}\big(n^2\big) = \pi\sqrt{2n^2} \cdot \lambda\big(2n^2, 4\big) \cdot \big(2\alpha\big(2n^2\big) - 1\big) = 8H_{(1,2,4)}(n) \cdot \frac{1}{2} = 4H_{(1,2,4)}(n);$$

and if $n \equiv 0 \pmod 4$, then

$$r_{(1,2,4)}\big(n^2\big) = \pi\sqrt{2n^2} \cdot \lambda\big(2n^2, 4\big) \cdot 2\alpha\big(2n^2\big) = 2^{e_2+2}H_{(1,2,4)}(n) \cdot \frac{3}{2^{e_2}} = 12H_{(1,2,4)}(n).$$

Case $(a, b, c) = (1, 4, 4)$. Note that

$$\pi\sqrt{n^2} \cdot \lambda\big(n^2, 4\big) = \pi n \cdot \frac{2^{e_2+1}}{\pi n} H_{(1,4,4)}(n) = 2^{e_2+1}H_{(1,4,4)}(n).$$

If n is odd, then

$$r_{(1,4,4)}\big(n^2\big) = \pi\sqrt{n^2} \cdot \lambda\big(n^2, 4\big) = 2H_{(1,4,4)}(n);$$

otherwise,

$$r_{(1,4,4)}\big(n^2\big) = 2\pi\sqrt{n^2} \cdot \lambda\big(n^2, 4\big) \cdot \alpha\big(n^2\big) = 2 \cdot 2^{e_2+1}H_{(1,4,4)}(n) \cdot \frac{3}{2^{e_2+1}} = 6H_{(1,4,4)}(n).$$

Case $(a, b, c) = (1, 1, 8)$. We have

$$\pi\sqrt{n^2} \cdot \lambda\big(2n^2, 4\big) = \pi n \cdot \frac{2^{e_2+1}\sqrt{2}}{\pi n} H_{(1,1,8)}(n) = 2^{e_2+1}\sqrt{2}H_{(1,1,8)}(n)$$

and $\alpha(\frac{n^2}{8}) = \frac{3}{2^{e_2-1}}$ when $e_2 \geqslant 2$.

If n is odd, then $n^2 \equiv 1 \pmod 4$, thus

$$r_{(1,1,8)}\big(n^2\big) = \sqrt{2}\pi\sqrt{n^2} \cdot \lambda\big(2n^2, 4\big) = 4H_{(1,1,8)}(n).$$

If $4 \mid n$, i.e., $e_2 \geqslant 2$, then $8 \mid n^2$, thus

$$\begin{aligned} r_{(1,1,8)}\big(n^2\big) &= \frac{1}{\sqrt{2}}\pi\sqrt{n^2} \cdot \lambda\big(2n^2, 4\big) \cdot \alpha\left(\frac{n^2}{8}\right) \\ &= 2^{e_2+1}H_{(1,1,8)}(n) \cdot \frac{3}{2^{e_2-1}} \\ &= 12H_{(1,1,8)}(n). \end{aligned}$$

If $n \equiv 2 \pmod 4$, i.e., $e_2 = 1$, then $n^2 \equiv 4 \pmod 8$, thus

$$r_{(1,1,8)}\big(n^2\big) = \frac{1}{\sqrt{2}}\pi\sqrt{n^2} \cdot \lambda\big(2n^2, 4\big) = 4H_{(1,1,8)}(n).$$

Case $(a, b, c) = (1, 4, 8)$. If n is odd, then $n^2 \equiv 1 \pmod 4$, thus

$$r_{(1,4,8)}\big(n^2\big) = \frac{1}{\sqrt{2}}\pi\sqrt{n^2} \cdot \lambda\big(2n^2, 4\big) = 2H_{(1,4,8)}(n).$$

If $4 \mid n$, then $8 \mid n^2$, thus

$$\begin{aligned} r_{(1,4,8)}\big(n^2\big) &= \frac{1}{\sqrt{2}}\pi\sqrt{n^2}\cdot\lambda\big(2n^2,4\big)\cdot\alpha\bigg(\frac{n^2}{8}\bigg) \\ &= 2^{e_2+1}H_{(1,4,8)}(n)\cdot\frac{3}{2^{e_2-1}} \\ &= 12H_{(1,4,8)}(n). \end{aligned}$$

If $n\equiv 2 \pmod 4$, then $n^2\equiv 4 \pmod 8$, hence

$$r_{(1,4,8)}\big(n^2\big) = \frac{1}{\sqrt{2}}\pi\sqrt{n^2}\cdot\lambda\big(2n^2,4\big) = 4H_{(1,4,8)}(n).$$

Case $(a,b,c)=(1,3,3)$.

$$\begin{aligned} r_{(1,3,3)}\big(n^2\big) &= 2\pi\sqrt{n^2}\cdot\lambda\big(n^2,12\big)\cdot\bigg(\frac{1}{3}-A\big(3,n^2\big)\bigg)\cdot\big(2-\alpha\big(n^2\big)\big) \\ &= 2\pi n\cdot\frac{2^{e_2}3^{e_3+1}}{\pi n}H_{(1,3,3)}(n)\cdot\frac{2}{3^{e_3+1}}\cdot\bigg(2-\frac{3}{2^{e_2+1}}\bigg) \\ &= 2\big(2^{e_2+2}-3\big)H_{(1,3,3)}(n). \end{aligned}$$

Case $(a,b,c)=(1,1,3)$.

$$\begin{aligned} r_{(1,1,3)}\big(n^2\big) &= 2\pi\sqrt{3n^2}\cdot\lambda\big(3n^2,12\big)\cdot\bigg(\frac{1}{3}-A\big(3,3n^2\big)\bigg)\cdot\big(2-\alpha\big(3n^2\big)\big) \\ &= 2\sqrt{3}\pi n\cdot\frac{2^{e_2-1}3^{e_3+1}\sqrt{3}}{\pi n}H_{(1,1,3)}(n)\cdot\frac{4}{3^{e_3+2}}\cdot\bigg(2-\frac{1}{2^{e_2}}\bigg) \\ &= 4\big(2^{e_2+1}-1\big)H_{(1,1,3)}(n). \end{aligned}$$

Case $(a,b,c)=(1,5,5)$.

$$\begin{aligned} r_{(1,5,5)}\big(n^2\big) &= 2\pi\sqrt{n^2}\cdot\lambda\big(n^2,20\big)\cdot\alpha\big(n^2\big)\cdot\bigg(A\big(5,n^2\big)+\frac{1}{5}\bigg) \\ &= 2\pi n\cdot\frac{2^{e_2}5^{e_5+1}}{3\pi n}H_{(1,5,5)}(n)\cdot\frac{3}{2^{e_2+1}}\cdot\frac{2}{5} \\ &= 2\cdot 5^{e_5}H_{(1,5,5)}(n). \end{aligned}$$

Case $(a,b,c)=(1,6,6)$. Note that

$$\begin{aligned} 2\pi\sqrt{n^2}\cdot\lambda\big(n^2,12\big)\cdot\bigg(\frac{1}{3}-A\big(3,n^2\big)\bigg) &= 2\pi n\cdot\frac{2^{e_2}3^{e_3+1}}{\pi n}H_{(1,6,6)}(n)\cdot\frac{2}{3^{e_3+1}} \\ &= 2^{e_2+2}H_{(1,6,6)}(n). \end{aligned}$$

Combining with $\alpha(n^2) = \frac{3}{2^{e_2+1}}$, we get

$$r_{(1,6,6)}\bigl(n^2\bigr) = \begin{cases} 2H_{(1,6,6)}(n), & \text{if } n \text{ is odd,} \\ 2(2^{e_2+1}-3)H_{(1,6,6)}(n), & \text{if } n \text{ is even.} \end{cases}$$

Case $(a,b,c) = (2,3,6)$. We have

$$\begin{aligned} 2\pi\sqrt{n^2}\cdot\lambda\bigl(n^2,12\bigr)\cdot\left(\frac{1}{3}+A\bigl(3,n^2\bigr)\right) &= 2\pi n\cdot\frac{2^{e_2}3^{e_3+1}}{\pi n}H_{(2,3,6)}(n)\cdot\left(\frac{2}{3}-\frac{2}{3^{e_3+1}}\right) \\ &= 2^{e_2+2}\bigl(3^{e_3}-1\bigr)H_{(2,3,6)}(n), \end{aligned}$$

which yields the result.

Case $(a,b,c) = (1,1,5)$.

$$\begin{aligned} r_{(1,1,5)}\bigl(n^2\bigr) &= 2\pi\sqrt{5n^2}\cdot\lambda\bigl(5n^2,20\bigr)\cdot\alpha\bigl(5n^2\bigr)\cdot\left(A\bigl(5,5n^2\bigr)+\frac{1}{5}\right) \\ &= 2\sqrt{5}\pi n\cdot\frac{2^{e_2}5^{e_5+1}\sqrt{5}}{3\pi n}H_{(1,1,5)}(n)\cdot\frac{3}{2^{e_2+1}}\cdot\left(\frac{2}{5}-\frac{6}{5^{e_5+2}}\right) \\ &= 2\bigl(5^{e_5+1}-3\bigr)H_{(1,1,5)}(n). \end{aligned}$$

Case $(a,b,c) = (2,3,3)$. Since $A(3,2n^2) = \frac{1}{3}$, we have $r_{(2,3,3)}(n^2) = 0$.

Case $(a,b,c) = (2,2,3)$. Since $n^2 \equiv 0,1 \pmod 4$, we have

$$\begin{aligned} r_{(2,2,3)}\bigl(n^2\bigr) &= 2\pi\sqrt{3n^2}\cdot\lambda\bigl(3n^2,12\bigr)\cdot\left(\frac{1}{3}-A\bigl(3,3n^2\bigr)\right)\cdot\bigl(1-\alpha\bigl(3n^2\bigr)\bigr) \\ &= 2\sqrt{3}\pi n\cdot\frac{2^{e_2-1}3^{e_3+1}\sqrt{3}}{\pi n}H_{(2,2,3)}(n)\cdot\frac{4}{3^{e_3+2}}\cdot\left(1-\frac{1}{2^{e_2}}\right) \\ &= 4\bigl(2^{e_2}-1\bigr)H_{(2,2,3)}(n). \end{aligned}$$

Case $(a,b,c) = (1,3,6)$. Note that $\alpha(2n^2) = \frac{3}{2^{e_2+1}}$ and

$$\begin{aligned} &2\pi\sqrt{2n^2}\cdot\lambda\bigl(2n^2,12\bigr)\cdot\left(\frac{1}{3}+A\bigl(3,2n^2\bigr)\right) \\ &\quad= 2\sqrt{2}\pi n\cdot\frac{2^{e_2-1}3^{e_3+1}\sqrt{2}}{\pi n}H_{(1,3,6)}(n)\cdot\frac{2}{3} \\ &\quad= 2^{e_2+2}\cdot 3^{e_3}H_{(1,3,6)}(n). \end{aligned}$$

Henceforth, if n is odd, then

$$r_{(1,3,6)}\bigl(n^2\bigr) = 4\cdot 3^{e_3}H_{(1,3,6)}(n)\cdot\left(\frac{3}{2}-1\right) = 2\cdot 3^{e_3}H_{(1,3,6)}(n);$$

and otherwise,

$$r_{(1,3,6)}\bigl(n^2\bigr) = 2^{e_2+2} \cdot 3^{e_3} H_{(1,3,6)}(n) \cdot \frac{3}{2^{e_2+1}} = 2 \cdot 3^{e_3+1} H_{(1,3,6)}(n).$$

Case $(a,b,c) = (1,2,6)$. Since $n^2 \equiv 0,1 \pmod 4$, we have

$$\begin{aligned} r_{(1,2,6)}\bigl(n^2\bigr) &= 2\pi\sqrt{3n^2} \cdot \lambda\bigl(3n^2,12\bigr) \cdot \left(\frac{1}{3} + A\bigl(3,3n^2\bigr)\right) \cdot \alpha\bigl(3n^2\bigr) \\ &= 2\sqrt{3}\pi n \cdot \frac{2^{e_2-1}3^{e_3+1}\sqrt{3}}{\pi n} H_{(1,2,6)}(n) \cdot \left(\frac{2}{3} - \frac{4}{3^{e_3+2}}\right) \cdot \frac{1}{2^{e_2}} \\ &= 2\bigl(3^{e_3+1} - 2\bigr) H_{(1,2,6)}(n). \end{aligned}$$

Case $(a,b,c) = (1,1,6)$. Since $A(3,6n^2) = \frac{1}{3} - \frac{4}{3^{e_3+2}}$, we have

$$\begin{aligned} &2\pi\sqrt{6n^2} \cdot \lambda\bigl(6n^2,12\bigr) \cdot \left(\frac{1}{3} - A\bigl(3,6n^2\bigr)\right) \\ &\quad = 2\sqrt{6}\pi n \cdot \frac{2^{e_2-1}3^{e_3+1}\sqrt{6}}{\pi n} H_{(1,1,6)}(n) \cdot \frac{4}{3^{e_3+2}} \\ &\quad = 2^{e_2+3} H_{(1,1,6)}(n). \end{aligned}$$

Then the result follows.

Case $(a,b,c) = (1,2,3)$. The result follows from

$$\begin{aligned} &2\pi\sqrt{6n^2} \cdot \lambda\bigl(6n^2,12\bigr) \cdot \left(\frac{1}{3} + A\bigl(3,6n^2\bigr)\right) \\ &\quad = 2\sqrt{6}\pi n \cdot \frac{2^{e_2-1}3^{e_3+1}\sqrt{6}}{\pi n} H_{(1,2,3)}(n) \cdot \left(\frac{2}{3} - \frac{4}{3^{e_3+2}}\right) \\ &\quad = 2^{e_2+2}\bigl(3^{e_3+1} - 2\bigr) H_{(1,2,3)}(n). \qquad \square \end{aligned}$$

5. The Hecke operators

We use the same notations as in the introduction. Let $f(x,y,z) = ax^2 + by^2 + cz^2 + ryz + szx + txy$ be a ternary quadratic form with integer coefficients. It is well known that the associated q-series $\theta_f(z)$ is a modular form in the space $\mathscr{M}(N, \chi_{2\det(A)})$, where

$$A = \begin{pmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{pmatrix}.$$

Let $\mathfrak{G}$ be the genus containing the ternary quadratic form Q. Recall that the *mass* of $\mathfrak{G}$ is by definition

$$M(\mathfrak{G}) := \sum_{Q \in \mathfrak{G}} \frac{1}{|\mathrm{Aut}(Q)|},$$

where the sum is over a complete representative system of equivalence classes of forms in $\mathfrak{G}$. Then the *theta series associated to* $\mathfrak{G}$ is defined to be

$$\theta_{\mathfrak{G}}(z) := \frac{1}{M(\mathfrak{G})} \sum_{Q \in \mathfrak{G}} \frac{\theta_Q(z)}{|\mathrm{Aut}(Q)|},$$

where $\theta_Q(z)$ is the usual theta series associated to the form Q.

It is shown in [16] that if Q is ternary form with discriminant d and level N then the function $\theta_{\mathrm{gen}(Q)}(z)$ is in the space $\mathscr{E}(N, \chi_d)$, and $\theta_{\mathrm{gen}(Q)}(z)$ has the same values as $\theta_Q(z)$ does at all cusps.

By [21, Theorem 10.2] we have the following result about the Hecke operator T_{p^2} on the space $\mathscr{M}(N, \chi_{2\det(A)})$:

Proposition 5.1. *If the class number of* $Q(x, y, z)$ *is* 1, *then for any prime* $p \nmid N$,

$$T_{p^2}\big(\theta_Q(z)\big) = (p+1)\theta_Q(z).$$

All the ternary diagonal quadratic forms of class number 1 have been listed on the web page http://www.kobepharma-u.ac.jp/~math/notes/note03.html, including all the cases $x^2 + by^2 + cz^2$ with b, c taking the values in Table 2. So Conjecture 1.5 is true. Moreover, Conjecture 1.5 is complete, that is, Table 2 contains exactly all the cases (b, c) such that

$$T_{p^2}\big(\theta_{x^2+by^2+cz^2}(z)\big) = (p+1)\theta_{x^2+by^2+cz^2}(z)$$

for any prime $p \nmid 2bc$.

6. Two conjectures of Sun

In this section we illustrate, by solving two conjectures of [20], that Pei's analytic formulas can be used to derive explicit formulas for representation numbers of other types of integers.

6.1. $S_{(1,1,3)}(p)$

Let $p \geqslant 5$ be a prime. Consider the representation number $S_{(1,1,3)}(p)$ which is contained in Section 3 of this paper,

$$S_{(1,1,3)}(p) = 2\pi\sqrt{3p} \cdot \lambda(3p, 12) \cdot \left(\frac{1}{3} - A(3, 3p)\right) \cdot \big(2 - \alpha(3p)\big).$$

It is clear that

$$\sigma(3p,12)=1,\qquad \frac{1}{3}-A(3,3p)=\frac{4}{9}$$

and

$$\alpha(3p)=\begin{cases}1, & \text{if } p\equiv 1 \pmod 8,\\ 0, & \text{if } p\equiv 5 \pmod 8,\\ 3/2, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

From Lemma 4.3 it follows that

$$\lambda(3p,12)=\frac{L_{12}(1,\chi_{-3p})}{L_{12}(2,\mathrm{id})}=\frac{4\cdot 3^2(1-\frac{\chi_{-3p}(2)}{2})}{3(3^2-1)}\cdot\frac{L(1,\chi_{-3p})}{\zeta(2)}$$
$$=\frac{9}{\pi^2}\cdot\left(1-\frac{\chi_{-3p}(2)}{2}\right)\cdot L(1,\chi_{-3p}).$$

Note that if $p\equiv 1 \pmod 4$, then

$$\chi_{-3p}(2)=\left(\frac{-3p}{2}\right)=\begin{cases}-1, & \text{if } p\equiv 1 \pmod 8,\\ 1, & \text{if } p\equiv 5 \pmod 8;\end{cases}$$

and if $p\equiv 3 \pmod 4$, then

$$\chi_{-3p}(2)=\left(\frac{-12p}{2}\right)=0.$$

Hence

$$1-\frac{\chi_{-3p}(2)}{2}=\begin{cases}3/2, & \text{if } p\equiv 1 \pmod 8,\\ 1/2, & \text{if } p\equiv 5 \pmod 8,\\ 1, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

So we are left to evaluate $L(1,\chi_{-3p})$. To do this we shall use the analytic class number formulas for imaginary quadratic fields.

Let d be a square-free integer and $h(d)$ the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$. Note that the discriminant of $\mathbb{Q}(\sqrt{d})$, denoted by $D(d)$, is d if $d\equiv 1 \pmod 4$ and $4d$ otherwise. The following formula can be seen on p. 424 of [11]:

Lemma 6.1. *If $d<0$, then*

$$h(d)=\frac{w\sqrt{|D(d)|}}{2\pi}\cdot L(1,\chi_{D(d)}),$$

where w is the number of roots of unity contained in $\mathbb{Q}(\sqrt{d})$.

If $p \equiv 1 \pmod 4$, then

$$D(-3p) = -3p.$$

Thus $L(1, \chi_{-3p}) = \frac{\pi h(-3p)}{\sqrt{3p}}$. Therefore $\lambda(3p, 3)$ is equal to $\frac{27h(-3p)}{2\pi\sqrt{3p}}$ if $p \equiv 1 \pmod 8$ and $\frac{9h(-3p)}{2\pi\sqrt{3p}}$ if $p \equiv 5 \pmod 8$.

If $p \equiv 3 \pmod 4$, then $D(-3p) = -12p$. Thus $L(1, \chi_{-3p}) = \frac{\pi h(-3p)}{\sqrt{12p}}$, and so $\lambda(3p, 12) = \frac{9h(-3p)}{2\pi\sqrt{3p}}$.

All told, we get

$$\lambda(3p, 12) = \nu_p \cdot \frac{h(-3p)}{\pi\sqrt{3p}},$$

where

$$\nu_p = \begin{cases} 27/2, & \text{if } p \equiv 1 \pmod 8, \\ 9/2, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} S_{(1,1,3)}(p) &= 2\pi\sqrt{3p} \cdot \lambda(3p, 12) \cdot \left(\frac{1}{3} - A(3, 3p)\right) \cdot \big(2 - \alpha(3p)\big) \\ &= 2\pi\sqrt{3p} \cdot \nu_p \cdot \frac{h(-3p)}{\pi\sqrt{3p}} \cdot \frac{4}{9} \cdot \big(2 - \alpha(3p)\big) \\ &= \frac{8h(-3p)}{9} \cdot \nu_p \cdot \big(2 - \alpha(3p)\big) \\ &= \begin{cases} 12h(-3p), & \text{if } p \equiv 1 \pmod 8, \\ 8h(-3p), & \text{if } p \equiv 5 \pmod 8, \\ 2h(-3p), & \text{if } p \equiv 3 \pmod 4. \end{cases} \end{aligned}$$

This proves Conjecture 18 of [20].

6.2. $S_{(1,1,3)}(3p)$

Let p be a prime with $p \equiv 1 \pmod 3$. Consider the representation number $r_{(1,1,3)}(3p)$:

$$S_{(1,1,3)}(3p) = 6\pi\sqrt{p} \cdot \lambda(9p, 12) \cdot \left(\frac{1}{3} - A(3, 9p)\right) \cdot \big(2 - \alpha(9p)\big).$$

By definition we have

$$\sigma(9p, 3) = 1, \qquad \frac{1}{3} - A(3, 9p) = \frac{2}{9}$$

and

$$\alpha(9p) = \begin{cases} 3/2, & \text{if } p \equiv 1 \pmod 4, \\ 1, & \text{if } p \equiv 3 \pmod 8, \\ 0, & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Then

$$\begin{aligned}\lambda(9p,12) &= \frac{L_{12}(1,\chi_{-p})}{L_{12}(2,\mathrm{id})} \\ &= \frac{4\cdot 3^2(1-\frac{\chi_{-p}(2)}{2}-\frac{\chi_{-p}(3)}{3}+\frac{\chi_{-p}(6)}{6})}{3(3^2-1)}\cdot\frac{L(1,\chi_{-p})}{\zeta(2)} \\ &= \frac{9}{\pi^2}\cdot\left(1-\frac{\chi_{-p}(2)}{2}-\frac{\chi_{-p}(3)}{3}+\frac{\chi_{-p}(6)}{6}\right)\cdot L(1,\chi_{-p}).\end{aligned}$$

If $p \equiv 1 \pmod 4$, then

$$\chi_{-p}(2) = \left(\frac{-4p}{2}\right) = 0, \qquad \chi_{-p}(3) = \left(\frac{-4p}{3}\right) = -1, \qquad \chi_{-p}(6) = \left(\frac{-4p}{6}\right) = 0;$$

and if $p \equiv 3 \pmod 4$, then

$$\chi_{-p}(2) = \left(\frac{-p}{2}\right) = \begin{cases} -1, & \text{if } p \equiv 3 \pmod 8, \\ 1, & \text{if } p \equiv 7 \pmod 8, \end{cases}$$

$\chi_{-p}(3) = (\frac{-p}{3}) = -1$, and

$$\chi_{-p}(6) = \left(\frac{-p}{6}\right) = \left(\frac{-p}{2}\right)\left(\frac{-p}{3}\right) = -\left(\frac{-p}{2}\right) = -\chi_{-p}(2).$$

Henceforth,

$$1-\frac{\chi_{-p}(2)}{2}-\frac{\chi_{-p}(3)}{3}+\frac{\chi_{-p}(6)}{6} = \begin{cases} 4/3, & \text{if } p \equiv 1 \pmod 4, \\ 2, & \text{if } p \equiv 3 \pmod 8, \\ 2/3, & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

On the other hand, thanks to Lemma 6.1, $L(1,\chi_{-p})$ can be expressed explicitly involving the class number $h(-p)$: If $p \equiv 1 \pmod 4$, then $D(-p) = -4p$, thus $L(1,\chi_{-p}) = \frac{\pi h(-p)}{2\sqrt{p}}$; and if $p \equiv 3 \pmod 4$, then $D(-p) = -p$, so $L(1,\chi_{-p}) = \frac{\pi h(-p)}{\sqrt{p}}$.

Now we obtain

$$\lambda(9p,12) = \nu_p \cdot \frac{h(-p)}{\pi\sqrt{p}},$$

where

$$\nu_p = \begin{cases} 18, & \text{if } p \equiv 3 \pmod 8, \\ 6, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned}
S_{(1,1,3)}(3p) &= 6\pi\sqrt{p}\cdot\lambda(9p,12)\cdot\left(\frac{1}{3}-A(3,9p)\right)\cdot\big(2-\alpha(9p)\big)\\
&= 6\pi\sqrt{p}\cdot\nu_p\cdot\frac{h(-p)}{\pi\sqrt{p}}\cdot\frac{2}{9}\cdot\big(2-\alpha(9p)\big)\\
&= \frac{4h(-p)}{3}\cdot\nu_p\cdot\big(2-\alpha(9p)\big)\\
&= \begin{cases} 4h(-p), & \text{if } p\equiv 1 \pmod 4,\\ 24h(-p), & \text{if } p\equiv 3 \pmod 8,\\ 16h(-p), & \text{if } p\equiv 7 \pmod 8.\end{cases}
\end{aligned}$$

This confirms Conjecture 19 of [20].

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