FPF RINGS AND THE AUT-PIC PROPERTY

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Abstract. We consider the non-commutative FPF ring and prove it has the Aut-Pic property. We also prove that R has Aut-Pic property if R/J does so, here J denotes the Jacobson radical of R. As an application, we give another proof of a result in [2]. In fact, we generalize that result.

Keywords: FPF ring, Aut-Pic property.

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0. Introduction

In [1], Chen and Tong defined the FPF commutative rings and got many good results. In fact, if the community is taken off, FPF ring also has some properties which is similar to the commutative cases.

In the second part of this paper, we prove the FPF ring has the Aut-pic property. This result generalized the Proposition 3.8 of [2]. For example, the integers ring $\mathbb{Z}$ is not a semiperfect ring, but it has the Aut-pic property. In fact, Proposition 3.8 in [2] becomes a corollary of Theorem 2.7 and Theorem 3.4.

1. FPF Rings

In [Co], the following three classes of rings are considered:

(a) For all $m$, $n$, $R^m \cong R^n$ implies $m = n$.

(b) For all $m$, $n$, $R^m \cong R^n \oplus K$ implies $m \geq n$.

(c) For all $n$, $R^m \cong R^n \oplus K$ implies $K = 0$.

and (c) implies (b), (b) implies (a). Next the ring satisfied (a), (b), or (c) as above will be called C1, C2, or C3 ring separatively. Throughout this paper, all rings are assumed to be C3 rings with identity. Recall that a ring $R$ is a PF rings if every finitely generated projective left $R$-module is free and a ring $R$ is PSF ring if $K_0(R) = Z$. A ring $R$ is defined to be an FPF ring, if it is the direct sum of finitely many PF rings. Here $R$ need not to be commutative. A ring
Proposition 1.2 The following two statements are equivalent for a C3 ring $k$:

1. $P \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_k)^{n_k}$.

Proof. If $e$ is a central idempotent of $R$, then $R = Re \oplus R(1 - e)$. Since $R$ is a PSF ring, $Re$ and $R(1 - e)$ are projective free $R$-modules. Assume

$$Re \oplus R^{k_1} \simeq R^k, \quad R(1 - e) \oplus R^{l_1} \simeq R^l, \quad k_1 \geq 0, \quad l_1, \quad l \geq 0,$$

then

$$Re \oplus R(1 - e) \oplus R^{k_1} \oplus R^{l_1} \simeq R^{k+l}.$$

i.e., $R \oplus R^{k_1} \oplus R^{l_1} \simeq R^{k+l}$. Since we have assumed $R$ is a C3 ring, so $k + l = 1 + k_1 + l_1$, and $k_1 \leq k, \quad l_1 \leq l$, so $k = k_1$ or $l_1 = l$, so $e = 0$ or $e = 1$. Q.E.D.

Proposition 1.2 The following two statements are equivalent for a C3 ring $R$:

1. $R$ is an FPSF ring.
2. There exist central idempotents $e_1, e_2, ..., e_m$ such that $1 = e_1 + e_2 + \cdots + e_m$ and $Re_i$ is a PSF ring, $1 \leq i \leq m$. Moreover $e_1, e_2, ..., e_m$ are unique, in the other word, if there exist central orthogonal idempotents $f_1, f_2, ..., f_n$ such that $1 = f_1 + f_2 + \cdots + f_n$ and $Rf_i$ is a PSF ring, then $n = m$ and $e_1, e_2, ..., e_m = f_1, f_2, ..., f_n$.

Proof. (1) $\Rightarrow$ (2). Assume $R = R_1 \oplus \cdots \oplus R_m, \quad R_i$ is a PF ring, $1 \leq i \leq m$. Let $e_i = (0, 0, 0, 1, 0, 0, 0, 0)$, (the $ith$ entry of $e_i$ is the identity of $R_i$ and the else are zeroes), then (2) is got easily.

(2) $\Rightarrow$ (1). Since $e_i \in \text{Center}(R)$, $e_ie_j = 0 = e_je_i (i \neq j)$, $R = Re_1 \oplus \cdots \oplus Re_m$ is an FPSF ring.

Now we come to prove the uniqueness of $e_i$. Since $1 = e_1 + e_2 + \cdots + e_m$, so $f_1 = f_1e_1 + \cdots + f_1e_m$. By Proposition 1.1, there exist $i_1$ such that $f_1e_{i_1} = f_1$, similarly we have $f_t = f_te_{i_t}$ for $t = 2, ..., n$. For $1 \leq t \neq t'$, since $e_{i_{t'}} = 0$, so $e_{i_t} \neq e_{i_{t'}}$, so $m \leq m$. For the same cause, $m \leq n$, so $n = m$. Thus $f_t = f_te_{i_t}, \quad t = 1, \ldots, n$ and $e_t = e_tf_{j_t}, \quad t = 1, \ldots, n$. Change the subscripts properly, we can assume $f_t = f_te_t$ and $e_s = e_sf_{j_s}$. So $f_s = f_s e_s = e_s f_s f_{j_s} = f_s f_{j_s}, \quad f_s = f_{j_s}$. So $f_s = e_s, \quad s = 1, \ldots, n$.

Recall that a central idempotent $e \in R$ is primitive if the fact $e = f_1 + f_2, \quad f_1, f_2$ are central orthogonal idempotents of $R$ implies $f_1 = 0$ or $f_2 = 0$. Q.E.D.
Proposition 1.3 Let $R$ be an FPSF and ring, and $e \neq 0$ is a central idempotent of $R$, then the following two statements are equivalent:

1. $e$ is primitive.
2. $Re$ is a PSF ring.

Proof. (1) $\Rightarrow$ (2). By Proposition 1.2, there exist orthogonal central idempotents $e_1, e_2, \ldots, e_m$, such that $1 = e_1 + e_2 + \cdots + e_m$ and $Re_i$ is a PF ring, $1 \leq i \leq m$. So $e = ee_1 + (ee_2 + \cdots + ee_m)$. Since $ee_1$ is orthogonal to $ee_2 + \cdots + ee_m$, according to Proposition 1.2, we have $e = ee_1$, or $e = ee_2 + \cdots + ee_m$. If $e = ee_2 + \cdots + ee_m$, then $e = ee_2$, or $e = ee_3 + \cdots + ee_m$, it follows that $e = ee_i$ for some $i$. Since $Re_i$ is a PSF ring and $ee_i$ is a central idempotent of $R$, by Proposition 1.2, $e = ee_i = e_i$ and $Re = Re_i$ is a PSF ring.

(2) $\Rightarrow$ (1). If $e = e_1 + e_2$, $e_1$, $e_2$ are central orthogonal idempotents of $R$, then $ee_1 = e_1^2 + e_2e_1 = e_1$ and so $e_1$ is an idempotent of $Re$. Since $Re$ is a PSF ring, by Proposition 1.1, $e_1 = e$ or $e_1 = 0$. So $e$ is primitive. Q.E.D.

The following Theorem gives the existence and uniqueness of direct sum decomposition of finitely generated projective left module for the FPSF ring.

Theorem 1.4 Let $R$ be a ring, the following two statements are equivalent:

1. $R$ is an FPSF ring.
2. There exist orthogonal central idempotents $e_1, e_2, \ldots, e_m \in R$, such that $1 = e_1 + e_2 + \cdots + e_m$, $Re_i$ is a PSF ring, $i = 1, \ldots, m$ and for any finitely generated projective left $R$-module $P$, there exist positive integers $n_1, n_2, \ldots, n_m, k$ such that

$$P \oplus R^k \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}, n_i \geq 0, 1 \leq i \leq m.$$ 

Moreover the above $n_i$ is uniquely determined by $P$ and $k$, $m$ is the rank of $K_0(R)$.

Proof. (1) $\Rightarrow$ (2). Assume that $R = R_1 \oplus \cdots \oplus R_m$, where $R_i$ is a PSF ring, $1 \leq i \leq m$. Let $e_i = (0, \ldots, 0, 1_i, 0, \ldots, 0)$, (the $i$th entry 1 of $e_i$ is the identity of $R_i$), the else are zeroes, then $1 = e_1 + e_2 + \cdots + e_m$ and $Re_i \simeq R_i$, $e_i$ is a central idempotent and $e_i e_j = 0$ ($i \neq j$). Then for any finitely generated projective left $R$-module $P$,

$$P \simeq R \otimes_R P \simeq (Re_1 \otimes_R P) \oplus (Re_2 \otimes_R P) \oplus \cdots \oplus (Re_m \otimes_R P).$$

Since $Re_i \otimes_R P$ is also a finitely generated projective left $Re_i$-module and $Re_i$ is a PSF ring ($1 \leq i \leq m$), there exists $n_i, k_i$ such that $Re_i \otimes_R P \oplus R^{k_i} \simeq (Re_i)^{n_i}$, it is also an $R$-module isomorphism, so

$$P \oplus R^k \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}, k = k_1 + k_2 + \cdots + k_n.$$ 

(2) $\Rightarrow$ (1). Since $1 = e_1 + e_2 + \cdots + e_m$ and $e_1, e_2, \ldots, e_m$ are orthogonal central idempotents, $R = Re_1 \oplus \cdots \oplus Re_m$. For any finitely generated projective left
Proposition 1.2, it is zero or the identity of \( R \) such that 1 = \( R \sum \) decomposition \( \text{Re} \).

Let \( e_0 \kappa, n \) for suitable \( \text{Re} \).

Since then \( \text{Re} \) is a PSF ring.

Hence \( \text{Re} \) is a PSF ring.

Finally, we prove the uniqueness of the above direct sum decomposition.

Obviously, \( m \) is unique, \( m \) is the rank of \( \text{K}_0(\text{R}) \). If there is another direct sum decomposition

\[
P \oplus R^k \simeq (\text{Re}_1)^{k_1} \oplus (\text{Re}_2)^{k_2} \oplus \cdots \oplus (\text{Re}_m)^{k_m},
\]

then

\[
\text{Re}_i \otimes_R \((\text{Re}_1)^{n_1} \oplus (\text{Re}_2)^{n_2} \oplus \cdots \oplus (\text{Re}_m)^{n_m})
\simeq \text{Re}_i \otimes_R \((\text{Re}_1)^{k_1} \oplus (\text{Re}_2)^{k_2} \oplus \cdots \oplus (\text{Re}_m)^{k_m}).
\]

Since \( e_i e_j = 0 = e_j e_i, (i \neq j) \), so \( (\text{Re}_i)^{n_i} \simeq (\text{Re}_i)^{k_i} \). Since \( \text{Re}_i \) is an IBN ring, \( n_i = k_i \).

Q.E.D.

Corollary 1.5 If \( \text{R} \) is an FPSF ring, \( \text{R} \) is a PT ring with finitely generated \( \text{K}_0 \) group.

Corollary 1.6 Let \( e_1, e_2, \ldots, e_m \) be orthogonal central idempotents of a ring \( \text{R} \) such that 1 = \( e_1 + e_2 + \cdots + e_m \), the following two statements are equivalent:

(1). \( \text{Re}_i \) is an FPSF ring, 1 \( \leq \) \( i \) \( \leq \) \( m \),

(2). \( \text{R} \) is an FPSF ring.

Proof. (1) \( \Rightarrow \) (2) is obvious.

(2) \( \Rightarrow \) (1). Assume \( \text{R} = R_1 \oplus R_2 \oplus \cdots \oplus R_k \), \( R_s \) is PSF ring, 1 \( \leq \) \( s \) \( \leq \) \( k \).

Let \( e_i' \) be the projection of \( e_i \) in \( R_s \), then \( e_i' \) is a central idempotent in \( R_s \), by Proposition 1.2, it is zero or the identity of \( R_s \). Assume \( e_{i_1}, \ldots, e_{i_j} \neq 0 \), then

\[
\text{Re}_i \simeq R_{i_1} \oplus R_{i_2} \oplus \cdots \oplus R_{i_j}, 1 \leq i_{m_1} < i_{m_2} < \cdots < i_{m_j} \leq s.
\]

So \( \text{Re}_i \) is an FPSF ring, 1 \( \leq \) \( i \) \( \leq \) \( m \).

Q.E.D.

Corollary 1.8 Let \( \text{R} \) be a ring, the following two statements are equivalent:

(1). \( \text{R} \) is an FPF ring.

(2). There exist orthogonal central idempotents \( e_1, e_2, \ldots, e_m \in \text{R} \), such that 1 = \( e_1 + e_2 + \cdots + e_m \), \( \text{Re}_i \) is a PF ring, \( i = 1, \ldots, m \) and for any finitely generated projective left \( \text{R} \)-module \( P \), there exist positive integers \( n_1, n_2, \ldots, n_m \), such that

\[
P \simeq (\text{Re}_1)^{n_1} \oplus (\text{Re}_2)^{n_2} \oplus \cdots \oplus (\text{Re}_m)^{n_m}, n_i \geq 0, 1 \leq i \leq m.
\]
Moreover the above $n_i$ is uniquely determined by $P$, $m$ is the rank of $K_0(R)$.

**Theorem 1.9** Let $R$ be an FPF ring. If a ring $S$ is Morita equivalent to $R$, there exist PF rings $R_1, R_2, ..., R_m$ such that

$$S \cong M_{n_1}(R_1) \oplus M_{n_2}(R_2) \oplus \cdots \oplus M_{n_m}(R_m), n_i \geq 0, 1 \leq i \leq m.$$ 

**Proof.** For any $S \cong R$, $S \cong \text{End}_R(P)$ for some finitely generated faithfully projective $R$-module $P$. According to Theorem 2.4 there exists orthogonal central idempotents $e_1, e_2, ..., e_m$, such that $1 = e_1 + e_2 + \cdots + e_m$ and

$$P \cong (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}, n_i \geq 0, 1 \leq i \leq m$$

, where $Re_i$ are PF rings. Since $e_1, e_2, ..., e_m$ are orthogonal central idempotents, $\text{Hom}_R(Re_i, Re_j) = 0$, and so

$$S \cong \text{End}_R((Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m})$$
$$\cong \text{End}_R((Re_1)^{n_1}) \oplus \text{End}_R((Re_2)^{n_2}) \oplus \cdots \oplus \text{End}_R((Re_m)^{n_m})$$
$$\cong M_{n_1}(R_1) \oplus M_{n_2}(R_2) \oplus \cdots \oplus M_{n_m}(R_m)$$

where $R_i \cong Re_i$ is a PF ring.

Q.E.D.

The inverse of this Theorem is not true, for example, $R = M_2(F)$, $F$ denotes a field, then If $S$ is similar to $R$, $S$ is similar to $F$ which is an FPF ring, but $R$ is not an FPF ring.

## 2. Aut-Pic Property

Let $M$ be an arbitrary right $R$-module and $f$ be an automorphism of $R(f \in \text{Aut}(R))$, $M_f$ denotes a right $R$-module which has the same elements and group structure as $M$, but if $m \in M$ and $\lambda \in R$, then element $m \cdot \lambda$ in $M_f$ is the element $mf(\lambda)$ in $M$.

By $\text{InAut}(R)$ it is meant the group of inner automorphisms of $R$. Let $\phi_R : \text{Aut}(R) \longrightarrow \text{Pic}(R)$ be a group homomorphism defined by $\phi_R(f) = (fR_1)$, here “1” means the identity automorphism of $R$, then there is an exact sequence

$$1 \longrightarrow \text{InAut}(R) \longrightarrow \text{Aut}(R) \xrightarrow{\phi_R} \text{Pic}(R).$$

(See [Ba] for details)

**Definition 3.1** A ring $R$ has the **Aut-Pic** property if $\phi_R$ is onto.

In [Bo], Bolla had proved that PF ring has the **Aut-Pic** property, but **Aut-Pic** is not always preserved when taking products of rings (examples can be
seen in [Bo]). Here we will prove that when these rings are PF rings, their direct product has the Aut-Pic property.

**Theorem 2.1.** FPF ring have the Aut-Pic property.

**Proof.** Assume that a ring $R$ is an FPF ring, that is $R = R_1 \oplus \cdots \oplus R_m$, where $R_i$ is a PF ring, $1 \leq i \leq m$. Let $P$ be an invertible $R$-$R$-bimodule then by Corollary 1.8, there exist orthogonal central idempotents $e_1, \ldots, e_m$, such that as left $R$-module

$$P \cong (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}, n_i \geq 0, 1 \leq i \leq m.$$  

Since $P$ is a projective generator, so there exist a positive integer $k$ such that $P^k \cong R \oplus K$ for suitable finitely generatedly projective $R$-module $K$. By the uniqueness of the decomposition of $P^k$, all the $n_i$ must be positive. So $P \cong R \oplus P_1$, where $P_1 \cong (Re_1)^{n_1-1} \oplus (Re_2)^{n_2-1} \oplus \cdots \oplus (Re_m)^{n_m-1}$. Suppose $Q$ be the inverse of $P$, similarly, there exist right module $Q_1$ such that $Q \cong R \oplus Q_1$ is a right $R$ module isomorphism. So

$$R \cong Q \otimes P \cong (R \oplus Q_1) \otimes (R \oplus P_1) \cong R \oplus (Re_1)^{n_1-1} \oplus (Re_2)^{n_2-1} \oplus \cdots \oplus (Re_m)^{n_m-1} \oplus M$$

where $M = Q_1 \oplus Q_1 P_1$. By the uniqueness, $(Re_1)^{n_1-1} \oplus (Re_2)^{n_2-1} \oplus \cdots \oplus (Re_m)^{n_m-1} = 0$, i.e., $n_1 = n_2 = \cdots = n_m = 1$. So $R \cong R$ as left $R$-module. By Proposition 1.2 in [2], there exist $f \in \text{Aut}(R)$ such that $P \cong (1Rf)$ is an $R$-$R$-bimodule isomorphism, so $R$ has the Aut-Pic property. Q.E.D.

**References**

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