FPF RINGS AND THE AUT-PIC PROPERTY

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Abstract. We consider the non-commutative FPF ring and prove it has the **Aut-Pic** property. We also prove that R has **Aut-Pic** property if R/J does so, here J denotes the Jacobson radical of R. As an application, we give another proof of a result in [2]. In fact, we generalize that result.

Keywords: FPF ring, Aut-Pic property .

AMR 20B10

0. Introduction

In [1], Chen and Tong defined the FPF commutative rings and got many good results. In fact, if the community is taken off, FPF ring also has some properties which is similar to the commutative cases.

In the second part of this paper, we prove the FPF ring has the *Aut-pic* property. This result generalized the Proposition 3.8 of [2]. For example, the integers ring Z is not a semiperfect ring, but it has the *Aut-pic* property. In fact, Proposition 3.8 in [2] becomes a corollary of Theorem 2.7 and Theorem 3.4.

1. FPF Rings

In [Co], the following three classes of rings are considered:

(a) For all m, n, $R^m \simeq R^n$ implies $m = n$.

(b) For all m, n, $R^m \simeq R^n \oplus K$ implies $m \geq n$.

(c) For all n, $R^m \simeq R^m \oplus K$ implies $K = 0$.

and (c) implies (b), (b) implies (a). Next the ring satisfied (a), (b), or (c) as above will be called C1, C2, or C3 ring separatively. Throughout this paper, all rings are assumed to be C3 rings with identity. Recall that a ring R is a PF rings if every finitely generated projective left R-module is free and a ring R is PSF ring if $K_0(R) = Z$. A ring R is defined to be an FPF ring, if it is the direct sum of finitely many PF rings. Here R need not to be commutative. A ring

R is defined to be an FPSF ring, if it is the direct sum of finitely many PSF rings. An idempotent $e \in R$ is called a central idempotent if $e \in Center(R)$. A ring R is called a PT ring, if for any finitely generated projective left R-module P, there exist finitely many central idempotents $e_1, e_2, ..., e_k \in R$, and positive integers $n_1, n_2, ..., n_k$ such that

$$P \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_k)^{n_k}.$$

Proposition 1.1 If ring R is a PSF and C3 ring, then 0 and 1 are the only central idempotents of R.

Proof. If e is a central idempotent of R, then $R = Re \oplus R(1-e)$. Since R is a PSF ring, Re and $R(1-e)$ are projective free R-modules. Assume

$$Re \oplus R^{k_1} \simeq R^k,\ R(1-e) \oplus R^{l_1} \simeq R^l,\ k_1,\ k \geq 0,\ l_1,\ l \geq 0,$$

then

$$Re \oplus R(1-e) \oplus R^{k_1} \oplus R^{l_1} \simeq R^{k+l}.$$

i.e., $R \oplus R^{k_1} \oplus R^{l_1} \simeq R^{k+l}$. Since we have assumed R is a C3 ring, so $k + l = 1 + k_1 + l_1$, and $k_1 \leq k$, $l_1 \leq l$, so $k = k_1$ or $l_1 = l$, so $e = 0$ or $e = 1$. Q.E.D.

Proposition 1.2 The following two statesments are equivalent for a C3 ring R:

(1). R is an FPSF ring.

(2). There exist central idempotents $e_1, e_2, ..., e_m$ such that $1 = e_1 + e_2 + \cdots + e_m$ and Re_i is a PSF ring, $1 \leq i \leq m$. Moreover $e_1, e_2, ..., e_m$ are unique, in the other word, if there exist central orthogonal idempotents $f_1, f_2, ..., f_n$ such that $1 = f_1 + f_2 + \cdots + f_n$ and Rf_i is a PSF ring, then $n = m$ and $e_1, e_2, ..., e_m = f_1, f_2, ..., f_n$.

Proof. (1) $\Rightarrow$ (2). Assume $R = R_1 \oplus \cdots \oplus R_m, R_i$ is a PF ring, $1 \leq i \leq m$. Let $e_i = (0, ..., 0, 1_i, 0, ..., 0)$, (the ith entry of e_i is the identity of R_i and the elses are zeroes), then (2) is got easily.

(2) $\Rightarrow$ (1). Since $e_i \in$ Center(R), $e_ie_j = 0 = e_je_i$ $(i \neq j)$, $R = Re_1 \oplus \cdots \oplus Re_m$ is an FPSF ring.

Now we come to prove the uniqueness of e_i. Since $1 = e_1 + e_2 + \cdots + e_m$, so $f_1 = f_1e_1 + \cdots + f_1e_m$. By Proposition 1.1, there exist i_1 such that $f_1e_{i_1} = f_1$, similarily we have $f_t = f_te_{i_t}$ for $t = 2, ..., n$. For $1 \leq t \neq t^{'}$, since $e_te_{t^{'}=0}$, so $e_{i_t} \neq e_{i_{t^{'}}}, so n \leq m$. For the same cause, $m \leq n$, so $n = m$. Thus $f_t = f_te_{i_t}$, $t = 1, ..., n$ and $e_t = e_tf_{j_t}$, $t = 1, ..., n$. Change the subscripts properly, we can assume $f_t = f_te_t$ and $e_s = e_sf_{j_s}$. So $f_s = f_se_s = e_sf_sf_{j_s} = f_sf_{j_s}$, $f_s = f_{j_s}$. So $f_s = e_s$, $s = 1, ..., n$. Q.E.D.

Recall that a central idempotent $e \in R$ is primitive if the fact $e = f_1 + f_2$, f_1, f_2 are central orthogonal idempotents of R implies $f_1 = 0$ or $f_2 = 0$.

Proposition 1.3 Let R be an FPSF and ring, and $e \neq 0$ is a central idmpotent of R, then the following two statesments are equivalent:

(1). e is primitive.

(2). Re is a PSF ring.

Proof. (1) $\Rightarrow$ (2). By Proposition 1.2, there exist orthogonal central idempotents $e_1, e_2, ..., e_m$, such that $1 = e_1 + e_2 + \cdots + e_m$ and Re_i is a PF ring, $1 \leq i \leq m$. So $e = ee_1 + (ee_2 + \cdots + ee_m)$. Since ee_1 is orthogonal to $ee_2 + \cdots + ee_m$, according to Proposition 1.2, we have $e = ee_1$, or $e = ee_2 + \cdots + ee_m$. If $e = ee_2 + \cdots + ee_m$, then $e = ee_2$, or $e = ee_3 + \cdots + ee_m$, it follows that $e = ee_i$ for some i. Since Re_i is PSF ring and ee_i is a central idempotent of R , by Proposition 1.2, $e = ee_i = e_i$ and $Re = Re_i$ is a PSF ring.

(2) $\Rightarrow$ (1). If $e = e_1 + e_2$, e_1, e_2 are central orthogonal idempotent of R, then $ee_1 = e_1^2 + e_2e_1 = e_1$ and so e_1 is an idempotent of Re. Since Re is a PSF ring , by Proposition 1.1, $e_1 = e$ or $e_1 = 0$. So e is primitive. Q.E.D.

The following Theorem gives the existence and uniqueness of direct sum decomposition of finitely generated projective left module for the FPF ring

Theorem 1.4 Let R be a ring, the following two statesments are equivalent:

(1). R is an FPSF ring.

(2). There exist orthogonal central idempotents $e_1, e_2, ..., e_m \in R$, such that $1 = e_1 + e_2 + \cdots + e_m$, Re_i is a PSF ring, $i = 1, ..., m$ and for any finitely generated projective left R-module P, there exist positive integers $n_1, n_2, ..., n_m, k$ such that

$$P \oplus R^k \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}, n_i \geq 0, 1 \leq i \leq m.$$

Moreover the above n_i is uniquely determined by P and k, m is the rank of $K_0(R)$.

Proof. (1) $\Rightarrow$ (2). Assume that $R = R_1 \oplus \cdots \oplus R_m$, where R_i is a PSF ring, $1 \leq i \leq m$. Let $e_i = (0, ..., 0, 1_i, 0, ..., 0)$, (the ith entry 1_i of e_i is the identity of R_i, the elses are zeroes, then $1 = e_1 + e_2 + \cdots + e_m$ and $Re_i \simeq R_i$, e_i is a central idempotent and $e_ie_j = 0$ $(i \neq j)$. Then for any finitely generated projective left R-module P,

$$P \simeq R \otimes_R P \simeq (Re_1 \otimes_R P) \oplus (Re_2 \otimes_R P) \oplus \cdots \oplus (Re_m \otimes_R P).$$

Since $Re_i \otimes_R P$ is also a finitely generated projective left Re_i-module and Re_i is a PSF ring $(1 \leq i \leq m)$, there exists n_i, k_i such that $Re_i \otimes_R P \oplus R^{k_i} \simeq (Re_i)^{n_i}$, it is also an R-module isomorphism, so

$$P \oplus R^k \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m},\ k = k_1 + k_2 + \cdots + k_n.$$

(2) $\Rightarrow$ (1). Since $1 = e_1 + e_2 + \cdots + e_m$ and $e_1, e_2, ..., e_m$ are orthogonal central idempotents, $R = Re_1 \oplus \cdots \oplus Re_m$. For any finitely generated projective left

Re_i-module P , $R \otimes_{Re_i} P$ is a finitely generated projective left R-module , So

$$R \otimes_{Re_i} P \oplus R^k \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}$$

for suitable $k,\ n_i \geq 0,\ 1 \leq i \leq m$. So

$$\begin{aligned} P \oplus (Re_i)^k &\simeq Re_i \otimes_R (R \otimes_{Re_i} P \oplus R^k) \\ &\simeq Re_i \otimes_R ((Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}) \simeq (Re_i)^{n_i}. \end{aligned}$$

Hence Re_i is a PSF ring.

Finally, we prove the uniqueness of the above direct sum decomposition.

Obviously, m is unique, m is the rank of $K_0(R)$. If there is another direct sum decomposition

$$P \oplus R^k \simeq (Re_1)^{k_1} \oplus (Re_2)^{k_2} \oplus ... \oplus (Re_m)^{k_m},$$

then

$$\begin{aligned} &Re_i \otimes_R ((Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}) \\ &\simeq Re_i \otimes_R ((Re_1)^{k_1} \oplus (Re_2)^{k_2} \oplus \cdots \oplus (Re_m)^{k_m}). \end{aligned}$$

Since $e_ie_j = 0 = e_je_i,(i \neq j)$, so $(Re_i)^{n_i} \simeq (Re_i)^{k_i}$. Since Re_i is an IBN ring, $n_i = k_i$. Q.E.D.

Corollary 1.5 If R is an FPSF ring, R is a PT ring with finitely generated K_0 group.

Corollary 1.6 Let $e_1, e_2, ..., e_m$ be orthogonal central idempotents of a ring R such that $1 = e_1 + e_2 + \cdots + e_m$, the following two statesments are equivalent:

(1). Re_i is an FPSF ring, $1 \leq i \leq m$,

(2). R is an FPSF ring.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Assume $R = R_1 \oplus R_2 \oplus \cdots \oplus R_k$, R_s is PSF ring, $1 \leq s \leq k$. Let e_{is} be the projection of e_i in R_s, then e_{is} is a central idempotent in R_s, by Proposition 1.2, it is zero or the identity of R_s. Assume $e_{i_{m_1}}, ..., e_{i_{m_j}} \neq 0$, then

$$Re_i \simeq R_{i_{m_1}} \oplus R_{i_{m_2}} \oplus \cdots \oplus R_{i_{m_j}}, 1 \leq i_{m_1} < i_{m_2} < \cdots < i_{m_j} \leq s.$$

So Re_i is an FPSF ring, $1 \leq i \leq m$. Q.E.D.

Corollary 1.8 Let R be a ring, the following two statesments are equivalent:

(1). R is an FPF ring.

(2). There exist orthogonal central idempotents $e_1, e_2, ..., e_m \in R$, such that $1 = e_1 + e_2 + \cdots + e_m$, Re_i is a PF ring, $i = 1, ..., m$ and for any finitely generated projective left R-module P, there exist positive integers $n_1, n_2, ..., n_m$, such that

$$P \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}, n_i \geq 0, 1 \leq i \leq m.$$

Moreover the above n_i is uniquely determined by P, m is the rank of $K_0(R)$.

Theorem 1.9 Let R be an FPF ring. If a ring S is Morita equivalent to R, there exist PF rings R_1, R_2, ..., R_m such that

$$S \simeq M_{n_1}(R_1) \oplus M_{n_2}(R_2) \oplus \cdots \oplus M_{n_m}(R_m), n_i \geq 0, 1 \leq i \leq m.$$

Proof. For any $S \approx R$, $S \simeq End_R(P)$ for some finitely generated faithfully projective R-module P. According to Theorem 2.4 there exists orthogonal central idempotents $e_1, e_2, ..., e_m$, such that $1 = e_1 + e_2 + \cdots + e_m$ and

$$P \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m},\ n_i \geq 0,\ 1 \leq i \leq m$$

, where Re_i are PF rings. Since $e_1, e_2, ..., e_m$ are orthogonal central idempotents, $Hom_R(Re_i, Re_j) = 0$, . and so

$$\begin{aligned} S &\simeq End_R((Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}) \\ &\simeq End_R((Re_1)^{n_1}) \oplus End_R((Re_2)^{n_2}) \oplus \cdots \oplus End_R((Re_m)^{n_m}) \\ &\simeq M_{n_1}(Re_1) \oplus M_{n_2}(Re_2) \oplus \cdots \oplus M_{n_m}(Re_m) \\ &M_{n_1}(R_1) \oplus M_{n_2}(R_2) \oplus \cdots \oplus M_{n_m}(R_m) \end{aligned},$$

where $R_i \simeq Re_i$ is a PF ring. Q.E.D.

The inverse of this Theorem is not true, for example, $R = M_2(F)$, F denotes a field, then If S is similar to R, S is similar to F which is an FPF ring, but R is not an FPF ring.

2. Aut-Pic Property

Let M be an arbitary right R-module and f be an automorphim of $R(f \in Aut(R))$, M_f denotes a right R-module which has the same elements and group structure as M, but if $m \in M$ and $\lambda \in R$, then then element $m \cdot \lambda$ in M_f is the element $mf(\lambda)$ in M.

By $InAut(R)$ it is meant the group of inner automorphisms of R. Let $\phi_R : Aut(R) \longrightarrow Pic(R)$ be a group homomorphism defined by $\phi_R(f) = ({}_f R_1)$, here "$1''$ means the identity automorphism of R, then there is an exact sequence

$$1 \longrightarrow InAut(R) \longrightarrow Aut(R) \xrightarrow{\phi_R} Pic(R).$$

(See [Ba] for details)

Definition 3.1 A ring R has the **Aut-Pic** property if ϕ_R is onto.

In [Bo], Bolla had proved that PF ring has the **Aut-Pic** property , but **Aut-Pic** is not always preserved when taking products of rings(examples can be seen in [Bo]). Here we will prove that when these rings are PF rings, their direct product has the **Aut-Pic** property.

Theorem 2.1. FPF ring have the **Aut-Pic** property.

Proof. Assume that a ring R is an FPF ring, that is $R = R_1 \oplus \cdots \oplus R_m$, where R_i is a PF ring, $1 \leq i \leq m$. Let P be an invertible R-R-bimodule then by Corollary 1.8, there exist orthogonal central idempotents $e_1, ..., e_m$, such that as left R-module

$$P \simeq (Re_1)^{n_1} \oplus (Re_2)^{n_2} \oplus \cdots \oplus (Re_m)^{n_m}, n_i \geq 0, 1 \leq i \leq m.$$

Since P is a projective generator, so there exsit a positive integer k such that $P^k \simeq R \oplus K$ for suituable finitely generatedly projective R-module K. By the uniqueness of the decomposition of P^k, all the n_i must be positive. So $P \simeq R \oplus P_1$, where $P_1 \simeq (Re_1)^{n_1-1} \oplus (Re_2)^{n_2-1} \oplus \cdots \oplus (Re_m)^{n_m-1}$. Suppose Q be the inverse of P, similarily, there exist right module Q_1 such that $Q \simeq R \oplus Q_1$ is a right R module isomorphism. So

$$\begin{aligned} R &\simeq Q \otimes P \simeq (R \oplus Q_1) \otimes (R \oplus P_1) \\ &\simeq R \oplus (Re_1)^{n_1-1} \oplus (Re_2)^{n_2-1} \oplus \cdots \oplus (Re_m)^{n_m-1} \oplus M \end{aligned}$$

where $M = Q_1 \oplus Q_1 P_1$. By the uniqueness, $(Re_1)^{n_1-1} \oplus (Re_2)^{n_2-1} \oplus \cdots \oplus (Re_m)^{n_m-1} = 0$, i.e., $n_1 = n_2 = \cdots = n_m = 1$. So ${}_R P \simeq_R R$ as left R-module. By Proposition 1.2 in [2], there exist $f \in Aut(R)$ such that $P \simeq ({}_1 R_f)$ is an R-R-bimodule isomorphism, so R has the **Aut-Pic** property. Q.E.D.

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