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Higher Class Groups of Locally Triangular Orders over Number Fields*

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Abstract. In this paper, we study the K-theory of triangular rings. As an application, we show that for a locally triangular order Λ, the p-torsion in the higher class group $Cl_{2n}(\Lambda)$ can only occur for primes p which lie under the prime ideals $\wp$ of $\mathcal{O}_F$, at which Λ is not maximal.

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1 Introduction

Let R be a ring and I a two-sided projective ideal of R. In [4], Keating studied the K-theory of a "triangular tiled" ring, i.e., a ring of the $n \times n$ matrix form

$$M = \begin{pmatrix} R & R & \cdots & R \\ I & R & \cdots & R \\ \vdots & \ddots & \ddots & \vdots \\ I & \cdots & I & R \end{pmatrix},$$

and proved that $K_*(M) \simeq K_*(R) \oplus (n-1)K_*(R/I)$.

It is interesting to study the K-theory of rings like

$$S = \begin{pmatrix} R & R & \cdots & R \\ I^{s_{21}} & R & \cdots & R \\ \vdots & \ddots & \ddots & \vdots \\ I^{s_{n1}} & \cdots & I^{s_{n\,n-1}} & R \end{pmatrix},$$

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where each s_{ij} is a positive integer. For any abelian group G and a rational integer s, let $G(1/s)$ be the group $G\bigotimes_{\mathbb{Z}}\mathbb{Z}[1/s]$. In this paper, we prove that if R is a $\mathbb{Z}_p$-algebra, then

$$K_*(S)(1/s) = K_*(R)(1/s) \oplus (n-1)K_*(R/I)(1/s),$$

where s is a rational integer such that $p\,|s$ (Proposition 2.3).

In Section 3, we give an application of this result. Let F be a number field and $\mathcal{O}_F$ the ring of integers in F. Let A be a semi-simple algebra over F and Λ an order in A. For any maximal ideal $\wp$ of $\mathcal{O}_F$, let $F_\wp$, $\mathcal{O}_{F_\wp}$, $A_\wp$, $\Lambda_\wp$ be the $\wp$-completions of F, $\mathcal{O}_F$, A, Λ, respectively. Let

$$S = \{p \in \mathbb{Z} \mid \text{for some maximal ideal } \wp \text{ of } \mathcal{O}_F \text{ such that } \wp|p,\ \Lambda_\wp \text{ is not maximal}\}.$$

Recall that for any integer $n \geq 1$, the higher class group of Λ is defined as $Cl_n(\Lambda) = \ker\left(SK_n(\Lambda) \to \bigoplus_\wp SK_n(\Lambda_\wp)\right)$, where $\wp$ runs through all maximal ideals of $\mathcal{O}_F$ and $SK_n(\Lambda) := \ker(K_n(\Lambda) \to K_n(A))$. By Theorems 1 and 2 in [3], $Cl_n(\Lambda)$ is trivial for maximal orders. Later, Kuku proved in [6] that $Cl_n(\Lambda)$ is finite for arbitrary orders. In [5], it is proved that the only p-torsion possible in $Cl_{2n+1}(\Lambda)$ is for those rational primes p which lie under the prime ideals of $\mathcal{O}_F$, at which Λ is not maximal. In [2], we prove that if the order is a generalized Eichler order, then the only p-torsion possible in $Cl_{2n+1}(\Lambda)$ is for those rational primes p which lie under the prime ideals of $\mathcal{O}_F$, at which Λ is not maximal. Locally, a generalized Eichler order has the same form with M in the beginning of this section.

In this paper, we consider locally triangular orders which are locally isomorphic to S as above. So locally triangular orders are a generalization of generalized Eichler orders. We prove in Theorem 3.3 that if Λ is a locally triangular order in a semi-simple algebra A over F, then the q-primary part of $Cl_{2n}(\Lambda)$ is trivial for $q \notin S$.

2 *K*-Theory of Triangular Rings

In this section, R is a $\mathbb{Z}_p$-algebra and I is a two-sided projective ideal of R. Assume that R/I^n is a finite ring for any positive integer n. Let s be an integer such that $p\,|s$. Let a, b, c be positive integers such that $a + b \geq c \geq b \geq a$. The last inequality is to assure that

$$A = \begin{pmatrix} R & R & R \\ I^a & R & R \\ I^c & I^b & R \end{pmatrix}$$

is a ring.

Proposition 2.1. *Let A be the triangular ring as above. Then*

$$K_*(A)(1/s) \simeq K_*(R)(1/s) \oplus K_*(R/I)(1/s) \oplus K_*(R/I)(1/s)\,.$$

Proof. Let

$$B_1 = \begin{pmatrix} R/I^b & R/I^b & R/I^b \\ I^a/I^b & R/I^b & R/I^b \\ I^b/I^c & 0 & R/I^b \end{pmatrix} \quad \text{and} \quad I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I^b/I^c & 0 & 0 \end{pmatrix}.$$

Note that since $2b \geq c$, I^b/I^c is an R/I^b-module. Hence, for any element $a \in I^b/I^c$ and $b \in R/I^b$, the product $ab \in I^b/I^c$ is well defined. So B_1 is a ring, and I_1 is a two-sided ideal of B_1. Let

$$A_1 = B_1/I = \begin{pmatrix} R/I^b & R/I^b & R/I^b \\ I^a/I^b & R/I^b & R/I^b \\ 0 & 0 & R/I^b \end{pmatrix}.$$

Then by Consequence 1.4 of [8], $K_*(B_1, I_1)$ is a $\mathbb{Z}[1/q]$-module for any prime $q \neq p$. So $K_*(B_1)(1/s) \xrightarrow{\sim} K_*(A_1)(1/s)$ for any integer s such that $p\,|s$. Note that the composition

$$A_1 \overset{\text{inclusion}}{\longrightarrow} B_1 \overset{\text{quotient}}{\longrightarrow} A_1$$

is an identity map of A_1. So the homomorphism $K_*(A_1)(1/s) \xrightarrow{\sim} K_*(B_1)(1/s)$ induced by inclusion is an isomorphism.

Let

$$A_2 = \begin{pmatrix} R/I^c & R/I^c & R/I^c \\ I^a/I^c & R/I^c & R/I^c \\ 0 & I^b/I^c & R/I^c \end{pmatrix}, \quad B_2 = \begin{pmatrix} R/I^c & R/I^c & R/I^c \\ I^a/I^c & R/I^c & R/I^c \\ I^b/I^c & I^b/I^c & R/I^c \end{pmatrix}$$

and

$$I_2 = \begin{pmatrix} I^b/I^c & I^b/I^c & I^b/I^c \\ I^b/I^c & I^b/I^c & I^b/I^c \\ 0 & I^b/I^c & I^b/I^c \end{pmatrix}.$$

Then $B_1 = B_2/I_2$ and $A_1 = A_2/I_2$. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} A_2 & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 \end{array}$$

This diagram is a pull back square. So by [7] or [1], this square has an associated $K_*(1/s)$ Mayer–Vietoris sequence

$$\cdots \to K_*(A_2)(1/s) \to K_*(B_2)(1/s) \oplus K_*(A_1)(1/s) \to K_*(B_1)(1/s) \to \cdots.$$

Since we have proved that $K_*(A_1)(1/s) \to K_*(B_1)(1/s)$ is an isomorphism, $K_*(A_2)(1/s) \to K_*(B_2)(1/s)$ must be an isomorphism too. Let

$$B = \begin{pmatrix} R & R & R \\ I^a & R & R \\ I^b & I^b & R \end{pmatrix} \quad \text{and} \quad I_3 = \begin{pmatrix} I^c & I^c & I^c \\ I^c & I^c & I^c \\ I^c & I^c & I^c \end{pmatrix}.$$

Then $A/I_3 = A_2$ and $B/I_3 = B_2$. Hence, we have a pull back square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 \end{array}$$

By the same arguments as above, the homomorphism $K_*(A)(1/s) \to K_*(B)(1/s)$ is an isomorphism. Let

$$C = \begin{pmatrix} R & R & R \\ I^a & R & R \\ I^a & I^a & R \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} I^b & I^b & I^b \\ I^b & I^b & I^b \\ I^b & I^b & I^b \end{pmatrix}.$$

Then

$$B/J = \begin{pmatrix} R/I^b & R/I^b & R/I^b \\ I^a/I^b & R/I^b & R/I^b \\ 0 & 0 & R/I^b \end{pmatrix} \quad \text{and} \quad C/J = \begin{pmatrix} R/I^b & R/I^b & R/I^b \\ I^a/I^b & R/I^b & R/I^b \\ I^a/I^b & I^a/I^b & R/I^b \end{pmatrix}.$$

We have a pull back square

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ B/J & \longrightarrow & C/J \end{array}$$

By the associated $K_*(1/s)$ Mayer–Vietoris sequence, $K_*(B)(1/s) \to K_*(C)(1/s)$ is an isomorphism if $K_*(B/J)(1/s) \to K_*(C/J)(1/s)$ is an isomorphism. Let

$$D = \begin{pmatrix} R/I & 0 & 0 \\ 0 & R/I & 0 \\ 0 & 0 & R/I \end{pmatrix}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} B/J & \xrightarrow{\;f\;} & C/J \\ & {}_{g}\searrow \quad \swarrow_{h} & \\ & D & \end{array}$$

where f is the inclusion, g and h are the obvious quotient homomorphisms. By Theorem A of [4] and Consequence 1.4 of [8], the induced maps $g_* : K_*(B/J) \to K_*(D)$ and $g_* : K_*(C/J) \to K_*(D)$ are isomorphisms. Thus, $f_* : K_*(B/J) \to K_*(C/J)$. Hence, $K_*(B)(1/s) \simeq K_*(C)(1/s)$. We have already proved

$$K_*(A)(1/s) \simeq K_*(B)(1/s).$$

So

$$\begin{aligned} K_*(A)(1/s) &\simeq K_*(C)(1/s) \\ &\simeq K_*(R)(1/s) \oplus K_*(R/I^a)(1/s) \oplus K_*(R/I^a)(1/s) \\ &\simeq K_*(R)(1/s) \oplus K_*(R/I)(1/s) \oplus K_*(R/I)(1/s), \end{aligned}$$

which completes the proof. □

Corollary 2.2. *Let $E = M_3(R)$ be the ring of all 3×3 matrices over R and $f : A \to E$ the natural inclusion. The induced homomorphism $f_* : K_*(A)(1/s) \to K_*(E)(1/s)$ is surjective.*

Proof. By the proof of Proposition 2.1, $K_*(A)(1/s) \to K_*(C)(1/s)$ is surjective. By [4], $K_*(C)(1/s) \to K_*(E)(1/s)$ is surjective. Hence, the induced homomorphism $f_*: K_*(A)(1/s) \to K_*(E)(1/s)$ is surjective. □

By the same arguments as in the proof of Proposition 2.1, we can prove the following proposition.

Proposition 2.3. *Let p be a rational prime number and R a $\mathbb{Z}_p$-algebra. Let*

$$S = \begin{pmatrix} R & R & \cdots & R \\ I^{s_{21}} & R & \cdots & R \\ \vdots & \ddots & \ddots & \vdots \\ I^{s_{n1}} & \dots & I^{s_{n\,n-1}} & R \end{pmatrix}$$

be a ring, where I is a two-sided projective ideal of R and each s_{ij} is a positive integer. Then

$$K_*(S)(1/s) = K_*(R)(1/s) \oplus (n-1)K_*(R/I)(1/s),$$

and the homomorphism $K_*(S) \to K_*(M_n(R))$ is surjective.

3 Application to Higher Class Groups of Locally Triangular Orders

Let F be a number field and $\mathcal{O}_F$ the ring of integers in F. Let A be a semi-simple algebra over F and Λ an order in A. Let $\wp$ be a maximal ideal of $\mathcal{O}_F$, and $F_\wp$, $\mathcal{O}_{F_\wp}$, $A_\wp$, $\Lambda_\wp$, S be as defined in Section 1. Let s be the product of prime numbers in S. If D is a division algebra, let $D_{(\wp)}$ be the division algebra such that $D_\wp \simeq M_k(D_{(\wp)})$.

If R is a ring, by $K_m^c(R)$ we denote the quotient of $K_m(R)$ by its divisible subgroup. Let $\mathcal{S}$ be the set of primes, at which $\Lambda_\wp$ is not maximal.

Lemma 3.1. *The higher class group $Cl_{2n}(\Lambda)$ is a homomorphic image of*

$$\operatorname{coker}\Big(\bigoplus_{\wp\in\mathcal{S}} K_{2n+1}^c(\Lambda_\wp) \to K_{2n+1}^c(\Gamma_\wp)\Big),$$

where Γ is a maximal order containing Λ.

Proof. This lemma follows from the proof of Lemma 2.1 of [2]. □

Definition 3.2. Let $A \simeq M_n(D)$, where D is a finite dimensional division algebra. We call an order Λ in A a *locally triangular order* if each $\Lambda_\wp$ has the form

$$\Lambda_\wp \simeq \begin{pmatrix} R & R & \cdots & R \\ I^{s_{21}} & R & \cdots & R \\ \vdots & \ddots & \ddots & \vdots \\ I^{s_{k1}} & \dots & I^{s_{k\,k-1}} & R \end{pmatrix},$$

where $s_{ij} \geq 1$, R is the unique maximal order in $D_{(\wp)}$ and I is the unique maximal ideal of R. If $A \simeq \bigoplus_i M_{n_i}(D_i)$ is a semi-simple algebra, then an order Λ in A is called a *locally triangular order* if $\Lambda \simeq \bigoplus_i \Lambda_i$, where Λ_i is a locally triangular order in $M_{n_i}(D_i)$.

Recall that if $\Gamma_\wp$ is a maximal order in the simple algebra $A_\wp$, then $\Gamma_\wp$ has the form

$$\Gamma_\wp \simeq \begin{pmatrix} R & R & \cdots & R \\ R & R & \cdots & R \\ \vdots & \ddots & \ddots & \vdots \\ R & \cdots & R & R \end{pmatrix}.$$

Theorem 3.3. *Let Λ be a locally triangular order in a semi-simple algebra A over F. For all $n \geq 1$, the q-primary part of $Cl_{2n}(\Lambda)$ is trivial for $q \notin S$.*

Proof. Since Λ can be expressed as a direct sum of locally triangular orders in the simple components of A, we may assume that A is simple. This theorem follows from Proposition 2.3 and Lemma 3.1. □

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