# HIGHER CLASS GROUPS OF GENERALIZED EICHLER ORDERS

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**Abstract:** In this paper we study the possible torsion in even dimensional higher class groups $Cl_{2n}(\Lambda)$ $(n \geq 1)$ of an order $\Lambda$ in a semi-simple algebra $A$ over a number field $F$ with ring of integers $\mathcal{O}_F$. We show that for certain orders $-$ called generalized Eichler orders $-$ $p$-torsion in $Cl_{2n}(\Lambda)$ can only occurs for primes $p$ dividing prime ideals $\wp$ of $\mathcal{O}_F$, at which $\Lambda$ is not maximal. In particular, the results apply to Eichler orders in quaternion algebras and to hereditary orders.

**Keywords:** higher class group, quaternion algebra, Eichler order, semi-simple algebra, hereditary order

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## 1. **Introduction**

Let $F$ be a number field and $\mathcal{O}_F$ the ring of integers in $F$. Let $A$ be a semi-simple algebra over $F$ and $\Lambda$ an $\mathcal{O}_F$-order in $A$. The higher class groups of $\Lambda$ are defined as

$$Cl_n(\Lambda) = ker(SK_n(\Lambda) \longrightarrow \bigoplus_{\wp} SK_n(\Lambda_\wp)),$$

where $\wp$ runs through all maximal ideals of $\mathcal{O}_F$ and

$$SK_n(\Lambda) := ker(K_n(\Lambda) \longrightarrow K_n(A)),$$

for all integers $n \geq 0$. By the Theorem 1 and 2 in [2], $Cl_n(\Lambda)$ are trivial for maximal orders. Later Kuku proved in [6] that $Cl_n(\Lambda)$ are finite for arbitrary orders. In [4], it is proved that the only $p$-torsion possible in $Cl_{2n+1}(\Lambda)$ is for those rational primes $p$ which lie under the prime ideals of $\mathcal{O}_F$ at which $\Lambda$ is not maximal. In this paper, we prove the analogous result for even dimensional higher class groups $Cl_{2n}(\Lambda)$ $(n \geq 1)$ in the case where $\Lambda$ is an Eichler order in a quaternion algebra or a hereditary order in a semi-simple algebra.

Note that in general, an Eichler order $\Lambda$ is not hereditary. An order is hereditary if and only if all the local orders $\Lambda_\wp$ is hereditary. So an Eichler $\Lambda$ is hereditary if and only if every $k_\wp \leq 1$ (See Section 2 for the definition of $k_\wp$).

To prove the above results simultaneously for Eichler and hereditary orders, it suffices for us to prove the result for orders $\Lambda$ in $A \simeq M_n(D)$, where $D$ is a finite dimensional algebra and locally at each prime $\wp$, $\Lambda$ has the form

$$\Lambda_\wp = \begin{pmatrix} (\triangle) & (\wp^{k_\wp}) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\wp^{k_\wp}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \end{pmatrix}^{(n_1,\, \ldots,\, n_r)},$$

for some $k_\wp \geq 1$, where $\triangle$ is the unique maximal order in the division algebra $D_{(\wp)}$ which satisfies $D \bigotimes_F F_\wp \simeq M_m(D_{(\wp)})$. Such orders we shall call generalized Eichler orders (see section 2 for definitions and notations).

If all $k_\wp$ are equal to 1, then $\Lambda$ is hereditary (Theorem 39.14 in [7]), whereas if $A$ is a quaternion algebra, then $\Lambda$ is an Eichler order as defined in [8].

In this paper, we use the same notations as in [4].

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## 2. **Higher class groups of generalized Eichler orders**

Let $F$ be a number field and $\mathcal{O}_F$ the ring of integers in $F$. Let $A$ be a semi-simple algebra over $F$ and $\Lambda$ an order in $A$. For any maximal ideal $\wp$ of $\mathcal{O}_F$, let $F_\wp$, $\mathcal{O}_{F_\wp}$, $A_\wp$, $\Lambda_\wp$ be the $\wp$-completions of $F$, $\mathcal{O}_F$, $A$, $\Lambda$ respectively. If $D$ is a division algebra, let $D_{(\wp)}$ be the division algebra such that $D_\wp \simeq M_k(D_{(\wp)})$.

If $R$ is any ring, we will denote the quotient of $K_m(R)$ by its divisible subgroup by $K_m^c(R)$. Let $S$ be the set of primes at which $\Lambda_\wp$ is not maximal. As in [4], we define $Cl_m(\Gamma, S)$ to be the cokernel of the map

$$K_{m+1}^c(A) \longrightarrow \bigoplus_{\wp \in S} K_{m+1}^c(A_\wp) \oplus \bigoplus_{\wp \notin S} K_{m+1}^c(A_\wp)/im(K_{m+1}^c(\Gamma_\wp)),$$

where $\Gamma$ is a maximal order containing $\Lambda$. Let $P_S$ denote the set of rational primes lying under the prime ideals in $S$.

**Lemma 2.1.** *The higher class group $Cl_{2n}(\Lambda)$ is a homomorphic image of*

$$coker(\bigoplus_{\wp\in S} K^c_{2n+1}(\Lambda_\wp) \longrightarrow K^c_{2n+1}(A_\wp)).$$

*Proof.* By Lemma 1.2 in [4],

$$Cl_{2n}(\Lambda) \simeq coker(\bigoplus_{\wp\in S} K^c_{2n+1}(\Lambda_\wp) \longrightarrow Cl_{2n}(\Gamma, S)).$$

By Theorem 1 in [2],

$$\bigoplus_{\wp\notin S} K^c_{2n+1}(A_\wp)/im(K^c_{2n+1}(\Gamma_\wp)) = 0.$$

So

$$Cl_{2n}(\Gamma, S) = coker(K^c_{2n+1}(A) \longrightarrow \bigoplus_{\wp\in S} K^c_{2n+1}(A_\wp)).$$

Hence $Cl_{2n}(\Lambda)$ is a homomorphic image of

$$coker(\bigoplus_{\wp\in S} K^c_{2n+1}(\Lambda_\wp) \longrightarrow \bigoplus_{\wp\in S} K^c_{2n+1}(A_\wp)).$$

□

**Definition 2.2** ([7], 39.2)**.** Let $R$ be a ring. For each ideal $I$ of $R$, let $(I)^{m\times n}$ denote the set of all $m \times n$ matrices with entries in $I$. If $\{I_{ij} :\ 1 \leq i,\ j \leq r\}$ is a set of ideals in $R$, we write

$$\Lambda = \begin{pmatrix} (I_{11}) & (I_{12}) & \cdots & (I_{1r}) \\ (I_{21}) & (I_{22}) & \cdots & (I_{2r}) \\ \cdots & \cdots & \cdots & \cdots \\ (I_{r1}) & (I_{r2}) & \cdots & (I_{rr}) \end{pmatrix}^{(n_1,\ \ldots,\ n_r)}$$

to indicate that $\Lambda$ is the set of all matrices $(T_{ij})_{1\leq i,j\leq r}$, where for each pair $(i, j)$, the matrix $T_{ij}$ ranges over all elements of $(I_{ij})^{n_i\times n_j}$.

**Definition 2.3.** Let $A \simeq M_n(D)$, where $D$ is a finite dimensional division algebra. We call an order $\Lambda$ in $A$ a generalized Eichler order if each $\Lambda_\wp$ has the form

$$\Lambda_\wp = \begin{pmatrix} (\triangle) & (\wp^{k_\wp}) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\wp^{k_\wp}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \end{pmatrix}^{(n_1,\ \ldots,\ n_r)},$$

where $k_\wp \geq 1$ and $\triangle$ is the unique maximal order in $D_{(\wp)}$. If $A \simeq \bigoplus_i M_{n_i}(D_i)$ is a semi-simple algebra, then an order $\Lambda$ in $A$ is called a generalized Eichler order if $\Lambda \simeq \bigoplus_i \Lambda_i$, where $\Lambda_i$ is a generalized Eichler order in $M_{n_i}(D_i)$.

**Proposition 2.4** ([3], Theorem A and 2.2)**.** *Let $R$ and $S$ be rings and $U$ an $R-S-$bimodule. Then the natural homomorphisms*

$$K_n(\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}) \longrightarrow K_n(\begin{pmatrix} R & U \\ 0 & S \end{pmatrix})$$

*are isomorphisms. Let $U^* = Hom(U,\ S)$, where $U$ is considered as a right $S$-module. If $R$ is the endomorphism ring $End(U)$ of the right $S$-module $U$, then the natural homomorphisms*

$$K_n(\begin{pmatrix} R & U \\ 0 & S \end{pmatrix}) \longrightarrow K_n(\begin{pmatrix} R & U \\ U^* & S \end{pmatrix})$$

*are surjective.*

**Lemma 2.5.** *Let $R$ be a ring,*

$$R_1 = \begin{pmatrix} (R) & (0) & (0) & \cdots & (0) \\ (R) & (R) & (0) & \cdots & (0) \\ (R) & (R) & (R) & \cdots & (0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (R) & (R) & (R) & \cdots & (R) \end{pmatrix}^{(n_1,\ \ldots,\ n_r)}$$

*and*

$$R_2 = \begin{pmatrix} (R) & (R) & (R) & \cdots & (R) \\ (R) & (R) & (R) & \cdots & (R) \\ (R) & (R) & (R) & \cdots & (R) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (R) & (R) & (R) & \cdots & (R) \end{pmatrix}^{(n_1,\ \ldots,\ n_r)} .$$

*Then the natural homomorphisms*

$$K_n(R_1) \longrightarrow K_n(R_2)$$

*are surjective.*

*Proof.* We will prove this lemma by induction. If $r = 1$, then $R_1 = R_2$. This is the trivial case.

Suppose that the lemma holds for $r-1$. Let

$$R_3^{'} = \begin{pmatrix} (R) & (0) & \cdots & (0) \\ (R) & (R) & \cdots & (0) \\ \cdots & \cdots & \cdots & \cdots \\ (R) & (R) & \cdots & (R) \end{pmatrix}^{(n_1, \ ..., \ n_{r-1})}$$

and

$$R_4^{'} = \begin{pmatrix} (R) & (R) & \cdots & (R) \\ (R) & (R) & \cdots & (R) \\ \cdots & \cdots & \cdots & \cdots \\ (R) & (R) & \cdots & (R) \end{pmatrix}^{(n_1, \ ..., \ n_{r-1})}.$$

By induction hypothesis, the homomorphisms

$$K_n(R_3^{'}) \longrightarrow K_n(R_4^{'})$$

are surjective.

Let

$$R_3 = \begin{pmatrix} R_3^{'} & 0 \\ 0 & (R)^{(n_r)} \end{pmatrix} \simeq R_3^{'} \times (R)^{(n_r)}$$

and

$$R_4 = \begin{pmatrix} R_4^{'} & 0 \\ 0 & (R)^{(n_r)} \end{pmatrix} \simeq R_4^{'} \times (R)^{(n_r)}.$$

Since $K_n(R_3) \simeq K_n(R_3^{'}) \bigoplus K_n((R)^{(n_r)})$ and $K_n(R_4) \simeq K_n(R_4^{'}) \bigoplus K_n((R)^{(n_r)})$, the homomorphisms

$$K_n(R_3) \longrightarrow K_n(R_4)$$

are surjective.

By Proposition 2.4, the homomorphisms

$$K_n(R_3) \longrightarrow K_n(R_1)$$

are surjective.

Let

$$R_5 = \begin{pmatrix} (R) & (R) & \cdots & (R) & (0) \\ (R) & (R) & \cdots & (R) & (0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (R) & (R) & \cdots & (R) & (0) \\ (R) & (R) & \cdots & (R) & (R) \end{pmatrix}^{(n_1, \ ..., \ n_r)}.$$

By Proposition 2.4, the homomorphisms

$$K_n(R_4) \longrightarrow K_n(R_5)$$

and

$$K_n(R_5) \longrightarrow K_n(R_2)$$

are surjective. Hence the compositions

$$f_{n*}: \qquad K_n(R_3) \longrightarrow K_n(R_4) \longrightarrow K_n(R_5) \longrightarrow K_n(R_2)$$

are surjective. Let $g_{n*}$ be the following composition

$$K_n(R_3) \longrightarrow K_n(R_1) \longrightarrow K_n(R_2).$$

Although $f_{n*}$ and $g_{n*}$ are obtained in different ways, they are both induced by the same natural ring inclusion

$$R_3 \longrightarrow R_2.$$

By the funtoriality of K-theory, $f_{n*} = g_{n*}$. Hence the maps $g_{n*}$ are surjective, which implies that the maps

$$K_n(R_1) \longrightarrow K_n(R_2)$$

are surjective. □

Let $A \simeq M_n(D)$ be a simple algebra and $\Lambda$ a generalized Eichler order in $A$. The local order $\Lambda_\wp$ is either maximal or isomorphic to some

$$\begin{pmatrix} (\triangle) & (\wp^{k_\wp}) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\wp^{k_\wp}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \end{pmatrix}^{(n_1,\ \ldots,\ n_r)},$$

where $k_\wp \geq 1$ and $\triangle$ is the unique maximal order in $D_{(\wp)}$. By the Skolem-Noether Theorem this isomorphism is given by an inner automorphism, hence there is an element $a_\wp \in A_\wp$ such that

$$\Lambda_\wp = a_\wp \begin{pmatrix} (\triangle) & (\wp^{k_\wp}) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\wp^{k_\wp}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \end{pmatrix}^{(n_1,\ \ldots,\ n_r)} a_\wp^{-1}.$$

We now define $\Gamma'_\wp = \Lambda_\wp$, if $\Lambda_\wp$ is maximal, and

$$\Gamma'_\wp = a_\wp \begin{pmatrix} (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \end{pmatrix}^{(n_1,\, \ldots,\, n_r)} a_\wp^{-1}$$

otherwise. By Theorem 5.3 in [7], there is a global maximal order $\Gamma$, so that $\Gamma_\wp = \Gamma'_\wp$ for all $\wp$.

Let

$$I = \prod_\wp \wp^{k_\wp}$$

throughout this section, where $\wp$ runs through all $\wp$ at which $\Lambda_\wp$ is not maximal.

**Lemma 2.6.** *For all $n \geq 1$, the natural homomorphisms*

$$K_n(\Lambda_\wp / I\Gamma_\wp) \longrightarrow K_n(\Gamma_\wp / I\Gamma_\wp)$$

*are surjective.*

*Proof.* If $\Lambda_\wp$ is maximal, then the lemma obviously holds. So we suppose that $\Lambda_\wp$ is not maximal. Without loss of generality, we assume

$$\Lambda_\wp = \begin{pmatrix} (\triangle) & (\wp^{k_\wp}) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\wp^{k_\wp}) & \cdots & (\wp^{k_\wp}) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\wp^{k_\wp}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \end{pmatrix}^{(n_1,\, \ldots,\, n_r)}$$

and

$$\Gamma_\wp = \begin{pmatrix} (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\triangle) & (\triangle) & (\triangle) & \cdots & (\triangle) \end{pmatrix}^{(n_1,\, \ldots,\, n_r)} ,$$

where $\triangle$ is the unique maximal order in $D_{(\wp)}$. Then

$$\Lambda_\wp / I\Gamma_\wp = \begin{pmatrix} (R) & (0) & (0) & \cdots & (0) \\ (R) & (R) & (0) & \cdots & (0) \\ (R) & (R) & (R) & \cdots & (0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (R) & (R) & (R) & \cdots & (R) \end{pmatrix}^{(n_1,\, \ldots,\, n_r)}$$

and

$$\Gamma_\wp/I\Gamma_\wp = \begin{pmatrix} (R) & (R) & (R) & \cdots & (R) \\ (R) & (R) & (R) & \cdots & (R) \\ (R) & (R) & (R) & \cdots & (R) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (R) & (R) & (R) & \cdots & (R) \end{pmatrix}^{(n_1,\ \ldots,\ n_r)},$$

where $R = \triangle/\wp^{k_\wp}$.

The lemma now follows from Lemma 2.5. $\square$

For any abelian group $G$, let $G(\frac{1}{s})$ be the group $G \bigotimes \mathbb{Z}[\frac{1}{s}]$. For any ring homomorphism

$$f: \ A \longrightarrow B,$$

we shall write $f_*$ for the induced homomorphism

$$K_n(A)(\frac{1}{s}) \longrightarrow K_n(B)(\frac{1}{s})$$

**Lemma 2.7.** *For all $n \geq 1$, the natural homomorphism*

$$f_{1*}: \qquad K_n(\Lambda_\wp)(\frac{1}{s}) \longrightarrow K_n(\Gamma_\wp)(\frac{1}{s})$$

*is surjective, where $s$ is the generator of $I \bigcap \mathbb{Z}$.*

*Proof.* The square

$$\text{(I).} \qquad \begin{array}{ccc} \Lambda_\wp & \xrightarrow{\ f_1\ } & \Gamma_\wp \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ \Lambda_\wp/I\Gamma_\wp & \xrightarrow[\ g_2\ ]{} & \Gamma_\wp/I\Gamma_\wp \end{array}$$

has an associated $K_*(\frac{1}{s})$ Mayer-Vietoris sequence

$$\cdots \longrightarrow K_n(\Lambda_\wp)(\frac{1}{s}) \stackrel{(f_{1*},\ f_{2*})}{\longrightarrow} K_n(\Gamma_\wp)(\frac{1}{s}) \bigoplus K_n(\Lambda_\wp/I\Gamma_\wp)(\frac{1}{s}) \stackrel{(g_{1*},\ g_{2*})}{\longrightarrow} K_n(\Gamma_\wp/I\Gamma_\wp)(\frac{1}{s}) \longrightarrow \cdots$$

by [1] or [9], where

$$(f_{1*},\ f_{2*})(x) = (f_{1*}(x),\ f_{2*}(x))$$

for $x \in K_n(\Lambda_\wp)(\frac{1}{s})$ and

$$(g_{1*},\ g_{2*})(a,\ b) = g_{1*}(a) - g_{2*}(b)$$

for $a \in K_n(\Gamma_\wp)(\frac{1}{s})$ and $b \in K_n(\Lambda_\wp/I\Gamma_\wp)(\frac{1}{s})$.

For any element $x \in K_n(\Gamma_\wp)(\frac{1}{s})$, we can find $y \in K_n(\Lambda_\wp/I\Gamma_\wp)(\frac{1}{s})$ such that

$$(g_{1*},\ g_{2*})(x,\ y) = g_{1*}(x) - g_{2*}(y) = 0$$

by Lemma 2.6. So

$$(x,\ y) \in ker(g_{1*},\ g_{2*}) = im(f_{1*},\ f_{2*}).$$

Hence $x \in im(f_{1*})$ which implies $f_{1*}$ is surjective. □

**Corollary 2.8.** *For all $n \geq 1$, the cokernel of*

$$K_n(\Lambda_\wp) \longrightarrow K_n(\Gamma_\wp)$$

*has no nontrivial $p$-torsion elements, where $p$ is an arbitrary rational prime which does not divide $s$.*

*Proof.* By Lemma 2.7, the cokernel of

$$K_n(\Lambda_\wp) \longrightarrow K_n(\Gamma_\wp)$$

is $s$-torsion. Hence the result follows. □

**Corollary 2.9.** *For all $n \geq 0$, the map*

$$f_* :\ K_{2n+1}(\Lambda_\wp)(\frac{1}{s}) \longrightarrow K_{2n+1}(A_\wp)(\frac{1}{s})$$

*is surjective, where $f_*$ is induced by the inclusion map $f :\ \Lambda_\wp \longrightarrow A_\wp$.*

*Proof.* The map

$$f :\ K_{2n+1}(\Lambda_\wp)(\frac{1}{s}) \longrightarrow K_{2n+1}(A_\wp)(\frac{1}{s})$$

is the composition

$$K_{2n+1}(\Lambda_\wp)(\frac{1}{s}) \xrightarrow{f_{1*}} K_{2n+1}(\Gamma_\wp)(\frac{1}{s}) \xrightarrow{h_*} K_{2n+1}(A_\wp)(\frac{1}{s}),$$

where $h_*$ is induced by the inclusion $h :\ \Gamma_\wp \longrightarrow A_\wp$. By Theorem 1 of [2], $h_*$ is surjective. Since $f_{1*}$ and $h_*$ are both surjective, $f$ is also surjective. □

**Theorem 2.10.** *Let $\Lambda$ be a generalized Eichler order in a semi-simple algebra $A$ over $F$. For all $n \geq 1$, the $q$-primary part of $Cl_{2n}(\Lambda)$ is trivial for $q \notin P_S$.*

*Proof.* Since $\Lambda$ can be expressed as the direct sum of generalized Eichler orders in the simple components of $A$, we may assume $A$ is simple.. Corollary 2.9 implies in this case that

$$coker(K^c_{2n+1}(\Lambda_\wp) \longrightarrow K^c_{2n+1}(A_\wp))(q) = 0$$

for $q \notin P_S$. So the $q$-primary part of $Cl_{2n}(\Lambda)$ is trivial for $q \notin P_S$ by Lemma 2.1. □

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