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Governing Fields of the 4-rank of $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{dp})}$ as p Varies

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Abstract

In this paper, we prove the existence of governing fields of the 4-rank of $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{dp})}$ as p varies. For some special d, we prove that the governing field of the 8-rank of $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{dp})}$ also exists.

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1 Introduction

Let n be a rational integer, $F = \mathbb{Q}(\sqrt{n})$, $\mathrm{Cl}(F)$ (or $\mathrm{Cl}(n)$) the class group of F, $h(n)$ the cardinality of $\mathrm{Cl}(F)$, $\mathrm{Cl}_2(n)$ (or $\mathrm{Cl}_2(F)$) the 2-Sylow subgroup of $\mathrm{Cl}(n)$ (or $\mathrm{Cl}(F)$). The following theorem is well known.

Theorem 1.1. ([4], (1.1)-(1.6)) *If p is a prime number, then*
(1) $2|h(-p)$ *if and only if p splits completely in* $\mathbb{Q}(i)$;
(2) $4|h(-p)$ *if and only if p splits completely in* $\mathbb{Q}(\zeta_8)$;
(3) $8|h(-p)$ *if and only if p splits completely in* $\mathbb{Q}(\zeta_8, \sqrt{1+\sqrt{2}})$.

One can see [4] for the history of this theorem. Later Stevenhagen gave another example in his Ph. D. thesis [21].

Theorem 1.2. (Stevenhagen, [21]) *Let p be a prime congruent to* 3 *modulo* 4. *Then the 4-rank of* $\mathrm{Cl}(-21p)$ *equals* 1 *unless* $p = 7$ *or* $\left(\frac{p}{3}\right) = -\left(\frac{p}{7}\right) = 1$, *when it is* 0. *And the 8-rank of* $\mathrm{Cl}(-21p)$ *is* 1 *if and only if p splits completely in one of*

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the following fields:

$$M_1 = \mathbb{Q}(\sqrt{-3},\ \sqrt{7},\ \sqrt{2-\sqrt{-3}}),$$
$$M_2 = \mathbb{Q}(\sqrt{3},\ \sqrt{7},\ \sqrt{2(7+\sqrt{21})}),$$
$$M_3 = \mathbb{Q}(\sqrt{-3},\ \sqrt{-7},\ \sqrt{-3+2\sqrt{-3}}).$$

These interesting examples suggest that the following conjecture raised by Cohn and Lagarias is true.

Conjecture $C_j(d)$**:** *Given an integer* $d \not\equiv 2 \pmod 4$*, there exists a normal extension* $K = K_j(d)$ *of* $\mathbb{Q}$ *having the following property* $P_j(d)$.

Property $P_j(d)$**:** *If* p_1 *and* p_2 *are primes such that* $[(K/\mathbb{Q})/(p_1)] = [(K/\mathbb{Q})/(p_2)]$ *then* $\mathrm{Cl}(dp_1)$ *and* $\mathrm{Cl}(dp_2)$ *have the same* 2^k*-rank for* $1 \le k \le j$*, where* $[(K/\mathbb{Q})/(p_i)]$ *is the Artin symbol in* $\mathrm{Gal}(K/\mathbb{Q})$, $1 \le i \le 2$.

Cohn and Lagarias proved in [4] that if there is a field with Property $P_j(d)$, then there exists a unique field $\Omega_j(d)$ of smallest degree with this property. Such a field $\Omega_j(d)$ is called a governing field.

The Conjecture $C_3(d)$ is finally proved by Stevenhagen in his thesis.

Let $K_2(dp) = K_2\mathcal{O}_{\mathbb{Q}(\sqrt{dp})}$. Inspired by the Conjecture $C_j(d)$, we raise the following K_2-analogue.

Conjecture $K_2(j,d)$**:** *Given an integer* $d \not\equiv 2 \pmod 4$*, there exists a normal extension* $L = L_j(d)$ *of* $\mathbb{Q}$ *having the following property* $\widetilde{P_j(d)}$.

Property $\widetilde{P_j(d)}$**:** *If* p_1 *and* p_2 *are primes such that* $[(L/\mathbb{Q})/(p_1)] = [(L/\mathbb{Q})/(p_2)]$ *then* $K_2(dp_1)$ *and* $K_2(dp_2)$ *have the same* 2^k*-rank for* $1 \le k \le j$*, where* $[(L/\mathbb{Q})/(p_i)]$ *is the Artin symbol in* $\mathrm{Gal}(L/\mathbb{Q})$, $1 \le i \le 2$.

One can see a similar idea in Kimura's talk [10].

For any finite abelian group G and positive integer k, let $\mathrm{rk}_{2^k}(G)$ be the 2^k-rank of G. Let F be a number field, r_1 the number of real embeddings of F, $g_2(F)$ the number of distinct prime ideals of $\mathcal{O}_F$ above 2, $\mathrm{C}_2(F)$ the subgroup of Cl generated by the prime ideals of $\mathcal{O}_F$ above 2. Then by Theorem 6.2 of [22],

$$\mathrm{rk}_2(K_2(\mathcal{O}_F)) = \mathrm{rk}_2(\mathrm{Cl}(F)/\mathrm{C}_2(F)) + g_2(F) + r_1 - 1.$$

One can also see [2] and [3] for more details. Hence Conjecture $K_2(1,d)$ is true.

In 1991, Qin gave a method to determine the 4-rank of $K_2\mathcal{O}_F$ in his Ph. D. thesis. By Qin's method, the 4-rank of $K_2\mathcal{O}_F$ can be obtained by considering the local Hilbert symbols. In [9], J.Hurrelbrink and M.Kolster introduced a kind of signs matrix to compute $\mathrm{rk}_4(K_2\mathcal{O}_F)$, which is via the local Hilbert symbols. One can also see [5] for a similar signs matrix to compute $\mathrm{rk}_4(K_2\mathcal{O}_F)$.

Theorem 1.3. (Qin, 15–17, [20]) *Let* $F = \mathbb{Q}(\sqrt{d}), d \in \mathbb{Z}$ *square-free. Suppose that* $m \mid d$ $(m > 0$ *if* $d > 0)$ *and write* $d = u^2 - 2w^2$ *with* $u, w \in \mathbb{Z}$ (*we take* $u > 0$ *if* $d > 0$) *if* $2 \in NF$. *Let* $S(d) = \{\pm 1,\ \pm 2\}$ *if* $d > 0$, *and* $\{1,\ 2\}$ *if* $d < 0$. *Then* $\{-1, m\} \in {K_2\mathcal{O}_F}^2$ *if and only if one can find an* $\varepsilon \in S(d)$ *such that for any odd prime* $p \mid d$,

$$\left(\frac{-d, m}{p}\right) = \left(\frac{\varepsilon}{p}\right),$$

and $\{-1, m(u+\sqrt{d})\} \in K_2\mathcal{O}_F{}^2$ *if and only if one can find a* $\delta \in S(d)$ *such that for any odd prime* $p \mid d$,

$$\left(\frac{-d, m}{p}\right) = \left(\frac{\delta(u+w)}{p}\right).$$

In [15–17], Qin gave tables of the 4-rank of tame kernels of $K_2\mathcal{O}_F$, where the number of odd prime factors of d is less than or equal to 3. In [19, 20, 23, 24], Qin, Yin and Zhu also determine the 4-rank of $K_2\mathcal{O}_F$ for arbitrary quadratic number fields F.

By Qin's theorem, one can easily get the relation between the 4-rank of $K_2\mathcal{O}_F$ and the 4-rank of the class group of $\mathcal{O}_F$. One can see [1] and [25] for such relations. Since the density of 4-rank of class group of $\mathcal{O}_F$ is already known, we can get the density of the 4-rank of $K_2\mathcal{O}_F$ by Qin's method (one can see [7] for details).

In this paper, we will show that the conjecture $K_2(2, d)$ is true. In general cases, in order to know the 8-rank of tame kernels $K_2\mathcal{O}_F$, one need to know the 16-rank of class groups by Qin's theorems in [18] and [20]. Hence we raise the following conjecture.

Conjecture. $C_{j+1}(d)$ *implies* $K_2(j, d)$.

Both of $C_{j+1}(d)$ and $K_2(j, d)$ are very difficult. Hence we will be satisfied if the above conjecture can be proved.

This paper is organized as follows. In Section 2, we prove that $K_2(j, d)$ is true for $j \leq 2$. In Section 3, we prove that for some special d, the governing field of the 8-rank of $K_2\mathcal{O}_{\mathbb{Q}(\sqrt{dp})}$ exists.

2 The governing field of the 4-rank of $K_2\mathcal{O}_F$

Let $F = \mathbb{Q}(\sqrt{D})$, where D is the discriminant of F. Let D' be the square free part of D, $\mathrm{Cl}^+(D)$ be the narrow class group of F.

It is well known that $2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$ if and only if all odd prime divisors of D are congruent to ± 1 modulo 8, i.e., the local Hilbert symbol

$$(2,\ D)_p = 1$$

for any prime numbers. We know that $2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$ if and only if

$$D \in \mathrm{Norm}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(\mathbb{Q}(\sqrt{2})^\times).$$

If this is the case, we assume that

$$D' = u^2 - 2w^2,\ u,\ w \in \mathbb{Q}.$$

Since $\mathbb{Z}[\sqrt{2}]$ is a principal ideal domain, we can assume further that $u,\ w \in \mathbb{N}$. Let

$$v = \begin{cases} u+w, & \text{if } 2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^{\times}), \\ 2, & \text{otherwise.} \end{cases}$$

Let S_f be the finite set of rational primes consisting of prime 2 and all finite primes that ramify in F, $|S_f|$ the cardinality of S_f. Let A be a matrix whose entries are local Hilbert symbols. Following Kolster's notation of [11], we can view A as a matrix $\varphi(A)$ over $\mathbb{F}_2$ if we replace 1 by 0 and -1 by 1. The rank of A is understood as the $\mathbb{F}_2$-rank of $\varphi(A)$. J. Hurrelbrink and M. Kolster proved in [9] the following theorem.

Theorem 2.1. *Let F be a quadratic number field with discriminant D and $p_1, \cdots, p_t$ the odd primes dividing D. Then*

$$\mathrm{rk}_4(K_2\mathcal{O}_F) = \begin{cases} |S_f| - \mathrm{rk}(M_D), & \text{if } D > 0 \\ |S_f| + 1 - \mathrm{rk}(\widetilde{M}_D), & \text{if } D < 0, \end{cases}$$

where $\mathrm{rk}(\cdot)$ *means the rank of a matrix,*

$$M_D = \begin{pmatrix} (-D,\ p_1)_2 & (-D,\ p_1)_{p_1} & \cdots & (-D,\ p_1)_{p_t} \\ (-D,\ p_2)_2 & (-D,\ p_2)_{p_1} & \cdots & (-D,\ p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-D,\ p_{t-1})_2 & (-D,\ p_{t-1})_{p_1} & \cdots & (-D,\ p_{t-1})_{p_t} \\ (-D,\ v)_2 & (-D,\ v)_{p_1} & \cdots & (-D,\ v)_{p_t} \\ (D,\ -1)_2 & (D,\ -1)_{p_1} & \cdots & (D,\ -1)_{p_t} \end{pmatrix}$$

and

$$\widetilde{M}_D = \begin{pmatrix} (-D,\ p_1)_2 & (-D,\ p_1)_{p_1} & \cdots & (-D,\ p_1)_{p_t} \\ (-D,\ p_2)_2 & (-D,\ p_2)_{p_1} & \cdots & (-D,\ p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-D,\ p_{t-1})_2 & (-D,\ p_{t-1})_{p_1} & \cdots & (-D,\ p_{t-1})_{p_t} \\ (-D,\ v)_2 & (-D,\ v)_{p_1} & \cdots & (-D,\ v)_{p_t} \\ (-D,\ -1)_2 & (-D,\ -1)_{p_1} & \cdots & (-D,\ -1)_{p_t} \end{pmatrix}.$$

At first, we study the tame kernel of imaginary quadratic number fields. Let $d = -2^a p_1 \cdots p_t$ be the discriminant of some quadratic number field, where p_i $(1 \le i \le t-1)$ is odd positive prime number and $a = 0,\ 2,$ or 3. Let p be an odd prime different from any p_i $(1 \le i \le t-1)$ such that $D = dp$ is the discriminant of $F = \mathbb{Q}(\sqrt{D})$. Let $E = \mathbb{Q}(\sqrt{-D})$. By the reciprocity law, there is an abelian number field L such that the first $t-1$ rows and the last row of M_{dp} depends only on the Artin symbol $[(L/\mathbb{Q})/(p)]$. Hence the problem now is reduced to if the row

$$((-D,\ v)_2,\ (-D,\ v)_{p_1},\ \cdots,\ (-D,\ v)_{p_t})$$

can be linearly expressed by the other rows, where $p_t = p$. If $v = 2$, there is nothing to do. So from now on in this section we will always assume that $v \neq 2$, i.e., $2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^{\times})$.

Recall that D' is the square free part of D. If 2 is ramified in E, by Corollary 2.3 of [11], the dyadic prime ideal $[J]$ is a square in the narrow class group of E. In fact this holds even if 2 is not ramified. Since $D' = u^2 - 2w^2$,

$$2(u+w)^2 = (u + 2w + \sqrt{-D'})(u + 2w - \sqrt{-D'}).$$

Note that there is no inert prime q dividing $u + 2w + \sqrt{-D'}$. Otherwise this inert prime q divides also $u + 2w - \sqrt{-D'}$ which implies that q divides $2\sqrt{-D'}$ which is impossible. Hence the principal ideal $(u+2w+\sqrt{-D'}) \subset \mathcal{O}_E$ has a decomposition

$$u + 2w + \sqrt{-D'} = JI^2,$$

where J is a dyadic prime ideal, I is a split ideal with norm $u + w$ and both of $u+w$ and $u+2w+\sqrt{-D'}$ are totally positive. Hence $[J]$ is a square in the narrow class group of E. In fact, Kolster's argument also works for the fourth power, which means that we have the following proposition.

Proposition 2.2. (Kolster [11]) *With notations as above. The dyadic ideal class $[J]$ is a fourth power in the narrow class group of E if and only if there is a positive divisor m of D such that $m(u+w) \in \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$.*

Recall that in [11], an integral ideal I is primitive if I has the form

$$I = I' \cdot [m]$$

with m a square free positive divisor of D and I' an integral ideal, such that I' is prime to $\overline{I'}$, where $\overline{I'}$ is the congugacy ideal of I'.

Theorem 2.3. *With notations as above. Let $F = \mathbb{Q}(\sqrt{D})$, where D is the discriminant of F, $E = \mathbb{Q}(\sqrt{-D})$. We assume that $2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$. In the matrix $\widetilde{M_D}$, the row vector*

$$\alpha = ((-D,\ v)_2,\ (-D,\ v)_{p_1},\ \cdots,\ (-D,\ v)_{p_t})$$

can be linearly expressed by the other rows if and only if the dyadic ideal class $[J]$ is a fourth power in the narrow class group of E.

Proof. If the ideal class $[J]$ is a fourth power in the narrow class group of E, then α can be linearly expressed by the other rows by the above Proposition.

Conversely, if α can be linearly expressed by the other rows, we know that there is an integer $m | p_1 \cdots p_{t-1}$ such that $vm \in \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$ or $-vm \in \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$. If $vm \in \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$, by the above Proposition, ideal class $[J]$ is a fourth power in the narrow class group of E.

Next we assume that $-vm \in \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$ but $vm \notin \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$. Since $-1 \notin \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$, some of the odd prime factors of D must be congruent to 3 modulo 4. Hence by Lemma 10 of [6], we have

$$\mathrm{Cl}(E) \times \mathbb{Z}/2\mathbb{Z} = \mathrm{Cl}^+(E)$$

which implies that $\mathrm{Cl}(E)^2 = \mathrm{Cl}^+(E)^2$ and $\mathrm{Cl}(E)^4 = \mathrm{Cl}^+(E)^4$. Hence in order to prove $[J] \in \mathrm{Cl}^+(E)^4$, it is sufficient to prove $[J] \in \mathrm{Cl}(E)^4$. Recall that $[J] = [I]^2 \in \mathrm{Cl}^+(E)^2 = \mathrm{Cl}(E)^2$. Since $-vm \in \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$, we can use the same argument as in the proof of Theorem 2.6 of [11] to prove that there is an ideal I' such that

$$[I] = {I'}^2 \in \mathrm{Cl}(E)^2 = \mathrm{Cl}^+(E)^2.$$

Hence the dyadic ideal class $[J] \in \mathrm{Cl}(E)^4 = \mathrm{Cl}^+(E)^4$. □

Theorem 2.4. *With notations as above. Assume that* $\delta \not\equiv 2 \pmod 4$ *is a non zero integer and* $a \in \{\pm 1\}$ *satisfies* $\delta a \equiv 0,\ 1 \pmod 4$. *Let* $E_1 = \mathbb{Q}(\sqrt{-\delta p_1})$, $E_2 = \mathbb{Q}(\sqrt{-\delta p_2})$. *We suppose that* $D_i = \delta p_i$ *is the discriminant of real quadratic number field* $F_i = \mathbb{Q}(\sqrt{D_i})$, *where* $i = 1,\ 2$. *Then there is a field* M *such that if* p_1 *and* p_2 *satisfy* $[(M/\mathbb{Q})/(p_1)] = (M/\mathbb{Q})/(p_2)]$, *then the dyadic ideal class* $[J] \in \mathrm{Cl}^+(F_1)^4$ *if and only if* $[J] \in \mathrm{Cl}^+(F_2)^4$.

Proof. Let c be a number of factors 2 in δ. We follow Stevenhagen's notation in [21]. Let

$$E = \mathbb{Q}(\sqrt{q_1},\ \cdots,\ \sqrt{q_k}),$$

where $q_1,\ \cdots,\ q_k$ are the odd prime factors of δ. Let

$$K = \begin{cases} E, & \text{if } c = 0, \\ E, & \text{if } c = 2 \text{ and } \dfrac{-\delta}{4} \equiv 1 \pmod 4, \\ E(\sqrt{-1}), & \text{if } c = 2 \text{ and } \dfrac{-\delta}{4} \equiv 1 \pmod 4, \\ E(\sqrt{2}), & \text{if } c = 3 \text{ and } \dfrac{-\delta}{8} \equiv 1 \pmod 4, \\ E(\sqrt{-2}), & \text{if } c = 3 \text{ and } \dfrac{-\delta}{8} \equiv -1 \pmod 4. \end{cases}$$

Let ∞ be the product of infinite primes. Let M be the maximal abelian extension of K that

1. is of exponent dividing 2;
2. unramified outside $\delta\infty$;
3. can locally at primes over 2 be obtained by adjoining square roots of local units in case D is odd.

Let L_i be the maximal abelian extension of K that has exponent 2 and conductor dividing $p_i\infty$. Then by the proof of Theorem 10.4 of Stevenhagen's thesis, if $2|\delta$ and if $p_1,\ p_2$ satisfy $[(M/\mathbb{Q})/(p_1)] = [(M/\mathbb{Q})/(p_2)]$, then we have

$$L_1 \bigotimes_{\mathbb{Q}} \mathbb{Q}_2 \simeq L_2 \bigotimes_{\mathbb{Q}} \mathbb{Q}_2.$$

Let H_i be the subfield of the Hilbert class field of F_i corresponding to $\mathrm{Cl}^+(F_i)/\mathrm{Cl}^+(F_i)^4$. Then one can see that $H_i \subset L_i$ and

$$H_1 \bigotimes_{\mathbb{Q}} \mathbb{Q}_2 \simeq H_2 \bigotimes_{\mathbb{Q}} \mathbb{Q}_2.$$

By the class field theory $[J_i] \in \mathrm{Cl}^+(F_i)^4$ if and only if $[J_i]$ is completely split in H_i. Hence if p_1 and p_2 satisfy $[(M/\mathbb{Q})/(p_1)] = (M/\mathbb{Q})/(p_1)]$, then $[J_1] \in \mathrm{Cl}^+(F_1)^4$ if and only if $[J_2] \in \mathrm{Cl}^+(F_2)^4$.

In the case that 2 does not divide δ, the above arguments still work by Theorem 8.1 of Stevenhagen's thesis [21].

So the dyadic ideal class $[J] \in \mathrm{Cl}^+(F_1)^4$ if and only if $[J] \in \mathrm{Cl}^+(F_2)^4$. □

Hence we get the following theorem.

Theorem 2.5. *With notations as above. Assume that* $\delta \not\equiv 2 \pmod 4$ *is a non-zero integer and* $a \in \{\pm 1\}$ *satisfies* $\delta a \equiv 0,\ 1 \pmod 4$. *Let* $E_1 = \mathbb{Q}(\sqrt{-\delta p_1})$, $E_2 = \mathbb{Q}(\sqrt{-\delta p_2})$. *We suppose that* $D_i = \delta p_i$ *is the discriminant of real quadratic number field* $F_i = \mathbb{Q}(\sqrt{D_i})$, *where* $i = 1,\ 2$. *Then there is a field* M *such that if* p_1 *and* p_2 *satisfy* $[(M/\mathbb{Q})/(p_1)] = [(M/\mathbb{Q})/(p_2)]$, *then*

$$\mathrm{rk}_4(K_2\mathcal{O}_{E_1}) = \mathrm{rk}_4(K_2\mathcal{O}_{E_2}).$$

By the similar method, we can prove that the above theorem hold also for real quadratic number fields.

3 The governing field of the 8-rank of $K_2\mathcal{O}_F$

Although the existence of governing fields of the 8-rank of the narrow class groups has already been proved by Stevenhagen, the existence of the governing fields of the 8-rank of K_2 is still open. While in some special cases, the existence of the governing fields of the 8-rank of K_2 can be deduced from Morton's Theorems in [12–14].

Let $s \le r$ be three nonnegative integers. Then there exists $r+1$ primes $p_1, \cdots, p_r, p_{r+1} = q$ such that

$$\begin{aligned} &(1)\ p_i \equiv 1 \pmod 8, \text{ for } 1 \le i \le r;\\ &(2)\ \left(\frac{p_i}{p_j}\right) = 1, \text{ for } 1 \le i \ne j \le r;\\ &(3)\ q \equiv 5 \pmod 8;\\ &(4)\ \left(\frac{p_i}{q}\right) = \begin{cases} 1, & \text{if } 1 \le i \le s,\\ -1, & \text{if } s+1 \le i \le r. \end{cases} \end{aligned} \tag{3.1}$$

The existence can be proved easily. Let $d = p_1 \cdots p_r$, $F = \mathbb{Q}(\sqrt{dq})$. Let $E = \mathbb{Q}(\sqrt{-dq})$.

For any $1 \le i \le s$, let K_i be the unique quartic cyclic extension of $\mathbb{Q}$ with conductor p_i. Note that $K_i \supset \mathbb{Q}(\sqrt{p_i})$. For any $i,\ j$ such that $1 \le i \ne j \le s$, let L_{ij} be the unique quartic cyclic extension of $\mathbb{Q}(\sqrt{p_i p_j})$ which is unramified at finite primes. Let $M = p_1 \cdots p_s$. Let $\overline{\Lambda}_M$ be the class field over $\mathbb{Q}(\sqrt{-M})$ corresponding to the subgroup $(\mathrm{Cl}(\mathbb{Q}(\sqrt{-M})))^4$ of fourth powers in $\mathrm{Cl}(\mathbb{Q}(\sqrt{-M}))$. Let

$$\begin{aligned} K_M &= \prod_{1 \le i \le s} K_i, \\ \Lambda_M &= \prod_{1 \le i \ne j \le s} L_{ij}, \\ \Sigma_M &= K_M \Lambda_M, \\ \overline{\Sigma}_M &= \Sigma_M \overline{\Lambda}_M. \end{aligned}$$

Theorem 3.1. (Guo and Qin [8]) *With notation as above, we have*

$$\begin{aligned} \mathrm{rk}_2(K_2(\mathcal{O}_E)) &= r, \\ \mathrm{rk}_4(K_2(\mathcal{O}_E)) &= s. \end{aligned}$$

And $\mathrm{rk}_8(K_2(\mathcal{O}_E))$ *is completely determined by the Artin symbol* $[(\overline{\Sigma}_M/\mathbb{Q})/(q)]$.

Next we will consider the tame kernels of certain real quadratic fields. Let $s,\ \widetilde{r}$ be two nonnegative integers such that $s \le \widetilde{r}$ and $\widetilde{r} \ge 2 + s$. Let $r = \widetilde{r} - 2$.

Let $d = p_1 \cdots p_r$, $F = \mathbb{Q}(\sqrt{dq})$ and $E = \mathbb{Q}(\sqrt{-dq})$.

We choose primes $p_1, \ \cdots, \ p_r$ and q (q will vary to create infinitely many real quadratic fields F) such that

$$\begin{aligned} &(1)\ p_i \equiv 1 \pmod 8, \text{ for } 1 \le i \le r; \\ &(2)\ \left(\frac{p_i}{p_j}\right) = 1, \text{ for } 1 \le i \ne j \le r; \\ &(3)\ \left(\frac{p_i}{p_j}\right)_4 \left(\frac{p_j}{p_i}\right)_4 = 1, \quad \text{for } i \ne j; \\ &(4)\ q \equiv 3 \pmod 8; \\ &(5)\ \left(\frac{p_i}{q}\right) = \begin{cases} 1, & \text{if } 1 \le i \le s, \\ -1, & \text{if } s+1 \le i \le r. \end{cases} \end{aligned} \tag{3.2}$$

For any $1 \le i \le s$, Let K_i be the unique quartic cyclic extension of $\mathbb{Q}$ with conductor p_i. Note that $K_i \supset \mathbb{Q}(\sqrt{p_i})$. For any $i,\ j$ such that $1 \le i \ne j \le s$, let L_{ij} be the unique quartic cyclic extension of $\mathbb{Q}(\sqrt{p_i p_j})$ which is unramified at finite primes. Let $\Sigma = \left(\prod_{1 \le i \le s}\right)\left(\prod_{1 \le i \ne j \le s} L_{ij}\right)$.

Theorem 3.2. *Let* $p_1, \ \cdots, \ p_r, q$ *be primes satisfying conditions* (1)–(5) *of* (3.2). *Let* $d = p_1 \cdots p_r q$, $F = \mathbb{Q}(\sqrt{dq})$, $E = \mathbb{Q}(\sqrt{-dq})$. *Then we have*

$$\begin{aligned} \mathrm{rk}_2(K_2\mathcal{O}_F) &= r + 2; \\ \mathrm{rk}_4(K_2\mathcal{O}_F) &= s. \end{aligned}$$

And $\mathrm{rk}_8(K_2\mathcal{O}_F)$ *is completely determined by the Artin symbol* $[(\Sigma/\mathbb{Q})/(q)]$.

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