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Even dimensional higher class groups of orders

Xuejun Guo

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Abstract Let F be a number field, and let $\mathcal{O}_F$ denote the ring of integers in F. Let A be a finite-dimensional central simple F-algebra, and let Λ be an $\mathcal{O}_F$-order in A. In this paper it is shown that the p-torsion of the even dimensional higher class group $Cl_{2n}(\Lambda)$ can only occur for primes p, which lie under prime ideals $\mathfrak{p}$, at which $\Lambda_{\mathfrak{p}}$ is not maximal, or which divide the dimension of A.

Keywords Higher class group · Semi-simple algebra · Order

Mathematics Subject Classification (2000) 19D50 · 19F27

1 Introduction

Let F be a number field and $\mathcal{O}_F$ the ring of integers in F. Let A be a semi-simple algebra over F and Λ an $\mathcal{O}_F$-order in A, $[A : F]$ the dimension of A over F. The higher class groups of Λ are defined as

$$Cl_n(\Lambda) = \ker\left(SK_n(\Lambda) \longrightarrow \bigoplus_{\mathfrak{p}} SK_n(\Lambda_{\mathfrak{p}})\right),$$

where $\mathfrak{p}$ runs through all prime ideals of $\mathcal{O}_F$ and

$$SK_n(\Lambda) := \ker\left(K_n(\Lambda) \longrightarrow K_n(A)\right),$$

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X. Guo (✉)
Department of Mathematics, Nanjing University, Nanjing, China
e-mail: guoxj@nju.edu.cn

X. Guo
Department of Mathematics, University College Dublin, Dublin, Ireland

for all integers $n \geq 1$. Kuku proved in [6,7] that the groups $Cl_n(\Lambda)$ are finite for arbitrary orders. Moreover, they vanish if Λ is maximal (cf [4, Theorems 1, 2]).

In [5], it is proved that the only p-torsion possible in $Cl_{2n+1}(\Lambda)$ is for those rational primes p which lie under the prime ideals of $\mathcal{O}_F$ at which Λ is not maximal. In [3] the analogous result for even-dimensional higher class groups is proved for Eichler orders.

In this paper it is shown that p-torsion in the even-dimensional higher class group $Cl_{2n}(\Lambda)$ can only occur if $p \mid [A : F]$ or p lies under some prime ideal of $\mathcal{O}_F$ at which Λ is not maximal. It turns out that $Cl_{2n}(\Lambda)$ can have non-trivial p-torsion for $p \mid [A : F]$ even if $\Lambda_{\mathfrak{p}}$ is maximal for all prime ideals $\mathfrak{p} \mid p$ (see Sect. 4).

2 K-theory of local orders

Lemma 2.1 *Let k be a finite field, $M_m(k)$ the ring of all $m \times m$ matrices over k. Let i be the natural inclusion $i : k \longrightarrow M_m(k)$, $a \mapsto aI_m$, where I_m is the identity matrix in $M_m(k)$. Then the composition of homomorphisms*

$$f : K_n(k) \xrightarrow{K_n(i)} K_n(M_m(k)) \xrightarrow{\sim} K_n(k)$$

maps x to x^m for any $n \geq 1$, where $K_n(i)$ is induced by i and the isomorphism $K_n(M_m(k)) \xrightarrow{\sim} K_n(k)$ is induced by Morita equivalence between the categories of finitely generated projective $M_m(k)$-modules and finitely generated projective k-modules.

Proof Let $\mathcal{P}(k)$, $\mathcal{P}(M_2(k))$ be the categories of finitely generated projective k-, $M_2(k)$-modules respectively. Then f is induced by the composition of functors

$$\begin{aligned} \mathcal{P}(k) &\longrightarrow \mathcal{P}(M_2(k)) &&\longrightarrow \mathcal{P}(k) \\ P &\longrightarrow M_m(k) \otimes_k P &&\longrightarrow P^m = P \oplus P \oplus \cdots \oplus P. \end{aligned}$$

Hence $f(x) = x^m$. □

Let K be a p-adic local field, $\mathcal{O}_K$ the ring of integers in K, k the residue field of $\mathcal{O}_K$. Let D be a finite dimensional central division algebra over K, $\mathcal{D}$ the unique maximal $\mathcal{O}_K$-order in D, d the residue field of $\mathcal{D}$. Then d is a finite extension of k.

In [4] Keating proved that the natural homomorphisms

$$g_n : K_n(\mathcal{D}) \longrightarrow K_n(d)$$

are surjective and constructed transfer maps

$$\tau_{2n+1} : K_{2n+1}(d) \longrightarrow K_{2n+1}(\mathcal{D})$$

in the following way. At first, choose a generator γ of the cyclic group $K_{2n+1}(d)$ and an element $\sigma\gamma$ in $K_{2n+1}(\mathcal{D})$ which maps onto γ. Let Π be a uniformizer of D such that Π^t is a uniformizer π of K for some t, and c the automorphism of $K_{2n+1}(\mathcal{D})$ induced by the conjugation by Π. The transfer map τ_{2n+1} is defined to be

$$\tau_{2n+1}(\gamma^j) = (\sigma\gamma)^j (c\sigma\gamma)^{-j}.$$

Compare Sect. 2 of [4] for details, and for a proof of the following result:

Lemma 2.2 [4, Sect. 2] *The composition*

$$g_{2n+1}\tau_{2n+1} : K_{2n+1}(d) \longrightarrow K_{2n+1}(\mathcal{D}) \longrightarrow K_{2n+1}(d)$$

is $1 - \mathcal{F}_*$, where $\mathcal{F}*$ is induced by the Frobenius automorphism $\mathcal{F}$ of d/k.

Proposition 2.3 *Let f be the natural inclusion of rings $k \longrightarrow M_m(d)$. Let*

$$f_{2n+1} : K_{2n+1}(k) \xrightarrow{K_{2n+1}(f)} K_{2n+1}(M_m(d)) \xrightarrow{\sim} K_{2n+1}(d)$$

be the homomorphism induced by f and the Morita equivalence between $K_{2n+1}(M_m(d))$ and $K_{2n+1}(d)$. Let $t = [d : k]$ be the degree of the finite fields extension d/k, q the number of elements in k, and $(mt,\ q^{n+1} - 1)$ the greatest common divisor of mt and $q^{n+1} - 1$. Then the subgroup of $K_{2n+1}(d)$ generated by the images of f_{2n+1} and $g_{2n+1}\tau_{2n+1}$ has index dividing $(mt,\ q^{n+1} - 1)$, where $g_{2n+1}\tau_{2n+1}$ is the same as in the above lemma.

Proof By Theorem 8 of [8], $K_{2n+1}(k) \simeq \mathbb{Z}/(q^{n+1} - 1)$ and $K_{2n+1}(d) \simeq \mathbb{Z}/(q^{t(n+1)} - 1)$. We assume $K_{2n+1}(d) = \mathbb{Z}/(q^{t(n+1)} - 1)$. Then $K_{2n+1}(k)$ can be seen as the subgroup of $K_{2n+1}(d)$ generated by $(q^{t(n+1)} - 1)/(q^{n+1} - 1)$.

By Lemma 2.1, the image of f_{2n+1} is the subgroup of $K_{2n+1}(d)$ generated by $m(q^{t(n+1)} - 1)/(q^{n+1} - 1)$. By [4], the image of $g_{2n+1}\tau_{2n+1}$ is isomorphic to $\mathbb{Z}/((q^{t(n+1)} - 1)/(q^{n+1} - 1))$. Hence the image of $g_{2n+1}\tau_{2n+1}$ in $K_{2n+1}(d)$ is generated by $q^{n+1} - 1$. So the subgroup of $K_{2n+1}(d)$ generated by the images of f_{2n+1} and $g_{2n+1}\tau_{2n+1}$ is generated by the greatest common divisor $(m(q^{t(n+1)} - 1)/(q^{n+1} - 1),\ q^{n+1} - 1))$.

Since

$$\begin{aligned} \frac{q^{t(n+1)} - 1}{q^{n+1} - 1} &= \frac{(q^{n+1} - 1 + 1)^t - 1}{q^{n+1} - 1} \\ &= \sum_{k=1}^{t} \binom{t}{k} (q^{n+1} - 1)^{k-1} \\ &= \sum_{k=2}^{t} \binom{t}{k} (q^{n+1} - 1)^{k-1} + t, \end{aligned}$$

it follows that the greatest common divisor $(q^{n+1} - 1,\ (q^{t(n+1)} - 1)/(q^{n+1} - 1))$ divides t.

Hence $(m(q^{t(n+1)} - 1)/(q^{n+1} - 1),\ q^{n+1} - 1)$ divides $(mt,\ q^{n+1} - 1)$. □

Let A be a finite dimensional central simple algebra over a p-adic field K, Λ an $\mathcal{O}_K$-order in A, Γ a maximal order containing Λ.

For any abelian group G and positive rational integer s, let $G(\frac{1}{s})$ be the group $G \bigotimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$. For any ring homomorphism $f : A \longrightarrow B$, we shall write $f_*^{(s)}$ for the induced homomorphism

$$K_n(A)\left(\frac{1}{s}\right) \longrightarrow K_n(B)\left(\frac{1}{s}\right).$$

Theorem 2.4 *The cokernel of the natural homomorphism*

$$h_{1*}^{(p)} : K_{2n+1}(\Lambda)\left(\frac{1}{p}\right) \longrightarrow K_{2n+1}(A)\left(\frac{1}{p}\right)$$

is annihilated by the greatest common divisor $([A : K],\ q^{n+1} - 1)$, where h_1 is the inclusion $\Lambda \longrightarrow A$.

Proof Since Λ and Γ are two orders, there is some positive rational integer m such that $p^m\Gamma \subset \Lambda$. The square

$$\begin{array}{ccc} \Lambda & \xrightarrow{f_1} & \Gamma \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ \Lambda/p^m\Gamma & \xrightarrow[g_2]{} & \Gamma/p^m\Gamma \end{array}$$

has an associated $K_*\left(\frac{1}{p}\right)$ Mayer-Vietoris sequence

$$\cdots \longrightarrow K_{2n+1}(\Lambda)\left(\frac{1}{p}\right) \overset{(f_{1*}^{(p)},\ f_{2*}^{(p)})}{\longrightarrow} K_{2n+1}(\Gamma)\left(\frac{1}{p}\right) \bigoplus K_{2n+1}(\Lambda/p^m\Gamma)\left(\frac{1}{p}\right)$$
$$\overset{(g_{1*}^{(p)},\ g_{2*}^{(p)})}{\longrightarrow} K_{2n+1}(\Gamma/p^m\Gamma)\left(\frac{1}{p}\right) \longrightarrow \cdots$$

by [2] or [11].

Since A is a central simple algebra, it is isomorphic to $M_{m'}(D)$ for some division algebra D and m'. Similarly, since Γ is a maximal order, it is isomorphic to $M_{m'}(\mathcal{D})$ for the maximal order $\mathcal{D}$ in D. Without loss of generality, we can assume $A = M_{m'}(D)$, $\Gamma = M_{m'}(\mathcal{D})$ for the maximal order $\mathcal{D}$ in D. Let d be the residue field of $\mathcal{D}$. By Proposition 1 of [4], the homomorphism $K_i(\Gamma) \longrightarrow K_i(M_{m'}(d))$ is surjective. This homomorphism can be decomposed into

$$K_i(\Gamma) \longrightarrow K_i(\Gamma/p^m\Gamma) \longrightarrow K_i(M_{m'}(d)).$$

By Corollary 5.4 of [11], the homomorphism $K_i(\Gamma/p^m\Gamma)(\frac{1}{p}) \longrightarrow K_i(M_{m'}(d))(\frac{1}{p})$ is an isomorphism. So the homomorphism $K_i(\Gamma)(\frac{1}{p}) \longrightarrow K_i(\Gamma/p^m\Gamma)(\frac{1}{p})$ is surjective. The homomorphism $g_{1*}^{(p)}$ induces an isomorphism

$$\overline{g_{1*}^{(p)}}: \quad \operatorname{coker} f_{1*}^{(p)} = \frac{K_{2n+1}(\Gamma)\left(\frac{1}{p}\right)}{f_{1*}^{(p)}\left(K_{2n+1}(\Lambda)\left(\frac{1}{p}\right)\right)}$$
$$\longrightarrow \frac{K_{2n+1}(\Gamma/p^m\Gamma)\left(\frac{1}{p}\right)}{g_{2*}^{(p)}\left(K_{2n+1}(\Lambda/p^m\Gamma)\left(\frac{1}{p}\right)\right)} = \operatorname{coker} g_{2*}^{(p)}.$$

By Corollary 5.4 of [11], we know that $K_{2n+2}(\Gamma/p^m\Gamma)$ is a p-group. Hence $f_{1*}^{(p)}$ is injective. So we have the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & K_{2n+1}(\Lambda)(\frac{1}{p}) & \xrightarrow{f_{1*}^{(p)}} & K_{2n+1}(\Gamma)(\frac{1}{p}) & \longrightarrow\!\!\!\!\rightarrow & \operatorname{coker} f_{1*}^{(p)} \\ & & {\scriptstyle h_{1*}^{(p)}}\downarrow & & \downarrow{\scriptstyle h_{2*}^{(p)}} & & \downarrow \\ 1 & \longrightarrow & K_{2n+1}(A)(\frac{1}{p}) & \xrightarrow[\text{identity}]{} & K_{2n+1}(A)(\frac{1}{p}) & \longrightarrow & 1 \end{array}$$

Hence by the Snake Lemma,

$$\begin{aligned}\mathrm{coker} h_{1*}^{(p)} &\simeq \mathrm{coker}\left(\mathrm{ker} h_{2*}^{(p)} \longrightarrow \mathrm{coker} f_{1*}^{(p)}\right)\\ &\simeq \frac{K_{2n+1}(\Gamma)\left(\frac{1}{p}\right)}{f_{1*}^{(p)}\left(K_{2n+1}(\Lambda)\left(\frac{1}{p}\right)\right)\mathrm{ker} h_{2*}^{(p)}}\\ &\simeq \frac{K_{2n+1}\left(\Gamma/p^m\Gamma\right)\left(\frac{1}{p}\right)}{(\mathrm{img}_{2*}^{(p)})\overline{g_{1*}^{(p)}}\,\overline{(\mathrm{ker} h_{2*}^{(p)})}}.\end{aligned}$$

By Theorem 1 of [4], we know that $\mathrm{ker} h_{2*}^{(p)}$ is the image of

$$\tau_{2n+1}: K_{2n+1}(M_{m'}(d)) \longrightarrow K_{2n+1}(\Gamma).$$

So the cokernel of $h_{1*}^{(p)}$ is isomorphic to the quotient group of $K_{2n+1}(\Gamma/p^m\Gamma)(\frac{1}{p})$ modulo the subgroup generated by the image of $g_{2*}^{(p)}$ and the image of

$$\begin{aligned}\varphi: \quad & K_{2n+1}(M_{m'}(d))\left(\frac{1}{p}\right) \longrightarrow K_{2n+1}(\Gamma)\left(\frac{1}{p}\right) \longrightarrow K_{2n+1}(M_{m'}(d))\left(\frac{1}{p}\right)\\ & \simeq K_{2n+1}(\Gamma/p^m\Gamma)\left(\frac{1}{p}\right).\end{aligned}$$

Note that $\Lambda/p^m\Gamma$ may not be k, however

$$K_{2n+1}(\Lambda/p^m\Gamma)\left(\frac{1}{p}\right) \simeq K_{2n+1}(M_{r_1}(k_1) \oplus \cdots \oplus M_{r_l}(k_l))\left(\frac{1}{p}\right)$$

for suitable finite fields $k_1,\ldots, k_l$ which are extensions of k, and subfields of d. Hence by Proposition 2.3, the cardinality of

$$\frac{K_{2n+1}(\Gamma/p^m\Gamma)\left(\frac{1}{p}\right)}{(\mathrm{img}_{2*}^{(p)})(\mathrm{im}\varphi)}$$

divides the greatest common divisor $(m^{'}t,\ q^{n+1}-1)$. Hence the cokernel of $h_{1*}^{(p)}$ is annihilated by $([A:K],\ q^{n+1}-1)$. □

Remark If $p \nmid [D:K]$, then the reduced norm $\mathrm{Nrd}: K_i(D) \longrightarrow K_i(K)$ is an isomorphism by Theorem 3 of [9]. We know that the reduced norm is multiplication by $[D:K]$ on $K_i(\mathcal{O}_K)$. So the cokernel of $K_i(\mathcal{O}_K) \longrightarrow K_i(D)$ is annihilated by $[D:K]$. By Lemma 2.1, the cokernel of $K_i(\mathcal{O}_K) \longrightarrow K_i(A)$ is annihilated by $[A:K]$. Hence the cokernel of $h_{1*}^{(p)}$ is annihilated by $[A:K]$. So we have a very simple proof of Theorem 2.4 in this special case. Using the same argument, it may be possible to give a proof of the general case by working with the K-groups up to p-torsion.

3 Even dimensional higher class groups of global orders

If R is a ring, we will denote the quotient of $K_m(R)$ by its divisible subgroup by $K_m^c(R)$. Let S be a finite set of prime ideals at which $\Lambda_{\mathfrak{p}}$ is not maximal. As in [5], we define $Cl_m(\Gamma, S)$ to be the cokernel of the map

$$K^c_{m+1}(A) \longrightarrow \bigoplus_{\mathfrak{p}\in S} K^c_{m+1}(A_{\mathfrak{p}}) \oplus \bigoplus_{\mathfrak{p}\notin S} K^c_{m+1}(A_{\mathfrak{p}})/\mathrm{im}(K^c_{m+1}(\Gamma_{\mathfrak{p}})),$$

where Γ is a maximal order containing Λ.

Note that Theorem 2.4 is for K_n not K^c_n. However since the maximal divisible subgroup of an abelian group is a direct summand, the proof works also for K^c_n.

Let A be a semi-simple algebra over a number field F. Let $\mathcal{O}_F$ be the ring of integers in F and let Λ be an $\mathcal{O}_F$-order in A. Let Γ be a maximal order containing Λ. Let S be the set of primes $\mathfrak{p}$ at which $\Lambda_{\mathfrak{p}}$ is not a local maximal order. By Lemma 1.2 in [5],

$$Cl_{2n}(\Lambda) \simeq \mathrm{coker}\left(\bigoplus_{\mathfrak{p}\in S} K^c_{2n+1}(\Lambda_p) \longrightarrow Cl_{2n}(\Gamma, S)\right).$$

By Theorem 1 in [4],

$$\bigoplus_{\mathfrak{p}\notin S} K^c_{2n+1}(A_{\mathfrak{p}})/\mathrm{im}(K^c_{2n+1}(\Gamma_{\mathfrak{p}})) = 0.$$

So

$$Cl_{2n}(\Gamma, S) = \mathrm{coker}\left(K^c_{2n+1}(A) \longrightarrow \bigoplus_{\mathfrak{p}\in S} K^c_{2n+1}(A_{\mathfrak{p}})\right).$$

Hence $Cl_{2n}(\Lambda)$ is the cokernel of

$$K_{2n+1}(A) \longrightarrow \mathrm{coker}\left(\bigoplus_{\mathfrak{p}\in S} K^c_{2n+1}(\Lambda_{\mathfrak{p}}) \longrightarrow \bigoplus_{\mathfrak{p}\in S} K^c_{2n+1}(A_{\mathfrak{p}})\right).$$

Theorem 3.1 *Let A be a finite-dimensional central simple algebra over a number field F. Let $\mathcal{O}_F$ be the ring of integers in F and let Λ be an $\mathcal{O}_F$-order in F. Let s denote the product of those rational primes p, for which $\Lambda_{\mathfrak{p}}$ is not maximal for some prime ideal $\mathfrak{p}$ dividing p. Then p-torsion of $Cl_{2n}(\Lambda)$ can only occur for $p|s$ or $p|[A:F]$.*

Proof This is obvious by Theorem 2.4 and the discussion above this proposition. □

4 Example

Let l be a prime dividing the dimension $[A : F]$. Next we will give an example to show that $Cl_{2n}(\Lambda)$ can have nontrivial l-torsion although $\Lambda_{\mathfrak{p}}$ is maximal for all prime ideals $\mathfrak{p}$ above l.

Let p be a rational prime such that $l|(p-1)$. By the density theorem, there are infinitely many such p. For any prime ideal $\mathfrak{p}|p$, $\mathfrak{p} \subset \mathcal{O}_F$, let $A_{\mathfrak{p}} = M_l(F_{\mathfrak{p}})$. Let $\Gamma_{\mathfrak{p}} = M_l(\mathcal{O}_{F_{\mathfrak{p}}})$ and $\Lambda_{\mathfrak{p}} \subset \Gamma_{\mathfrak{p}}$ the order so that $\Lambda_{\mathfrak{p}}/\pi\Gamma_{\mathfrak{p}} \simeq k_{\mathfrak{p}}$, where π is a uniformizer of $F_{\mathfrak{p}}$.

Lemma 4.1 *For any $n \geq 1$ the cokernel of the natural homomorphism*

$$f_{1*}^{(\mathfrak{p})} : \quad K_{2n+1}(\Lambda_{\mathfrak{p}})\left(\frac{1}{p}\right) \longrightarrow K_{2n+1}(\Gamma_{\mathfrak{p}})\left(\frac{1}{p}\right)$$

has a non-trivial l-part.

Proof The square

$$\begin{tikzcd} \Lambda_{\mathfrak{p}} \arrow[r, "f_1"] \arrow[d, "f_2"'] & \Gamma_{\mathfrak{p}} \arrow[d, "g_1"] \\ \Lambda_{\mathfrak{p}}/p\Gamma_{\mathfrak{p}} \arrow[r, "g_2"'] & \Gamma_{\mathfrak{p}}/p\Gamma_{\mathfrak{p}} \end{tikzcd}$$

has an associated $K_*(\frac{1}{p})$ Mayer-Vietoris sequence

$$\cdots \longrightarrow K_{2n+1}(\Lambda_{\mathfrak{p}})\left(\frac{1}{p}\right) \overset{(f_{1*}^{(\mathfrak{p})},\ f_{2*}^{(\mathfrak{p})})}{\longrightarrow} K_{2n+1}(\Gamma_{\mathfrak{p}})\left(\frac{1}{p}\right) \bigoplus K_{2n+1}(\Lambda_{\mathfrak{p}}/p\Gamma_{\mathfrak{p}})\left(\frac{1}{p}\right)$$
$$\overset{(g_{1*}^{(\mathfrak{p})},\ g_{2*}^{(\mathfrak{p})})}{\longrightarrow} K_{2n+1}(\Gamma_{\mathfrak{p}}/p\Gamma_{\mathfrak{p}})\left(\frac{1}{p}\right) \longrightarrow \cdots$$

by [2] or [11].

Let $z \in K_{2n+1}(\Gamma_{\mathfrak{p}}/p\Gamma_{\mathfrak{p}})(\frac{1}{p})$ be a generator of the p-Sylow subgroup of $K_{2n+1}(\Gamma_{\mathfrak{p}}/p\Gamma_{\mathfrak{p}})$ $(\frac{1}{p})$ and $g_{1*}^{(\mathfrak{p})^{-1}}(z)$ an inverse image of z. Since $K_{2n+1}(\Gamma_{\mathfrak{p}}/p\Gamma_{\mathfrak{p}}) \simeq K_{2n+1}(M_l(k_{\mathfrak{p}}))$ and $K_{2n+1}(\Lambda_p/p\Gamma_p) \simeq K_{2n+1}(k_{\mathfrak{p}})$, z is not in the image of $g_{2*}^{(p)}$ by Lemma 2.1. Hence $g_{1*}^{(p)^{-1}}(z)$ is not in the image of $f_{1*}^{(p)}$. However $(g_{1*}^{(p)^{-1}}(z))^l$ is in the image of f_{1*} for $z^l \in (K_{2n+1}(\Gamma_p/p\Gamma_p))^l(\frac{1}{p})$ while the last term is just the image of $g_{2*}^{(p)}$. So the cokernel of $f_{1*}^{(p)}$ has a non-trivial l-part. □

Note that if we can construct a division algebra $A_{\mathfrak{p}}$, a non-maximal order $\Lambda_{\mathfrak{p}}$ such that the cokernel of $g_{2*}^{(p)}$ has a no-trivial l-part, then Lemma 3.2 also holds. Such examples can be found in quaternion algebras. One can see the details of the explicit construction in Proposition 5.6 (b) of [1].

Lemma 4.2 *The cokernel of*

$$K_{2n+1}(\Lambda_{\mathfrak{p}})\left(\frac{1}{p}\right) \longrightarrow K_{2n+1}(A_{\mathfrak{p}})\left(\frac{1}{p}\right)$$

has a non-trivial l-part.

Proof Since $\Gamma_{\mathfrak{p}}$ is a maximal order in a split local algebra $A_{\mathfrak{p}}$, the residue field of $\Gamma_{\mathfrak{p}}$ is the ring of $l \times l$ matrices over $k_{\mathfrak{p}}$. By Theorem 1 of [4], the natural homomorphism $K_{2n+1}(\Gamma_{\mathfrak{p}}) \longrightarrow K_{2n+1}(A_{\mathfrak{p}})$ is an isomorphism. So the result follows from Theorem 2.4. □

Let A be a division algebra over F with $[A : F] = l^2$. By [7], $K_{2n+1}(A)$ is finitely generated. Let Γ be a maximal order in A. Let m be a natural number which is greater than the number of generators of $K_{2n+1}(A)$. By the Density Theorem and the fact that $A_{\mathfrak{p}}$ is split for almost all $\mathfrak{p}$, we can find m different primes $p_1, \ldots, p_m$ such that $l|(p_i - 1)$ for any $1 \le i \le m$, and A is completely split at $\mathfrak{p}_i$ for any $\mathfrak{p}_i|p_i$. Let $S = \{p_1, \ldots, p_m\}$. Let $\Lambda_{\mathfrak{p}}$ be the order such that $\Lambda_{\mathfrak{p}}/\pi\Gamma_{\mathfrak{p}} \simeq k_{\mathfrak{p}}$ if p is equal to some p_i, $1 \le i \le m$. Otherwise, let $\Lambda_{\mathfrak{p}}$ be the completion of Γ at $\mathfrak{p}$. So for any prime $\mathfrak{p}$, we have a local order Λ_p and for almost all p, Λ_p is maximal. Hence by Proposition 5.1 in Chap. 5 of [10], there is a global order Λ such that the completion of Λ at p is Λ_p.

Proposition 4.3 *Let Λ be an order defined as later. Then the even dimensional higher class group $Cl_{2n}(\Lambda)$ has a non-trivial l-part, although the localization $\Lambda_{\mathfrak{p}}$ is maximal for any $\mathfrak{p}|l$.*

Proof Recall that $Cl_{2n}(\Lambda)$ is the cokernel of

$$K_{2n+1}(A) \longrightarrow \operatorname{coker}\left(\bigoplus_{\mathfrak{p}\in S} K^c_{2n+1}(\Lambda_{\mathfrak{p}}) \longrightarrow \bigoplus_{\mathfrak{p}\in S} K^c_{2n+1}(A_{\mathfrak{p}})\right).$$

By assumption, $K_{2n+1}(A)$ is generated by less than m elements. Since the l-part of $\left(\operatorname{coker}\left(\bigoplus_{p\in S} K^c_{2n+1}(\Lambda_p) \longrightarrow \bigoplus_{p\in S} K^c_{2n+1}(A_p)\right)\right)$ can not be generated by less than m generators, $Cl_{2n}(\Lambda)$ must have a non-trivial l-part. □

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References

1. Brzezinski, J.: On orders in quaternion algebras. Commun. Algebra **11**(5), 501–522 (1983)
2. Charney, R.M.: A note on excision in K-theory, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), 47–54, Lecture Notes in Math. **1046**. Springer, Berlin (1984)
3. Guo, X., Kuku, A.O.: Higher class groups of the generalized Eichler orders. Commun. Algebra **33**(3), 709–718 (2005)
4. Keating, M.E.: A transfer map in K-theory. J. Lond. Math. Soc. (2) **16**(1), 38–42 (1977)
5. Kolster, M., Laubenbacher, R.: On higher class groups of orders. Math. Z. **228**(2), 229–246 (1998)
6. Kuku, A.O.: K_n, SK_n of integral group-rings and orders. Contemp. Math. **55**, 333–338 (1986)
7. Kuku, A.O.: Some finiteness results in the higher K-theory of orders and group-rings. Topol. Appl. **25**(2), 185–191 (1987)
8. Quillen, D.: On the cohomology and K-theory of the general linear groups over a finite field. Ann. Math. (2) **96**, 552–586 (1972)
9. Suslin, A.A., Yufryakov, A.V.: K-theory of local division algebras. Soviet Math. Dokl. **33**, 794–798 (1986)
10. Vignéras, M.F.: Arithmétique des algèbres de quaternions, Lecture Notes in Mathematics, vol. **800**. Springer, Berlin (1980)
11. Weibel, C.A.: Mayer-Vietoris sequences and module structures on NK_*, Algebraic K-theory, Evanston 1980. In: Proceedings of conference, Northwestern University, Evanston (1980), pp. 466–493, Lecture Notes in Mathematics, vol. **854**. Springer, Berlin (1981)