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Three-variable Mahler measures and special values of L-functions of modular forms

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Abstract
In this paper we study the Mahler measures of two families of Laurent polynomials. We prove several three-variable Mahler measure formulas initially conjectured by D. Samart.

Keywords Mahler measure · L-function · Newform

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1 Introduction

The logarithmic Mahler measure of a Laurent polynomial $P(x_1, \dots, x_n)$ in n variables with complex coefficients is defined by

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}^n$ denotes the n-torus

$$S^1 \times \cdots \times S^1 = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1 \}.$$

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There are interesting connections between the Mahler measures of certain types of Laurent polynomials and special values of L-functions. The first example, which is due to Smyth [16], is

$$m(x+y+1)=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)=L'(\chi_{-3},1),$$

where χ_D denotes the Jacobi symbol $\left(\frac{D}{\bullet}\right)$. This phenomenon inspired a great deal of investigations and there have been a considerable amount of relevant literature in this direction. One can see a review on this topic in [8].

If a two-variable Laurent polynomial $P(x,\ y)=0$ defines an elliptic curve E over $\mathbb{Q}$ and $P(x,\ y)$ is a tempered polynomial, then $m(P)$ is often related to $L(E,\ 2)$ by the Beilinson's regulator map. One can see the definition of "tempered polynomial" in [10]. Deninger first noticed this relation and conjectured in [6] that

$$m(1+X+1/X+Y+1/Y)=\frac{15}{4\pi^2}L(E_{15},2),$$

where E_{15} is an elliptic curve with conductor 15. His work inspired Boyd to raise hundreds of conjectures in this direction by numerical computation in [4]. Deninger's Conjecture is proved by Rogers and Zudilin in [12].

In 2008, Bertin [1,2] studied the Mahler measures of certain three-variable Laurent polynomials, $m(x+x^{-1}+y+y^{-1}+z+z^{-1}+k)$. Later, Rogers [11] considered the following functions:

$$\begin{aligned}f_2(k) :=&2m\left(\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)+\sqrt{k}\right),\\ f_4(k) :=&4m(x^4+y^4+z^4+1+k^{1/4}xyz),\end{aligned}$$

where k is a complex number. The special values of the above two functions are related to the special values of L-functions of certain particular modular forms (c.f. [14])—specifically, Samart computed in [14] $f_2(64)$, $f_2(256)$, $f_3(216)$, $f_3(1458)$, $f_4(648)$, $f_4(2304)$, $f_4(20736)$ and $f_4(614656)$, and showed that each of them is a rational linear combination of derivatives of L-functions of some modular forms and Dirichlet L-functions. Samart found many more relations between three-variable Mahler measures and special values of L-functions of modular forms and Dirichlet L-functions in [14,15]. See also comprehensive lecture notes [19] by Zudilin on Mahler measures and L-functions.

Samart then conjectured in [14], based on computational results that the following formulas hold:

$$\begin{aligned}f_2(-64)&=2L'(g_{32},0)+2L'(\chi_{-4},-1),\\ f_2(-512)&=L'(g_{64},0)+L'(\chi_{-8},-1),\\ f_4(-1024)&=\frac{8}{5}(5L'(g_{20},0)+2L'(\chi_{-4},-1)),\end{aligned}$$

$$f_4(-12288) = \frac{40}{9}(L'(g_{36}, 0) + 2L'(\chi_{-3}, -1)),$$
$$f_4(-82944) = \frac{40}{13}(L'(g_{52}, 0) + 2L'(\chi_{-4}, -1)), \tag{1.1}$$

where g_N denotes a newform with rational coefficients in the space of cusp forms of weight 3 and level N.

Note that the Mahler measures $f_2(k)$ and $f_4(k)$ can be written as $f_2(s_2(q))$ and $f_4(s_4(q))$, where $s_2(q)$ and $s_4(q)$ ($q = e^{2\pi i\tau}$) are some modular functions whose definitions can be found in Sect. 2. All the τ's in the identities proven by Samart in [14] are purely imaginary. However none of the τ's in the above conjectural list is purely imaginary. Hence we cannot use Weber's table in [17] directly. We will use the algebraic equations of the modular functions to get what we need.

In 2015, Samart conjectured in [15] that

$$f_2(280 + 198\sqrt{2}) = \frac{1}{8}(36M_{16} + 4M_{16\otimes 8} + 13d_4 + 4d_8),$$
$$f_2(280 - 198\sqrt{2}) = \frac{1}{2}(36M_{16} - 4M_{16\otimes 8} - 13d_4 + 4d_8). \tag{1.2}$$

Here the notation means that

$$d_k := L'(\chi_{-k}, -1),\ M_N := L'(g_N, 0),\ M_{N\otimes D} := L'(g_N \otimes \chi_D, 0),$$

where g_N is a normalized newform with rational Fourier coefficients in $S_3(\Gamma_0(N), \chi_{-N})$ and $g_N \otimes \chi_D$ is the quadratic twist of g_N by χ_D. One should note that Samart used a simplified notation in [15]. In this paper, we quote his conjectures in the form consistent with the original notation.

In fact, there are three tables in [15] containing many more conjectures. We choose the two conjectures above of [15] just because the values of k are not rational. We will prove the two identities of (1.2) in Sect. 3.3. Huimin Zheng, together with the first and the third authors of this paper, proved recently the following conjectural identities of [15] in a preprint by similar arguments:

$$f_2(16) = 8M_{12},$$
$$f_2(4096) = \frac{4}{7}(M_{7\otimes(-4)} + 8d_4),$$
$$f_2\left(\frac{1+\sqrt{-7}}{2}\right) = \frac{1}{14}(4M_{7\otimes(-4)} + 384M_7 + 32d_4 + 11d_7),$$
$$f_2\left(\frac{7+\sqrt{-7}}{14}\right) = \frac{1}{2}(4M_{7\otimes(-4)} - 384M_7 - 32d_4 + 11d_7),$$
$$f_2(-104 - 60\sqrt{3}) = \frac{1}{6}(4M_{12\otimes(-4)} + 36M_{12} + 15d_3 + 8d_4),$$
$$f_2(-104 + 60\sqrt{3}) = \frac{1}{2}(4M_{12\otimes(-4)} - 36M_{12} + 15d_3 - 8d_4). \tag{1.3}$$

We believe that all the other conjectures of [15] could be proven by similar arguments.

2 Modular functions

Following the usual notation in the theory of modular forms, we denote the nome $q = e^{2\pi i\tau}$ with the variable τ in the upper half complex plane and $\eta(\tau)$ the Dedekind eta function, that is,

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{2.1}$$

Then the modular discriminant Δ can be expressed as $\Delta(\tau) = \eta(\tau)^{24}$. Define

$$s_2(q(\tau)) := -\frac{\Delta(\tau+\frac{1}{2})}{\Delta(2\tau+1)}.$$

It turns out that $s_2(q)$ is exactly $\mathfrak{f}(2\tau)^{24}$, where $\mathfrak{f}(\tau)$ denotes the classical Weber modular function (cf. [14, Lemma 2.2]). Recall that

$$\begin{aligned} \mathfrak{f}(\tau) &= e^{-\frac{\pi i}{24}}\frac{\eta((\tau+1)/2)}{\eta(\tau)} = \frac{\mathfrak{f}_1(2\tau)}{\mathfrak{f}_1(\tau)}, & \mathfrak{f}_1(\tau) &= \frac{\eta(\tau/2)}{\eta(\tau)}, \\ \mathfrak{f}_2(\tau) &= \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)}, & \mathfrak{f}(\tau)\mathfrak{f}_1(\tau)\mathfrak{f}_2(\tau) &= \sqrt{2}. \end{aligned} \tag{2.2}$$

Note that $-\mathfrak{f}(\tau)^{24}$, $\mathfrak{f}_1(\tau)^{24}$, $\mathfrak{f}_2(\tau)^{24}$ are exactly the three roots of the cubic equation

$$(x+16)^3 - j(\tau)\,x = 0, \tag{2.3}$$

where $j(\tau)$ is the classical j-invariant. One can see more details on [3, p. 288] or in [18, Sect. 1].

Lemma 2.1 *We have the following special values of* $\mathfrak{f}(\tau)^{24}$*:*

(1) $s_2(q(-\frac{1}{2}+\frac{\sqrt{2}}{2}i)) = \mathfrak{f}(-1+\sqrt{2}i)^{24} = -64.$
(2) $s_2(q(-\frac{1}{2}+i)) = \mathfrak{f}(-1+2i)^{24} = -512.$
(3) $s_2(q(i)) = \mathfrak{f}(2i)^{24} = 280+198\sqrt{2}.$
(4) $s_2(q(\frac{1}{2}+\frac{i}{4})) = \mathfrak{f}(1+\frac{1}{2}i)^{24} = 280-198\sqrt{2}.$

Proof (1) We have

$$\begin{pmatrix}\mathfrak{f}(\tau+1)\\ \mathfrak{f}_1(\tau+1)\\ \mathfrak{f}_2(\tau+1)\end{pmatrix} = \begin{pmatrix}0 & \zeta^{-1} & 0\\ \zeta^{-1} & 0 & 0\\ 0 & 0 & \zeta^2\end{pmatrix}\begin{pmatrix}\mathfrak{f}(\tau)\\ \mathfrak{f}_1(\tau)\\ \mathfrak{f}_2(\tau)\end{pmatrix}, \tag{2.4}$$

where $\zeta = e^{\frac{\pi i}{24}}$ (cf. [18, p. 1647]). Combining the above identity and the fact that $\mathfrak{f}_1(\sqrt{2}i)^{24} = 64$ (this value was listed by Weber in [17], p. 721), we can see that $\mathfrak{f}(-1+\sqrt{2}i)^{24} = -64$.

(2) In Weber's Table, $\mathfrak{f}_1(2i)$ is wrongly written as $\sqrt[3]{8}$. The correct value $\mathfrak{f}_1(2i) = \sqrt[8]{8}$ given by Brillhart and Morton [5, p. 1379]. By the same argument,

$$\mathfrak{f}(-1+2i)^{24} = -\mathfrak{f}_1(2i)^{24} = -(\sqrt[8]{8})^{24} = -512. \tag{2.5}$$

(3) and (4) Notice that $-\mathfrak{f}(\tau)^{24},\ \mathfrak{f}_1(\tau)^{24},\ \mathfrak{f}_2(\tau)^{24}$ are exactly the three roots of the cubic equation

$$(x+16)^3 - j(\tau)\,x = 0. \tag{2.6}$$

Since $\mathfrak{f}_1(2i)^{24} = 512$ by (2.5) and $j(2i) = 2^3 3^3 11^3$, we can see that

$$\mathfrak{f}(2i)^{24} = 280 + 198\sqrt{2},\ \mathfrak{f}_2(2i)^{24} = 280 - 198\sqrt{2}$$

by solving the equation (2.6). Note that the transformation $\tau \mapsto -\frac{1}{\tau}$ fixes $\mathfrak{f}$, and exchanges $\mathfrak{f}_1,\ \mathfrak{f}_2$. Hence $\mathfrak{f}(\frac{1}{2}i) = \mathfrak{f}(2i)$ which implies that

$$\mathfrak{f}\left(\frac{1}{2}i\right)^{24} = 280 + 198\sqrt{2},\ \mathfrak{f}_1\left(\frac{1}{2}i\right)^{24} = 280 - 198\sqrt{2}.$$

By (2.4),

$$\mathfrak{f}\left(1+\frac{1}{2}i\right)^{24} = -\mathfrak{f}_1\left(\frac{1}{2}i\right)^{24} = 280 - 198\sqrt{2}.$$

□

Define

$$s_4(q(\tau)) := \frac{\Delta(2\tau)}{\Delta(\tau)}\left(16\left(\frac{\eta(\tau)\eta(4\tau)^2}{\eta(2\tau)^3}\right)^4 + \left(\frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)^2}\right)^4\right)^4.$$

Then by (2.2), we have

$$s_4(q(\tau)) = \frac{1}{\mathfrak{f}_1(2\tau)^{24}}\left(16\left(\frac{1}{\mathfrak{f}_1(4\tau)\mathfrak{f}(2\tau)}\right)^4 + (\mathfrak{f}_1(4\tau)\mathfrak{f}(2\tau))^4\right)^4.$$

Lemma 2.2 *We have the following special values of $s_4(q(\tau))$:*

(1) $s_4\left(q\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}i\right)\right) = -1024$.
(2) $s_4\left(q\left(-\frac{1}{2}+\frac{3}{2}i\right)\right) = -12288$.

(3) $s_4\left(q\left(-\frac{1}{2}+\frac{\sqrt{13}}{2}i\right)\right)=-82944.$

Proof In fact, we found these values with the aid of Sagemath [13]. However all these identities can be proved by Table VI of [17] and the modular equations of Weber modular functions. We will give the proof of the first one. The proofs of the other two are similar. Hence we omit them.

(1) By the Table of [17], we have

$$\mathfrak{f}(\sqrt{5}i)^4=1+\sqrt{5},\ \mathfrak{f}_1(2\sqrt{5}i)^4=\sqrt{8}x,\ x^2=\frac{1+\sqrt{5}}{2}(2x+1).$$

Hence

$$\mathfrak{f}_1(\sqrt{5}i)^4=\frac{\mathfrak{f}_1(2\sqrt{5}i)^4}{\mathfrak{f}(\sqrt{5}i)^4}=\frac{\sqrt{8}x}{1+\sqrt{5}}.$$

Let $\tau=-\frac{1}{2}+\frac{\sqrt{5}}{2}i$. Then

$$\begin{aligned}\mathfrak{f}_1(2\tau)^{24}&=\mathfrak{f}_1(-1+\sqrt{5}i)^{24}=-\mathfrak{f}(\sqrt{5}i)^{24}=-(1+\sqrt{5})^6,\\ \mathfrak{f}_1(4\tau)^4&=\mathfrak{f}_1(-2+2\sqrt{5}i)^4=\zeta^8\mathfrak{f}_1(2\sqrt{5}i)^4=\zeta^8\sqrt{8}x,\\ \mathfrak{f}(2\tau)^4&=\mathfrak{f}(-1+\sqrt{5}i)^4=\zeta^4\mathfrak{f}_1(\sqrt{5}i)^4=\zeta^4\frac{\sqrt{8}x}{1+\sqrt{5}}.\end{aligned}$$

Hence

$$s_4(q(\tau))=\frac{1}{\mathfrak{f}_1(2\tau)^{24}}\left(16\left(\frac{1}{\mathfrak{f}_1(4\tau)\mathfrak{f}(2\tau)}\right)^4+(\mathfrak{f}_1(4\tau)\mathfrak{f}(2\tau))^4\right)^4=-1024.$$

□

Samart established an explicit formula for $f_2(s_2(q))$ and $f_4(s_4(q))$ as follows (see the proof of [14, Prop.2.1]). We use the notation $\sum'_{m,n\in\mathbb{Z}}$ to denote the summation over $m,n\in\mathbb{Z}$ with $(m,\ n)\neq(0,\ 0)$.

Lemma 2.3 ([14] Proposition 2.1 (i) and (iii)).

(1) *If* $\mathrm{Im}(\tau)\geqslant\frac{1}{2}$, *then* $f_2(s_2(q))=\frac{2\mathrm{Im}(\tau)}{\pi^3}(-A+16B)$, *where*

$$A=\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{4(m\mathrm{Re}(\tau)+n)^2}{|m\tau+n|^6}-\frac{1}{|m\tau+n|^4}\right),\ B=\sum_{m,n\in\mathbb{Z}}{}'\left(\frac{4(4m\mathrm{Re}(\tau)+n)^2}{|4m\tau+n|^6}-\frac{1}{|4m\tau+n|^4}\right).$$

(2) *If* $\mathrm{Im}(\tau) \geqslant \frac{1}{\sqrt{2}}$, *then* $f_4(s_4(q)) = \frac{10\mathrm{Im}(\tau)}{\pi^3}(-A+4B)$, *where*

$$A = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(m\mathrm{Re}(\tau)+n)^2}{|m\tau+n|^6} - \frac{1}{|m\tau+n|^4} \right), \quad B = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(2m\mathrm{Re}(\tau)+n)^2}{|2m\tau+n|^6} - \frac{1}{|2m\tau+n|^4} \right).$$

This suggests an effective way to compute $f_i(k)$ $(i = 2,\ 4)$: For a given k, if we find a pre-image τ of k through $s_i(q)$, that is, write $k = s_i(q(\tau))$, then we can use the above formula to derive the value of $f_i(k)$.

The lower bound of $\mathrm{Im}(\tau)$ is unnecessary. In fact, both $f_2(s_2(q))$, $f_4(s_4(q))$ and the lattice sums are the real parts of holomorphic functions. Hence they are equal everywhere which implies that the corresponding Mahler measures still can be computed by the formulas of the above lemma even if the imaginary parts of τ do not satisfy the condition in Lemma 2.3.

For a quadratic Dirichlet character χ the L-function with respect to χ is denoted by $L(s,\chi)$. Let $\chi_{-4}(\cdot) = (\frac{-1}{\cdot})$, $\chi_{-8}(\cdot) = (\frac{-2}{\cdot})$, $\chi_8(\cdot) = (\frac{2}{\cdot})$ and $\chi_{-3}(\cdot) = (\frac{-3}{\cdot})$, etc.

We will use the notation as in [7]. Let $L_1(s) = \zeta(s)$ be the Riemann ζ-function, $L_n(s) = L(s,\chi_n)$ the Dirichlet function. The key of our computation is the following technical lemma, which gives an analytic formula for a series of lattice sums.

Lemma 2.4 (Glasser & Zucker [7], Table VI). *For any complex number* s *with* $\mathrm{Re}(s) > 1$,

(1) $\displaystyle\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^s} = 4L_1(s)L_{-4}(s);$

(2) $\displaystyle\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+2n^2)^s} = 2L_1(s)L_{-8}(s);$

(3) $\displaystyle\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+4n^2)^s} = 2(1-2^{-s}+2^{1-2s})L_1(s)L_{-4}(s);$

(4) $\displaystyle\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+5n^2)^s} = L_1(s)L_{-20}(s) + L_5(s)L_{-4}(s);$

(5) $\displaystyle\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+8n^2)^s} = (1-2^{-s}+2^{1-2s})L_1(s)L_{-8}(s) + L_{-4}(s)L_8(s);$

(6) $\displaystyle\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+9n^2)^s} = (1+3^{1-2s})L_1(s)L_{-4}(s) + L_{-3}(s)L_{12}(s);$

(7) $\displaystyle\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+16n^2)^s} = (1-2^{-s}+2^{1-2s}-2^{1-3s}+2^{2-4s})L_1(s)L_{-4}(s) + L_{-8}(s)L_8(s).$

3 Computation of $f_2(k)$ for $k = -64,\ -512,\ 280 \pm 198\sqrt{2}$

Now we come to the computation of $f_2(-64)$ and $f_2(-512)$. We shall first express $f_2(-64)$ or $f_2(-512)$ as a lattice sum by Lemma 2.3, then construct a particular modular form in the light of Lemma 2.4, which will turn out to be the exact form we want.

3.1 $f_2(-64)$

For $k = -64$, by Lemma 2.1, we find that $\tau = -\frac{1}{2} + \frac{\sqrt{2}}{2}i$ satisfies $s_2(q(\tau)) = -64$, hence by Samart's formula we get $f_2(-64) = \frac{\sqrt{2}}{\pi^3}(-A + 16B)$, where

$$\begin{aligned} -A &= \sum_{m,n\in\mathbb{Z}}{}' \frac{-64(2n-m)^2}{((2n-m)^2+2m^2)^3} + \sum_{m,n\in\mathbb{Z}}{}' \frac{16}{((2n-m)^2+2m^2)^2}, \\ 16B &= \sum_{m,n\in\mathbb{Z}}{}' \frac{64(n-2m)^2}{((n-2m)^2+8m^2)^3} - \sum_{m,n\in\mathbb{Z}}{}' \frac{16}{((n-2m)^2+8m^2)^2}. \end{aligned}$$

In order to use Lemma 2.4, we need to change the index in the summation of A and B and then twist them by some Dirichlet characters. We denote $a \equiv b \pmod r$ by $a \equiv b\,(r)$.

Let $\alpha = 2n - m$, $\beta = m$. Then

$$-A = \sum_{\alpha+\beta\equiv 0\,(2)}{}' \frac{-64\alpha^2}{(\alpha^2+2\beta^2)^3} + \sum_{\alpha+\beta\equiv 0\,(2)}{}' \frac{16}{(\alpha^2+2\beta^2)^2}.$$

Consider the function $f(\tau) := \eta(\tau)^2\,\eta(2\tau)\,\eta(4\tau)\,\eta(8\tau)^2$. This is a modular form of weight 3 with level 8. It has q-expansion (cf. [14, Lemma 2.7])

$$f(\tau) = \sum_{m,n\in\mathbb{Z}} \frac{m^2-2n^2}{2} q^{m^2+2n^2}.$$

From $f(\tau)$ we get another modular form by twisting it with χ_{-4}, that is,

$$\begin{aligned} f\otimes\chi_{-4}(\tau) &= \sum_{m,n\in\mathbb{Z}} \frac{m^2-2n^2}{2}\chi_{-4}(m^2+2n^2)\,q^{m^2+2n^2} \\ &= \sum_{\substack{m\equiv 1\,(2)\\ n\equiv 0\,(2)}} \frac{m^2-2n^2}{2} q^{m^2+2n^2} - \sum_{\substack{m\equiv 1\,(2)\\ n\equiv 1\,(2)}} \frac{m^2-2n^2}{2} q^{m^2+2n^2}. \end{aligned}$$

We can also twist $f(\tau)$ by χ_{-8},

$$
\begin{aligned}
f\otimes\chi_{-8}(\tau) &= \sum_{m,n\in\mathbb{Z}} \frac{m^2-2n^2}{2}\chi_{-8}(m^2+2n^2)\,q^{m^2+2n^2} \\
&= \sum_{\substack{m\equiv 1\,(2)\\ n\equiv 0\,(2)}} \frac{m^2-2n^2}{2}\,q^{m^2+2n^2} + \sum_{\substack{m\equiv 1\,(2)\\ n\equiv 1\,(2)}} \frac{m^2-2n^2}{2}\,q^{m^2+2n^2}.
\end{aligned}
$$

Hence

$$
\sum_{\substack{m\equiv 1\,(2)\\ n\equiv 1\,(2)}} \frac{m^2-2n^2}{2}\,q^{m^2+2n^2} = \frac{1}{2}(f\otimes\chi_{-8}(\tau)-f\otimes\chi_{-4}(\tau)).
$$

On the other hand, one easily sees that

$$
\sum_{\substack{m\equiv 0\,(2)\\ n\equiv 0\,(2)}} \frac{m^2-2n^2}{2}\,q^{m^2+2n^2} = 4f(4\tau).
$$

Thus we have

$$
f_A(\tau) = \sum_{m+n\equiv 0\,(2)} \frac{m^2-2n^2}{2}\,q^{m^2+2n^2} = 4f(4\tau)+\frac{1}{2}(f\otimes\chi_{-8}(\tau)-f\otimes\chi_{-4}(\tau)).
$$

Since

$$
\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv n\,(2)}} = \sum_{m,n\in\mathbb{Z}} - \sum_{\substack{m\equiv 0\,(2)\\ n\in\mathbb{Z}}} - \sum_{\substack{m\in\mathbb{Z}\\ n\equiv 0\,(2)}} + 2\sum_{\substack{m\equiv 0\,(2)\\ n\equiv 0\,(2)}}, \tag{3.1}
$$

we have

$$
\begin{aligned}
&\sideset{}{'}\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv n\,(2)}} \frac{1}{(m^2+2n^2)^2} \\
&\quad = \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(m^2+2n^2)^2} - \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(4m^2+2n^2)^2} - \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(m^2+8n^2)^2} \\
&\qquad + 2\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{16(m^2+2n^2)^2} \\
&\quad = \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(m^2+2n^2)^2} - \frac{1}{4}\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(2m^2+n^2)^2} - \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(m^2+8n^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+2n^2)^2}\\
&=\frac{7}{8}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+2n^2)^2}-\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+8n^2)^2}.
\end{aligned}
$$

By Lemma 2.4,

$$
\begin{aligned}
\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv n\ (2)}}{}' \frac{1}{(m^2+2n^2)^2} &= \frac{7}{8}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+2n^2)^2}-\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+8n^2)^2}\\
&=\frac{7}{8}\cdot 2L_1(2)L_{-8}(2)-\frac{7}{8}L_1(2)L_{-8}(2)-L_{-4}(2)L_8(2)\\
&=\frac{7}{8}L_1(2)L_{-8}(2)-L_{-4}(2)L_8(2).
\end{aligned}
$$

Hence

$$
\begin{aligned}
-A &= \sum_{\alpha+\beta\in2\mathbb{Z}}{}' \frac{-64\alpha^2}{(\alpha^2+2\beta^2)^3}+\sum_{\alpha+\beta\in2\mathbb{Z}}{}' \frac{16}{(\alpha^2+2\beta^2)^2}\\
&=\sum_{\alpha+\beta\in2\mathbb{Z}}{}' \frac{-64(\alpha^2-2\beta^2)}{2(\alpha^2+2\beta^2)^3}-\sum_{\alpha+\beta\in2\mathbb{Z}}{}' \frac{16}{(\alpha^2+2\beta^2)^2}\\
&=-64L(f_A,3)-16\left(\frac{7}{8}L_1(2)L_{-8}(2)-L_{-4}(2)L_8(2)\right).
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
16B &= \sum_{m,n\in\mathbb{Z}}{}' \frac{64(n-2m)^2}{\left((n-2m)^2+8m^2\right)^3}-\sum_{m,n\in\mathbb{Z}}{}' \frac{16}{\left((n-2m)^2+8m^2\right)^2}\\
&=\sum_{x+y\equiv0\ (2)}{}' \frac{64x^2}{(x^2+8y^2)^3}-\sum_{x+y\equiv0\ (2)}{}' \frac{16}{(x^2+8y^2)^2}\qquad (x=2n-m,\ \ y=m)\\
&=\sum_{x+y\equiv0\ (2)}{}' \frac{64(x^2-8y^2)}{2(x^2+8y^2)^3}+\sum_{x+y\equiv0\ (2)}{}' \frac{16}{(x^2+8y^2)^2}.
\end{aligned}
$$

Let

$$
f_B(\tau)=\sum_{x,y\in\mathbb{Z}}\frac{x^2-8y^2}{2}q^{x^2+8y^2}.
$$

Then $f_B(\tau)$ is also a modular form of weight 3. By the same argument as in the case concerning A, we can prove that

$$f_B(\tau) = 4f(4\tau) + \frac{1}{2}(f \otimes \chi_{-8}(\tau) + f \otimes \chi_{-4}(\tau)).$$

By Lemma 2.4,

$$\sum_{x+y\in 2\mathbb{Z}}' \frac{16}{(x^2+8y^2)^2} = 16\left(\frac{7}{8}L_1(2)L_{-8}(2) + L_{-4}(2)L_8(2)\right).$$

So

$$16B = 64L(f_B, 3) + 16\left(\frac{7}{8}L_1(2)L_{-8}(2) + L_{-4}(2)L_8(2)\right).$$

Therefore,

$$\begin{aligned} f_2(-64) =& \frac{\sqrt{2}}{\pi^3}(-A+16B) \\ =& \frac{\sqrt{2}}{\pi^3}\left(-64L(f_A,3) - 16\left(\frac{7}{8}L_1(2)L_{-8}(2) - L_{-4}(2)L_8(2)\right)\right) \\ &+ \frac{\sqrt{2}}{\pi^3}\left(64L(f_B,3) + 16\left(\frac{7}{8}L_1(2)L_{-8}(2) + L_{-4}(2)L_8(2)\right)\right) \\ =& \frac{64\sqrt{2}}{\pi^3}L(f_B - f_A, 3) + \frac{4}{\pi}L_{-4}(2), \end{aligned}$$

wherein we have used the identity

$$L_8(2) = \sqrt{2}\pi^2/16 \tag{3.2}$$

to substitute $L_8(2)$.

Let $g_{32} := f_B - f_A = f \otimes \chi_{-4}$. Although we have used the lower index "32", we need to show that the level of g_{32} is indeed 32. We can prove it by Sagemath [13]. We know that $f \otimes \chi_{-4}$ is a weight 3 cusp form whose level divides 128. Since the first few Fourier coefficients of $f \otimes \chi_{-4}$ agree with those of a weight 3 cusp form of level 32 computed in Sagemath, we have that they are the same cusp by Sturm's Theorem.

By the functional equations of L-functions,

$$L(g_{32}, 3) = \frac{\sqrt{2}\pi^3}{64}L'(g_{32}, 0), \ \ L_{-4}(2) = \frac{\pi}{2}L'_{-4}(-1), \tag{3.3}$$

we get the equality

$$f_2(-64) = 2L'(g_{32}, 0) + 2L'_{-4}(-1).$$

3.2 $f_2(-512)$

For $k = -512$, we find $\tau = -\frac{1}{2} + i$ such that $s_2(q(\tau)) = -512$. For any integers m, n,

$$\begin{aligned}|m\tau + n|^2 &= \left(n - \frac{m}{2}\right)^2 + m^2,\\ |4m\tau + n|^2 &= (n-2m)^2 + 16m^2,\\ m\mathrm{Re}(\tau) + n &= n - \frac{m}{2},\\ 4m\mathrm{Re}(\tau) + n &= n - 2m.\end{aligned}$$

Therefore, in Samart's formula, $f_2(-512) = \frac{2}{\pi^2}(-A + 16B)$, where

$$\begin{aligned}A &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(m\mathrm{Re}(\tau)+n)^2}{|m\tau+n|^6} - \frac{1}{|m\tau+n|^4}\right)\\ &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4\left(n-\frac{m}{2}\right)^2}{\left(\left(n-\frac{m}{2}\right)^2+m^2\right)^3} - \frac{1}{\left(\left(n-\frac{m}{2}\right)^2+m^2\right)^2}\right),\\ B &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(4m\mathrm{Re}(\tau)+n)^2}{|4m\tau+n|^6} - \frac{1}{|4m\tau+n|^4}\right)\\ &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(n-2m)^2}{\left((n-2m)^2+16m^2\right)^3} - \frac{1}{\left((n-2m)^2+16m^2\right)^2}\right).\end{aligned}$$

For A, we have

$$\begin{aligned}A &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{64(2n-m)^2}{\left((2n-m)^2+4m^2\right)^3} - \frac{16}{\left((2n-m)^2+4m^2\right)^2}\right)\\ &= \sum_{\substack{m,\ell\in\mathbb{Z}\\ m\equiv\ell\ (2)}}{}' \left(\frac{64\,\ell^2}{(\ell^2+4m^2)^3} - \frac{16}{(\ell^2+4m^2)^2}\right).\end{aligned}$$

From $64\ell^2 = 32\left((\ell^2+4m^2) + (\ell^2-4m^2)\right)$ it follows that

$$A = \sum_{\substack{m,\ell\in\mathbb{Z}\\ m\equiv\ell\ (2)}}{}' \left(\frac{32(\ell^2-4m^2)}{(\ell^2+4m^2)^3} + \frac{16}{(\ell^2+4m^2)^2}\right).$$

Let $h(\tau) = \sum_{m,n\in\mathbb{Z}} \frac{m^2-4n^2}{2} q^{m^2+4n^2} = \eta(4\tau)^6$ and

$$h_A(\tau) := \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv n\ (2)}} \frac{m^2-4n^2}{2} q^{m^2+4n^2} = 4h(4\tau) + \frac{1}{2}(h\otimes\chi_{-4} - h\otimes\chi_8)(\tau).$$

Then

$$\begin{aligned} A &= \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv n\ (2)}}{}' \left(\frac{32(m^2-4n^2)}{(m^2+4n^2)^3} + \frac{16}{(m^2+4n^2)^2}\right) \\ &= 64L(h_A,3) + 16 \sum_{m,n\in\mathbb{Z} m\equiv n\ (2)}{}' \frac{1}{(m^2+4n^2)^2}. \end{aligned}$$

By (3.1) and Lemma 2.4, we have

$$\begin{aligned} &\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv n\ (2)}}{}' \frac{1}{(m^2+4n^2)^2} \\ &= \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+4n^2)^2} - \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(4m^2+4n^2)^2} - \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+16n^2)^2} \\ &\quad + 2\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{16(m^2+4n^2)^2} \\ &= \frac{9}{8}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+4n^2)^2} - \frac{1}{16}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2} - \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+16n^2)^2} \\ &= \frac{9}{8}\cdot\frac{7}{4}L_1(2)L_{-4}(2) - \frac{1}{16}\cdot 4L_1(2)L_{-4}(2) - \left(\frac{55}{64}L_1(2)L_{-4}(2) + L_{-8}(2)L_8(2)\right) \\ &= \frac{55}{64}L_1(2)L_{-4}(2) - L_{-8}(2)L_8(2) \end{aligned}$$

and so $A = 64L(h_A,3) + 16\left(\frac{55}{64}L_1(2)L_{-4}(2) - L_{-8}(2)L_8(2)\right)$.

For B, we have

$$\begin{aligned} B &= \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{4(n-2m)^2}{\left((n-2m)^2+16m^2\right)^3} - \frac{1}{\left((n-2m)^2+16m^2\right)^2}\right) \\ &= \sum_{m,\ell\in\mathbb{Z}}{}' \left(\frac{4\,\ell^2}{(\ell^2+16m^2)^3} - \frac{1}{(\ell^2+16m^2)^2}\right). \end{aligned}$$

Write $4\ell^2 = 2\left((\ell^2+16m^2)+(\ell^2-16m^2)\right)$, then we get

$$B = \sum_{m,\ell\in\mathbb{Z}}{}' \left(\frac{2(\ell^2-16m^2)}{(\ell^2+16m^2)^3} + \frac{1}{(\ell^2+16m^2)^2}\right).$$

If we put $h_B(\tau) := \sum_{m,n\in\mathbb{Z}} \frac{m^2-16n^2}{2} q^{m^2+16n^2}$, then $h_B(\tau) = 4h(4\tau) + \frac{1}{2}(h\otimes\chi_{-4} + h\otimes\chi_8)(\tau)$ and

$$\begin{aligned} B &= 4L(h_B,3) + \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+16n^2)^2} \\ &= 4L(h_B,3) + \left(\frac{55\pi^2}{384}L_{-4}(2) + L_{-8}(2)\frac{\sqrt{2}\pi^2}{16}\right) \end{aligned}$$

by Lemma 2.4 (7), identity (3.2) and $L_1(2) = \zeta(2) = \pi^2/6$.

Now we obtain

$$\begin{aligned} -A + 16B =& -64L(h_A,3) - 16\left(\frac{55}{64}L_1(2)L_{-4}(2) - L_{-8}(2)L_8(2)\right) \\ &+ 64L(h_B,3) + 16\left(\frac{55}{64}L_1(2)L_{-4}(2) + L_{-8}(2)L_8(2)\right) \\ =& 64L(h_B-h_A,3) + 2\sqrt{2}L_{-8}(2)\pi^2 \end{aligned}$$

by identity (3.2) and $L_1(2) = \zeta(2) = \pi^2/6$. So

$$f_2(-512) = \frac{2}{\pi^3}(-A+16B) = \frac{128}{\pi^3}L(h_B-h_A,3) + \frac{4\sqrt{2}}{\pi}L_{-8}(2).$$

Note that the function $g_{64} := h_B - h_A = f\otimes\chi_8$ is a new form of weight 3, and Sagemath [13] shows that its level is indeed 64.

Finally, with the aid of the functional equation of L-functions,

$$L(g_{64},3) = \frac{\pi^3}{128}L'(g_{64},0),\ \ L_{-8}(2) = \frac{\sqrt{2}\pi}{8}L'_{-8}(-1), \tag{3.4}$$

we get

$$f_2(-512) = L'(g_{64},0) + L'_{-8}(-1).$$

3.3 $f_2(280 \pm 198\sqrt{2})$

Since these two cases are similar, we give only details of the computation of $f_2(280 + 198\sqrt{2})$. For $k = 280 + 198\sqrt{2}$, the corresponding $\tau = i$. Hence by Samart's formula we get $f_2(280+198\sqrt{2})=\frac{2}{\pi^3}(-A+16B)$, where

$$\begin{aligned}
-A &= -\sum_{m,n\in\mathbb{Z}}{}' \frac{4(m\mathrm{Re}(\tau)+n)^2}{|m\tau+n|^6} + \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{|m\tau+n|^4} \\
&= \sum_{m,n\in\mathbb{Z}}{}' \frac{-4n^2}{(m^2+n^2)^3} + \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2} \\
&= \sum_{m,n\in\mathbb{Z}}{}' \frac{2m^2-2n^2}{(m^2+n^2)^3} - \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2} = -\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2}, \\
16B &= \sum_{m,n\in\mathbb{Z}}{}' \frac{64(4m\mathrm{Re}(\tau)+n)^2}{|4m\tau+n|^6} - \sum_{m,n\in\mathbb{Z}}{}' \frac{16}{|4m\tau+n|^4} \\
&= \sum_{m,n\in\mathbb{Z}}{}' \frac{64n^2}{((4m)^2+n^2)^3} - \sum_{m,n\in\mathbb{Z}}{}' \frac{16}{((4m)^2+n^2)^2} \\
&= -\sum_{m,n\in\mathbb{Z}}{}' \frac{32((4m)^2-n^2)}{((4m)^2+n^2)^3} + \sum_{m,n\in\mathbb{Z}}{}' \frac{32((4m)^2+n^2)}{((4m)^2+n^2)^3} - \sum_{m,n\in\mathbb{Z}}{}' \frac{16}{((4m)^2+n^2)^2} \\
&= -\sum_{m,n\in\mathbb{Z}}{}' \frac{32((4m)^2-n^2)}{((4m)^2+n^2)^3} + \sum_{m,n\in\mathbb{Z}}{}' \frac{16}{((4m)^2+n^2)^2}.
\end{aligned}$$

Consider the function $h(\tau)$ = Newforms(Gamma1(16),3,names='a')[0]. We call an ideal I of $\mathbb{Z}[i]$ "odd" if the norm of I is odd. Then

$$h(\tau)=\sum_{I \text{ odd}} \psi(I)q^{N(I)} = q-6q^5+9q^9+\dots,$$

where I is an odd ideal of $\mathbb{Z}[i]$ and

$$\psi(x+yi)=(x+yi)^2,\ x \text{ odd},\ y \text{ even}.$$

Hence $x^2+y^2\equiv 1,\ 5 \pmod 8$. Since $\chi_8(5)=-1$,

$$(h\otimes\chi_8+h)(\tau)=2\sum_{N(I)\equiv 1(8)}\psi(I)q^{N(I)} = \sum_{m\in\mathbb{Z},\ n\equiv 1\,(2)}{}' (-(4m)^2+n^2)q^{(4m)^2+n^2}.$$

Note that $h(\tau)$ is the same with that in [14, Lemma 2.7]. It has another form of

$$h(\tau)=\eta(4\tau)^6=\sum_{m,n\in\mathbb{Z}}\frac{m^2-4n^2}{2}q^{m^2+4n^2}.$$

Therefore

$$L(h,3)+L(h\otimes\chi_8,3)=-\sum_{m\in\mathbb{Z},n\equiv 1\ (2)}{}' \frac{(4m)^2-n^2}{((4m)^2+n^2)^3}.$$

Note that

$$\begin{aligned}-\sum_{m,n\in\mathbb{Z}}{}' \frac{(4m)^2-n^2}{((4m)^2+n^2)^3}&=-\sum_{m\in\mathbb{Z},n\ \text{odd}}{}' \frac{(4m)^2-n^2}{((4m)^2+n^2)^3}-\sum_{m,n\in\mathbb{Z}}{}' \frac{(2m)^2-n^2}{16((2m)^2+n^2)^3}\\&=L(h\otimes\chi_8,3)+\frac{9}{8}L(h,3).\end{aligned}$$

By Lemma 2.4 and (3.2), we have

$$\begin{aligned}&f_2(280+198\sqrt{2})\\&=\frac{2}{\pi^3}(-A+16B)\\&=\frac{2}{\pi^3}\left(-\sum_{m,n\in\mathbb{Z}}{}' \frac{32((4m)^2-n^2)}{((4m)^2+n^2)^3}-\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+n^2)^2}+\sum_{m,n\in\mathbb{Z}}{}' \frac{16}{((4m)^2+n^2)^2}\right)\\&=\frac{2}{\pi^3}\left(36L(h,3)+32\cdot L(h\otimes\chi_8,3)+\frac{13\pi^2}{8}L_{-4}(2)+\sqrt{2}\pi^2L_{-8}(2)\right).\end{aligned}$$

The level of $h\otimes\chi_8$ is 64 by Sagemath [13]. One can verify this by the same arguments as in the end of Sect. 3.1.

$$\begin{aligned}f_2(280+198\sqrt{2})&=\frac{2}{\pi^3}(-A+16B)\\&=\frac{9}{2}L'(h,0)+\frac{1}{2}L'(h\otimes\chi_8,0)+\frac{13}{8}L_{-4}(2)+\frac{1}{2}L_{-8}(2)\\&=\frac{1}{8}\left(36M_{16}+4M_{16\otimes 8}+13d_4+4d_8\right).\end{aligned}$$

4 Computation of $f_4(k)$ for $k=-1024,\ -12288,\ -82994$

4.1 $f_4(-1024)$

In this case, $\tau=-\frac{1}{2}+\frac{\sqrt{5}}{2}i$. So $f_4(s_4(q))=\frac{5\sqrt{5}}{\pi^3}(-A+4B)$, where

$$-A=\sum_{m,n\in\mathbb{Z}}{}' \left(-\frac{64(2n-m)^2}{\left((2n-m)^2+5m^2\right)^3}+\frac{16}{\left((2n-m)^2+5m^2\right)^2}\right),$$

$$4B=\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{16(n-m)^2}{\left((n-m)^2+5m^2\right)^3}-\frac{4}{\left((n-m)^2+5m^2\right)^2}\right).$$

Let $\alpha=2n-m$ and $\beta=m$, then $-A+4B=S_1+S_2$, where

$$\begin{aligned}S_1&=\sum_{\alpha\equiv\beta(2)}{}' \frac{-32(\alpha^2-5\beta^2)}{(\alpha^2+5\beta^2)^3}+\sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{8(\alpha^2-5\beta^2)}{(\alpha^2+5\beta^2)^3}\\&=\frac{-128}{5}\sum_{\alpha\equiv\beta\equiv1(2)}{}' \frac{\alpha^2-5\beta^2}{(\alpha^2+5\beta^2)^3}+\frac{32}{5}\sum_{\alpha\not\equiv\beta(2)}{}' \frac{\alpha^2-5\beta^2}{(\alpha^2+5\beta^2)^3},\\S_2&=\sum_{\alpha\equiv\beta(2)}{}' \frac{-16}{(\alpha^2+5\beta^2)^2}+\sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{4}{(\alpha^2+5\beta^2)^2}.\end{aligned}$$

Let $K=\mathbb{Q}(\sqrt{-5})$, $\Lambda=(2)=\mathfrak{p}^2$, where $\mathfrak{p}=(2,\ 1+\sqrt{-5})$ a non-principal prime ideal of $\mathcal{O}_K$. Then the class group of $\mathcal{O}_K$ is of order 2. Hence any ideal I is either principal, or $\mathfrak{p}I$ is principal. We can define a Hecke character ϕ by

$$\phi(I)=\begin{cases}x^2, & \text{if } I=(x) \text{ is principal;}\\ x^2/2, & \text{if } \mathfrak{p}I=(x).\end{cases}$$

One can easily prove that ϕ defines a homomorphism on the group of integral ideals. For example, if $I,\ J$ are two non-principal ideals and $\mathfrak{p}I=(x),\ \mathfrak{p}J=(y)$. Then $IJ=(\frac{xy}{2})$ and

$$\phi(IJ)=\phi\left(\frac{xy}{2}\right)=\frac{(xy)^2}{4}=\frac{x^2}{2}\frac{y^2}{2}=\phi(I)\phi(J).$$

Similarly, we can also define a Hecke character $\hat{\phi}$ by

$$\hat{\phi}(I)=\begin{cases}x^2, & \text{if } I=(x) \text{ is principal;}\\ -x^2/2, & \text{if } \mathfrak{p}I=(x).\end{cases}$$

One can also prove directly that $\hat{\phi}$ defines a homomorphism on the group of ideals.

By Theorem 1.31 of [9], we have

$$\begin{aligned}g(\tau)&=\sum_{I\lhd\mathcal{O}_K}\phi(I)q^{N(I)}\\&=\sum_{\alpha,\beta\in\mathbb{Z}}\frac{\alpha^2-5\beta^2}{2}q^{\alpha^2+5\beta^2}+\frac{1}{2}\sum_{\alpha\equiv\beta(2)}\frac{\alpha^2-5\beta^2}{2}q^{(\alpha^2+5\beta^2)/2},\end{aligned}$$

$$
\begin{aligned}
h(\tau) &= \sum_{I \lhd \mathcal{O}_K} \hat{\phi}(I) q^{N(I)} \\
&= \sum_{\alpha,\beta\in\mathbb{Z}} \frac{\alpha^2-5\beta^2}{2} q^{\alpha^2+5\beta^2} - \frac{1}{2} \sum_{\alpha\equiv\beta(2)} \frac{\alpha^2-5\beta^2}{2} q^{(\alpha^2+5\beta^2)/2}
\end{aligned}
$$

are newforms of level 20 and weight 3. In fact h and g are Newforms(Gamma1(20),3, names='a')[i] for $i = 0, 1$ respectively in Sagemath [13],

$$
\begin{aligned}
g =& q + 2q^2 - 4q^3 + 4q^4 - 5q^5 - 8q^6 + 4q^7 + 8q^8 + 7q^9 - 10q^{10} \\
&- 16q^{12} + 8q^{14} + 20q^{15} + 16q^{16} + 14q^{18} + O(q^{20}), \\
h =& q - 2q^2 + 4q^3 + 4q^4 - 5q^5 - 8q^6 - 4q^7 - 8q^8 + 7q^9 + 10q^{10} \\
&+ 16q^{12} + 8q^{14} - 20q^{15} + 16q^{16} - 14q^{18} + O(q^{20}).
\end{aligned}
$$

We can restrict ϕ to the group of ideals coprime to $\mathfrak{p}$ to get a new Hecke character and a new modular form. By Theorem 1.31 of [9], we have

$$
\begin{aligned}
h_1(\tau) &= \sum_{\mathfrak{p}\nmid I} \phi(I) q^{N(I)} \\
&= \sum_{\mathfrak{p}\nmid I,\, I \text{ principal}} \phi(I) q^{N(I)} + \sum_{\mathfrak{p}\nmid I,\, \mathfrak{p}I \text{ principal}} \phi(I) q^{N(I)} \\
&= \sum_{\alpha\not\equiv\beta(2)} \frac{\alpha^2-5\beta^2}{2} q^{\alpha^2+5\beta^2} + \frac{1}{2} \sum_{\alpha\equiv\beta\equiv 1(2)} \frac{\alpha^2-5\beta^2}{2} q^{(\alpha^2+5\beta^2)/2} \\
&= h \otimes \chi_{-4}
\end{aligned}
$$

is a newform of level $r|80$ and weight 3. Let

$$
\begin{aligned}
g_1 &= g \otimes \chi_{-4} \\
&= \sum_{\alpha\not\equiv\beta(2)} \frac{\alpha^2-5\beta^2}{2} q^{\alpha^2+5\beta^2} - \sum_{\alpha\equiv\beta\equiv 1(2)} \frac{\alpha^2-5\beta^2}{4} q^{(\alpha^2+5\beta^2)/2}, \\
f_1 &= g_1 + h_1 = \sum_{\alpha\not\equiv\beta(2)} (\alpha^2-5\beta^2) q^{\alpha^2+5\beta^2}, \\
f_2 &= -g_1 + h_1 = \sum_{\alpha\equiv\beta\equiv 1(2)} \frac{\alpha^2-5\beta^2}{2} q^{(\alpha^2+5\beta^2)/2}.
\end{aligned}
$$

Hence

$$S_1 = \frac{16}{5} \sum_{\alpha\equiv\beta\equiv 1(2)}' \frac{-(\alpha^2-5\beta^2)}{(\frac{\alpha^2+5\beta^2}{2})^3} + \frac{32}{5} \sum_{\alpha\not\equiv\beta(2)}' \frac{\alpha^2-5\beta^2}{(\alpha^2+5\beta^2)^3}$$
$$= \frac{32}{5}(L(f_1,3)-L(f_2,3)) = \frac{32}{5}(L(f_1-f_2,3)).$$

So

$$\frac{5\sqrt{5}}{\pi^3}(-A+4B) = \frac{5\sqrt{5}}{\pi^3}(S_1+S_2) = \frac{64\sqrt{5}}{\pi^3}L(g_1,3) + \frac{5\sqrt{5}}{\pi^3}S_2.$$

Since

$$h(\tau) = \sum_{\alpha,\beta\in\mathbb{Z}} \frac{\alpha^2-5\beta^2}{2} q^{\alpha^2+5\beta^2} - \sum_{\alpha\equiv\beta(2)} \frac{\alpha^2-5\beta^2}{4} q^{(\alpha^2+5\beta^2)/2},$$
$$g_1(\tau) = \sum_{\alpha\not\equiv\beta(2)} \frac{\alpha^2-5\beta^2}{2} q^{\alpha^2+5\beta^2} - \sum_{\alpha\equiv\beta\equiv 1(2)} \frac{\alpha^2-5\beta^2}{4} q^{(\alpha^2+5\beta^2)/2},$$

we can see that g_1 is the odd exponential part of h and $f_1 - f_2 = 2g_1$. This is a modular form of weight 3, level may be greater than 20. Similarly,

$$-2h(2\tau) = -\sum_{\alpha,\beta\in\mathbb{Z}} \frac{(2\alpha)^2-5(2\beta)^2}{4} q^{\frac{(2\alpha)^2+5(2\beta)^2}{2}} + \sum_{\alpha\equiv\beta(2)} \frac{\alpha^2-5\beta^2}{2} q^{(\alpha^2+5\beta^2)}$$

is exactly the even exponential part of h. Hence $-2h(2\tau) + g_1 = h$ which implies $L(g_1,3) = L(h,3) + \frac{1}{4}L(h,3) = \frac{5}{4}L(h,3)$.

Now we begin to compute S_2,

$$S_2 = \sum_{\alpha\equiv\beta(2)}' \frac{-16}{(\alpha^2+5\beta^2)^2} + \sum_{\alpha,\beta\in\mathbb{Z}}' \frac{4}{(\alpha^2+5\beta^2)^2}$$
$$= \sum_{\alpha\equiv\beta\equiv 1(2)}' \frac{-16}{(\alpha^2+5\beta^2)^2} + \sum_{\alpha,\beta\in\mathbb{Z}}' \frac{3}{(\alpha^2+5\beta^2)^2}$$
$$= \sum_{\text{non principal odd } I\subseteq\mathcal{O}_K} \frac{-32}{N(\mathfrak{p}I)^2} + \sum_{\text{principal } I\subseteq\mathcal{O}_K} \frac{6}{N(I)^2}$$
$$= \sum_{\text{non principal odd } I\subseteq\mathcal{O}_K} \frac{-8}{N(I)^2} + \sum_{\text{principal } I\subseteq\mathcal{O}_K} \frac{6}{N(I)^2}.$$

Let

$$t = \sum_{I\subseteq\mathcal{O}_K} \frac{1}{N(I)^2} = \prod_{\mathfrak{q}} \left(1 - \frac{1}{N(\mathfrak{q})^2}\right)^{-1} = t_1 + t_2 + t_3 + t_4,$$

$$\begin{aligned} t_1 &= \sum_{\text{principalodd } I\subseteq\mathcal{O}_K} \frac{1}{N(I)^2}, \\ t_2 &= \sum_{\text{non principal odd } I\subseteq\mathcal{O}_K} \frac{1}{N(I)^2}, \\ t_3 &= \sum_{\text{principal even } I\subseteq\mathcal{O}_K} \frac{1}{N(I)^2} = \frac{1}{4}t_2 + \frac{1}{16}(t_1+t_3), \\ t_4 &= \sum_{\text{non principal even } I\subseteq\mathcal{O}_K} \frac{1}{N(I)^2}. \end{aligned} \tag{4.1}$$

By Lemma 2.4 and the value of Dedekind zeta function $\zeta_K(s)$ at 2, we have

$$\begin{aligned} t &= L_1(2)L_{-20}(2), \\ t_1+t_3 &= \frac{1}{2}\sum_{m,n\in\mathbb{Z}}' \frac{1}{(m^2+5n^2)^2} = \frac{1}{2}(L_1(2)L_{-20}(2)+L_{-4}(2)L_5(2)), \\ t_1+t_2 &= \prod_{\mathfrak{q}\neq\mathfrak{p}} \left(1-\frac{1}{N(\mathfrak{q})^2}\right)^{-1} = \frac{3}{4}t = \frac{3}{4}L_1(2)L_{-20}(2). \end{aligned} \tag{4.2}$$

Hence we have

$$S_2 = -8t_2 + 6(t_1+t_3) = 16(t_1+t_3) - \frac{32}{3}(t_1+t_2) = 8L_{-4}(2)L_5(2).$$

Note that $L_5(2) = \frac{4\sqrt{5}\pi^2}{125}$. Therefore

$$\begin{aligned} f_4(s_4(q)) &= \frac{10\mathrm{Im}(\tau)}{\pi^3}(-A+4B) = \frac{80\sqrt{5}}{\pi^3}L(g,3) + \frac{32}{5\pi}L_{-4}(2) \\ &= \frac{8}{5}(5L'(g,0)+2L'(\chi_{-4},-1)) \end{aligned}$$

by the functional equations of L-functions (see [14, Eq. (2.28)]), where g is a new form in $S_3(\Gamma_1(20))$ with rational coefficients.

4.2 $f_4(-12288)$

In this case, $\tau = -\frac{1}{2} + \frac{3}{2}i$. So $f_4(s_4(q)) = \frac{15}{\pi^3}(-A+4B)$, where

$$\begin{aligned} -A &= \sum_{m,n\in\mathbb{Z}}' \left(-\frac{64(2n-m)^2}{\left((2n-m)^2+9m^2\right)^3} + \frac{16}{\left((2n-m)^2+9m^2\right)^2} \right), \\ 4B &= \sum_{m,n\in\mathbb{Z}}' \left(\frac{16(n-m)^2}{\left((n-m)^2+9m^2\right)^3} - \frac{4}{\left((n-m)^2+9m^2\right)^2} \right). \end{aligned}$$

Let $\alpha = 2n - m$, and $\beta = n - m$, then

$$-\frac{64(2n-m)^2}{\left((2n-m)^2+9m^2\right)^3} + \frac{16}{\left((2n-m)^2+9m^2\right)^2} = -\frac{64\alpha^2}{(\alpha^2+9m^2)^3} + \frac{16}{(\alpha^2+9m^2)^2},$$
$$\frac{16(n-m)^2}{\left((n-m)^2+9m^2\right)^3} - \frac{4}{\left((n-m)^2+9m^2\right)^2} = \frac{16\beta^2}{(\beta^2+9m^2)^3} - \frac{4}{(\beta^2+9m^2)^2}.$$

Since $2n - m \equiv m \pmod 2$, we have

$$-A = \sum_{\alpha\equiv m(2)}{}' \left(-\frac{32(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3} - \frac{16}{(\alpha^2+9m^2)^2}\right),$$
$$4B = \sum_{\beta,m\in\mathbb{Z}}{}' \left(\frac{8(\beta^2-9m^2)}{(\beta^2+9m^2)^3} + \frac{4}{(\beta^2+9m^2)^2}\right).$$

Let

$$S_1 = \sum_{\alpha\equiv m(2)}{}' -\frac{32(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3} + \sum_{\alpha,m\in\mathbb{Z}}{}' \frac{8(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3},$$
$$S_2 = \sum_{\alpha\equiv m(2)}{}' \frac{-16}{(\alpha^2+9m^2)^2} + \sum_{\alpha,m\in\mathbb{Z}}{}' \frac{4}{(\alpha^2+9m^2)^2}.$$

Then

$$-A + 4B = S_1 + S_2.$$

Now we will use Theorem 1.31 of [9] to compute the first half of the RHS of the above formula. Let $K = \mathbb{Q}(\sqrt{-4})$, $\Lambda = (3)$ an ideal. The residue field of Λ is $\mathbb{F}_9$. Note that $\pm 1, \pm i$ are the square elements of $\mathbb{F}_9^*$. Any ideal $\mathfrak{a}$ prime to Λ is of the form

$$\mathfrak{a} = (a),\ a \equiv 1 \text{ or } 1 + i \pmod{\Lambda}.$$

Note that $\mathfrak{a}$ has a generator $a \equiv 1 \pmod{\Lambda}$ if and only if $N(\mathfrak{a}) \equiv 1 \pmod{\Lambda}$.

Then the map ϕ defined by

$$\phi(\mathfrak{a}) = \phi(a) = \begin{cases} a^2, & \text{if } a \equiv 1 \pmod{\Lambda}, \\ a^2 i, & \text{if } a \equiv 1 + i \pmod{\Lambda}, \end{cases}$$

is a Hecke character. One can verify that ϕ is a homomorphism by direct computation. For example, if $\mathfrak{a} = (a)$, $\mathfrak{b} = (b)$ with $a \equiv 1 + i \pmod{\Lambda}$, $b \equiv 1 + i \pmod{\Lambda}$, then $\mathfrak{ab} = (ab) = (abi)$, $abi \equiv -2 \equiv 1 \pmod{\Lambda}$ which implies that

$$\phi(\mathfrak{ab}) = (abi)^2 = -(ab)^2 = a^2 i \cdot b^2 i = \phi(\mathfrak{a})\phi(\mathfrak{b}).$$

Similarly, we can define another Hecke character $\hat{\phi}$ by

$$\hat{\phi}(\mathfrak{a}) = \phi((a)) = \begin{cases} a^2, & \text{if } a \equiv 1 \pmod{\Lambda}; \\ -a^2 i, & \text{if } a \equiv 1+i \pmod{\Lambda}. \end{cases}$$

Hence by Theorem 1.31 of [9], we have

$$g = \sum_{3 \nmid \mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})},$$
$$h = \sum_{3 \nmid \mathfrak{a}} \hat{\phi}(\mathfrak{a}) q^{N(\mathfrak{a})}$$

are new forms of level 36 and weight 3 by Sagemath [13].

They have Fourier expansions

$$g = q - 2q^2 + 4q^4 + 8q^5 - 8q^8 - 16q^{10} - 10q^{13} + 16q^{16} - 16q^{17} + O(q^{20}),$$
$$h = q + 2q^2 + 4q^4 - 8q^5 + 8q^8 - 16q^{10} - 10q^{13} + 16q^{16} + 16q^{17} + O(q^{20}).$$

Let $g_1 = g(\tau) \otimes \chi_{-3}(\tau) + g(\tau)$. Then

$$g_1(\tau) = 2 \sum_{N(\mathfrak{a}) \equiv 1 (3)} \phi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{m,n \in \mathbb{Z},\ 3 \nmid m} (m^2 - 9n^2) q^{m^2+9n^2}$$
$$= g + h.$$

So we have

$$(g_1 - g_1 \otimes \chi_{-4})(\tau) = \sum_{m \equiv n (2)}{}' (m^2 - 9n^2) q^{m^2+9n^2} = -2g(2\tau) + 2h(2\tau).$$

Note that we can omit the condition $3 \nmid m$ for

$$\sum_{m,n \in \mathbb{Z}, 3 | m} (m^2 - 9n^2) q^{m^2+9n^2} = 0.$$

Let $g_2 = g_1 - g_1 \otimes \chi_{-4}$, then

$$L(g_2, 3) = \sum_{\alpha \equiv m (2)}{}' \frac{(\alpha^2 - 9m^2)}{(\alpha^2 + 9m^2)^3} = \frac{1}{4} L(h - g, 3).$$

Now we need to know the value of

$$\sum_{\alpha, m \in \mathbb{Z}}{}' \frac{(\alpha^2 - 9m^2)}{(\alpha^2 + 9m^2)^3}.$$

We will compute this part:

$$\sum_{\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3} = \sum_{3|\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3} + \sum_{3\nmid\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3}$$
$$= \frac{1}{81}\sum_{\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-m^2)}{(\alpha^2+m^2)^3} + \sum_{3\nmid\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3}.$$

Note that $\frac{1}{81}\sum_{\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-m^2)}{(\alpha^2+m^2)^3} = 0$. Hence

$$\sum_{\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3} = \sum_{3\nmid\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3}.$$

So

$$L(g+h,3) = \sum_{\alpha,m\in\mathbb{Z}}{}' \frac{(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3}.$$

Hence

$$S_1 = \sum_{\alpha\equiv m(2)}{}' - \frac{32(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3} + \sum_{\alpha,m\in\mathbb{Z}}{}' \frac{8(\alpha^2-9m^2)}{(\alpha^2+9m^2)^3}$$
$$= -8L(h-g,3) + 8L(g+h,3)$$
$$= 16L(g,3).$$

Therefore $\frac{15}{\pi^3}S_1 = \frac{40}{9}L'(g,0)$ by the functional equation of L-functions. Now we begin to compute

$$S_2 = \sum_{\alpha\equiv m(2)}{}' \frac{-16}{(\alpha^2+9m^2)^2} + \sum_{\alpha,m\in\mathbb{Z}}{}' \frac{4}{(\alpha^2+9m^2)^2}.$$

Note that

$$\sum_{\alpha\equiv m(2)}{}' \frac{-16}{(\alpha^2+9m^2)^2} = \sum_{\alpha\equiv m(2)}{}' \frac{-16}{N(\alpha+3mi)^2} = 4\sum_{\alpha\equiv m(2)}{}' \frac{-16}{N((1+i)(\alpha+3mi))^2}$$
$$= \sum_{\alpha\equiv m(2)}{}' \frac{-64}{(N((\alpha-3m)+(\alpha+3m)i))^2}.$$

Let

$$\begin{cases} x &= \frac{\alpha-3m}{2}, \\ y &= \frac{\alpha+3m}{2}. \end{cases}$$

Then $x \equiv y\ (3)$ and

$$\sum_{\alpha\equiv m(2)}' \frac{-64}{(N((\alpha-3m)+(\alpha+3m)i))^2} = \sum_{x\equiv y(3)}' \frac{-4}{(x^2+y^2)^2}.$$

Note that

$$\sum_{x\equiv y(3)}' \frac{-4}{(x^2+y^2)^2} = \sum_{x\equiv -y(3)}' \frac{-4}{(x^2+y^2)^2}.$$

Since

$$\sum_{x\equiv y(3)}' + \sum_{x\not\equiv y(3)}' + \sum_{xy\equiv 0(3)}' = \sum_{x,y\in\mathbb{Z}}' + 2\sum_{x\equiv y\equiv 0(3)}',$$

the following identity holds:

$$\begin{aligned}
&\sum_{x\equiv y(3)}' \frac{-4}{(x^2+y^2)^2} + \sum_{x\equiv -y(3)}' \frac{-4}{(x^2+y^2)^2} + 2\sum_{x,y\in\mathbb{Z}}' \frac{-4}{(x^2+9y^2)^2} \\
&\quad = \sum_{x,y\in\mathbb{Z}}' \frac{-4}{(x^2+y^2)^2} + \sum_{x,y\in\mathbb{Z}}' \frac{-4}{27(x^2+y^2)^2} = \sum_{x,y\in\mathbb{Z}}' \frac{-112}{27(x^2+y^2)^2}.
\end{aligned}$$

Hence

$$\sum_{x\equiv y(3)}' \frac{-4}{(x^2+y^2)^2} = \sum_{x,y\in\mathbb{Z}}' \frac{-56}{27(x^2+y^2)^2} + \sum_{x,y\in\mathbb{Z}}' \frac{4}{(x^2+9y^2)^2}.$$

By Lemma 2.4, we have

$$\begin{aligned}
S_2 &= \sum_{x,y\in\mathbb{Z}}' \frac{-56}{27(x^2+y^2)^2} + \sum_{x,y\in\mathbb{Z}}' \frac{8}{(x^2+9y^2)^2} \\
&= \frac{-56}{27} 4L_1(2)L_{-4}(2) + 8\left(\frac{28}{27} L_1(2)L_{-4}(2) + L_{-3}(2)L_{12}(2)\right) \\
&= 8L_{-3}(2)L_{12}(2) = 8L_{-3}(2)\left(\frac{\pi^2\sqrt{3}}{18}\right) = \frac{4\pi^2\sqrt{3}}{9} L_{-3}(2).
\end{aligned}$$

By the functional equation of Dirichlet L-functions, we have $L_{-3}(2) = \frac{4\pi}{3\sqrt{3}} L'_{-3}(-1)$. So $\frac{15}{\pi^3} S_2 = \frac{80}{9} L'_{-3}(-1)$ and

$$f_4(s_4(q)) = \frac{15}{\pi^3}(S_1+S_2) = \frac{40}{9} L'(g,0) + \frac{80}{9} L'_{-3}(-1),$$

where g is a new form with rational coefficients of weight 3 and level 36.

4.3 $f_4(-82994)$

Note that the argument of this section is quite similar to Sect. 4.1, the computation of $f_4(-1024)$.

In this case, $\tau = -\frac{1}{2} + \frac{\sqrt{13}}{2}i$. So $f_4(s_4(q)) = \frac{5\sqrt{13}}{\pi^3}(-A+4B)$, where

$$-A = \sum_{m,n\in\mathbb{Z}}{}' \left(-\frac{64(2n-m)^2}{\left((2n-m)^2+13m^2\right)^3} + \frac{16}{\left((2n-m)^2+13m^2\right)^2} \right),$$
$$4B = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{16(n-m)^2}{\left((n-m)^2+13m^2\right)^3} - \frac{4}{\left((n-m)^2+13m^2\right)^2} \right).$$

Hence

$$-A = \sum_{\alpha\equiv\beta(2)}{}' \left(-\frac{32(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3} - \frac{16}{(\alpha^2+13\beta^2)^2} \right),$$
$$4B = \sum_{\alpha,\beta\in\mathbb{Z}}{}' \left(\frac{8(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3} + \frac{4}{(\alpha^2+13\beta^2)^2} \right).$$

So

$$-A+4B = \sum_{\alpha\equiv\beta(2)}{}' \frac{-32(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3} + \sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{8(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3}$$
$$+ \sum_{\alpha\equiv\beta(2)}{}' \frac{-16}{(\alpha^2+13\beta^2)^2} + \sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{4}{(\alpha^2+13\beta^2)^2}.$$

Let

$$S_1 = \sum_{\alpha\equiv\beta(2)}{}' \frac{-32(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3} + \sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{8(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3}$$
$$S_2 = \sum_{\alpha\equiv\beta(2)}{}' \frac{-16}{(\alpha^2+13\beta^2)^2} + \sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{4}{(\alpha^2+13\beta^2)^2}.$$

Then

$$S_1 = \sum_{\alpha\equiv\beta(2)}{}' \frac{-24(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3} + \sum_{\alpha\not\equiv\beta(2)}{}' \frac{8(\alpha^2-13\beta^2)}{(\alpha^2+13\beta^2)^3}$$
$$= \frac{-128}{5} \sum_{\alpha\equiv\beta\equiv1(2)}{}' \frac{\alpha^2-13\beta^2}{(\alpha^2+13\beta^2)^3} + \frac{32}{5} \sum_{\alpha\not\equiv\beta(2)}{}' \frac{\alpha^2-13\beta^2}{(\alpha^2+13\beta^2)^3}.$$

Let $K=\mathbb{Q}(\sqrt{-13})$, $\Lambda=(2)=\mathfrak{p}^2$ an ideal of $\mathcal{O}_K$. Then the class group of $\mathcal{O}_K$ is of order 2. Hence any ideal I is either principal, or $\mathfrak{p}I$ is principal. Then we can define a Hecke character ϕ by

$$\phi(I)=\begin{cases}x^2, & \text{if } I=(x) \text{ is principal,}\\ x^2/2, & \text{if } \mathfrak{p}I=(x).\end{cases}$$

One can verify directly that ϕ is a Hecke character as in Sect. 4.1. Similarly, we can also define a Hecke character $\hat{\phi}$ by

$$\hat{\phi}(I)=\begin{cases}x^2, & \text{if } I=(x) \text{ is principal,}\\ -x^2/2, & \text{if } \mathfrak{p}I=(x).\end{cases}$$

By Theorem 1.31 of [9], we know that

$$\begin{aligned}h(\tau)&=\sum_I \phi(I)q^{N(I)}\\&=\sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{\alpha^2-13\beta^2}{2}q^{\alpha^2+13\beta^2}+\frac{1}{2}\sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{\alpha^2-13\beta^2}{2}q^{(\alpha^2+13\beta^2)/2}\\ g(\tau)&=\sum_I \hat{\phi}(I)q^{N(I)}\\&=\sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{\alpha^2-13\beta^2}{2}q^{\alpha^2+13\beta^2}-\frac{1}{2}\sum_{\alpha,\beta\in\mathbb{Z}}{}' \frac{\alpha^2-13\beta^2}{2}q^{(\alpha^2+13\beta^2)/2}\end{aligned}$$

are newforms of level 52 and weight 3, and

$$\begin{aligned}f(\tau)&=\sum_{\mathfrak{p}\nmid I}\phi(I)q^{N(I)}=\sum_{\mathfrak{p}\nmid I,I\ \text{principal}}\phi(I)q^{N(I)}+\sum_{\mathfrak{p}\nmid I,\mathfrak{p}I\ \text{principal}}\phi(I)q^{N(I)}\\&=\sum_{\alpha\not\equiv\beta(2)}{}' \frac{\alpha^2-13\beta^2}{2}q^{\alpha^2+13\beta^2}+\frac{1}{2}\sum_{\alpha\equiv\beta\equiv1(2)}{}' \frac{\alpha^2-13\beta^2}{2}q^{(\alpha^2+13\beta^2)/2}\end{aligned}$$

is a newform of level 208 and weight 3.

The new forms g and h are Newforms(Gamma1(52),3,names='a')[i] for $i=0,1$ respectively in Sagemath [13],

$$\begin{aligned}g&=q-2q^2+4q^4+12q^7-8q^8+9q^9+4q^{11}-13q^{13}-24q^{14}\\&\quad+16q^{16}-18q^{17}-18q^{18}-12q^{19}+O(q^{20})\\ h&=q+2q^2+4q^4-12q^7+8q^8+9q^9-4q^{11}-13q^{13}-24q^{14}\\&\quad+16q^{16}-18q^{17}+18q^{18}+12q^{19}+O(q^{20}).\end{aligned}$$

Let

$$\begin{aligned} g_1 &= g \otimes \chi_{-4} \\ &= \sum_{\alpha \not\equiv \beta (2)}' \frac{\alpha^2 - 13\beta^2}{2} q^{\alpha^2+13\beta^2} + \sum_{\alpha \equiv \beta \equiv 1(2)}' \frac{\alpha^2 - 13\beta^2}{4} q^{(\alpha^2+13\beta^2)/2} = f, \\ h_1 &= h \otimes \chi_{-4} = \sum_{\alpha \not\equiv \beta (2)}' \frac{\alpha^2 - 13\beta^2}{2} q^{\alpha^2+13\beta^2} - \sum_{\alpha \equiv \beta \equiv 1(2)}' \frac{\alpha^2 - 13\beta^2}{4} q^{(\alpha^2+13\beta^2)/2}. \end{aligned}$$

Then

$$\begin{aligned} f_1 = g_1 + h_1 &= \sum_{\alpha \not\equiv \beta (2)}' (\alpha^2 - 13\beta^2) q^{\alpha^2+13\beta^2}, \\ f_2 = g_1 - h_1 &= \sum_{\alpha \equiv \beta \equiv 1(2)}' \frac{\alpha^2 - 13\beta^2}{2} q^{(\alpha^2+13\beta^2)/2}. \end{aligned}$$

Hence

$$\begin{aligned} S_1 =& \frac{16}{5} \sum_{\alpha \equiv \beta \equiv 1(2)}' \frac{-(\alpha^2 - 13\beta^2)}{(\frac{\alpha^2+13\beta^2}{2})^3} + \frac{32}{5} \sum_{\alpha \not\equiv \beta (2)}' \frac{(\alpha^2 - 13\beta^2)}{(\alpha^2 + 13\beta^2)^3} \\ =& \frac{32}{5}(L(f_1, 3) - L(f_2, 3)) = \frac{32}{5} L(f_1 - f_2, 3). \end{aligned}$$

So

$$\frac{5\sqrt{13}}{\pi^3}(-A + 4B) = \frac{5\sqrt{13}}{\pi^3}(S_1 + S_2) = \frac{64\sqrt{13}}{\pi^3} L(h_1, 3) + \frac{5\sqrt{13}}{\pi^3} S_2,$$

where $f_1 - f_2 = 2h_1$ is the odd exponential part of $2g$, and $-2g(2\tau)$ is the even exponential part of g by similar argument as in Sect. 4.1. Hence $-2g(2\tau) + h_1 = g$ which implies $L(h_1, 3) = L(g, 3) + \frac{1}{4} L(g, 3) = \frac{5}{4} L(g, 3)$. Now we begin to compute S_2,

$$\begin{aligned} S_2 &= \sum_{\alpha \equiv \beta (2)}' \frac{-16}{(\alpha^2 + 13\beta^2)^2} + \sum_{\alpha, \beta \in \mathbb{Z}}' \frac{4}{(\alpha^2 + 13\beta^2)^2} \\ &= \sum_{\alpha \equiv \beta \equiv 1(2)}' \frac{-16}{(\alpha^2 + 13\beta^2)^2} + \sum_{\alpha, \beta \in \mathbb{Z}}' \frac{3}{(\alpha^2 + 13\beta^2)^2} \\ &= \sum_{\text{non principal odd } I} \frac{-32}{N(\mathfrak{p} I)^2} + \sum_{\text{principal } I} \frac{6}{N(I)^2} \\ &= \sum_{\text{non principal odd } I} \frac{-8}{N(I)^2} + \sum_{\text{principal } I} \frac{6}{N(I)^2}. \end{aligned}$$

Let

$$
\begin{aligned}
t &= \sum_{I\subseteq\mathcal{O}_K} \frac{1}{N(I)^2} = t_1+t_2+t_3+t_4,\\
t_1 &= \sum_{\text{principalodd } I} \frac{1}{N(I)^2},\\
t_2 &= \sum_{\text{non principal odd } I} \frac{1}{N(I)^2},\\
t_3 &= \sum_{\text{principal even } I} \frac{1}{N(I)^2} = \frac{1}{4}t_2 + \frac{1}{16}(t_1+t_3),\\
t_4 &= \sum_{\text{non principal even } I} \frac{1}{N(I)^2}.
\end{aligned}
$$

We know that

$$
\begin{aligned}
t &= L_1(2)L_{-52}(2),\\
t_1+t_3 &= \frac{1}{2}(L_1(2)L_{-52}(2)+L_{-4}(2)L_{13}(2)),\\
t_1+t_2 &= \frac{3}{4}t = \frac{3}{4}L_1(2)L_{-52}(2).
\end{aligned}
$$

Hence we have

$$
S_2 = -8t_2 + 6(t_1+t_3) = 16(t_1+t_3) - \frac{32}{3}(t_1+t_2) = 8L_{-4}(2)L_{13}(2).
$$

Note that $L_{13}(2) = \frac{4\pi^2}{13\sqrt{13}}$. Hence

$$
\begin{aligned}
f_4(s_4(q)) &= \frac{10\mathrm{Im}(\tau)}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' (-A+4B) = \frac{80\sqrt{13}}{\pi^3}L(g,3) + \frac{160}{13\pi}L_{-4}(2)\\
&= \frac{40}{13}(L'(g,0)+2L'(\chi_{-4},-1))
\end{aligned}
$$

by the functional equations of L-functions (see [14, Eq. (2.28)]). By Sagemath [13], we know g is a new form with rational coefficients of level 52 and weight 3.

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