Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Number Theory 107 (2004) 207–214

JOURNAL OF Number Theory

http://www.elsevier.com/locate/jnt

An embedding theorem for Eichler orders ☆

Xuejun Guo[a,b,*] and Hourong Qin[a,b]

[a] *Department of Mathematics, Nanjing University, 22 Hankou Road, Nanjing, Jiangsu 210093, People's Republic of China*
[b] *Morningside Center of Mathematics, Beijing 100080, China*

Received 23 January 2002; revised 20 November 2003

Communicated by M.-F. Vigneras

Abstract

Let B be a quaternion algebra over number field K. Assume that B satisfies the Eichler condition (i.e., there is at least one archimedean place which is unramified in B). Let Ω be an order in a quadratic extension L of K. The Eichler orders of B which admit an embedding of Ω are determined. This is a generalization of Chinburg and Friedman's embedding theorem for maximal orders.
© 2004 Elsevier Inc. All rights reserved.

MSC: 11R52; 20G30; 11S45

Keywords: Quaternion algebra; Eichler orders

1. Introduction

Let B be a quaternion algebra over a number field K and let L be a quadratic field extension of K. The classical Hasse–Brauer–Noether–Albert Theorem says that there is an embedding of L into B over K if and only if no place of K which ramifies in B splits in L. In [2], Chinburg and Friedman prove an integral version of this theorem.

☆ Supported by National Distinguished Young Science Foundation of China Grant and the 973 Grant.
* Corresponding author. Department of Mathematics, Nanjing University, 22 Hankou Road, Nanjing, Jiangsu 210093, People's Republic of China. Fax: +86-25-359-6468.
E-mail address: guoxj@nju.edu.cn (X. Guo).

0022-314X/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jnt.2004.04.008

Theorem 1.1 (Chinburg and Friedman [2, Theorem 3.3]). *Let B be a quaternion algebra over a number field K and let $\mathcal{O}_K$ be the ring of integers of K. Let $\Omega \subset B$ be a commutative $\mathcal{O}_K$-order and assume B satisfies the Eichler condition (i.e., there is at least one archimedean place which is unramified in B). Then every maximal order of B contains a conjugate (by B^*) of Ω except when the following three conditions hold*:

(1) *Ω is an integral domain and its quotient field $L \subset B$ is a quadratic extension of K.*
(2) *The extension L/K and the algebra B are unramified at all finite places and ramify at exactly the same (possibly empty) set of real places of K.*
(3) *All prime ideals of K dividing the relative discriminant ideal* $\operatorname{disc}(\Omega/\mathcal{O}_K)$ *of Ω split in L/K.*

Suppose now that (1)–(3) *hold. Then B has an even number of conjugacy classes of maximal orders. The maximal orders $\mathcal{D}$ containing some conjugate of Ω make up exactly half of these conjugacy classes. If $\mathcal{D}$ and $\mathcal{E}$ are maximal orders and $\mathcal{E}$ contains a conjugate of Ω, then $\mathcal{D}$ contains a conjugate of Ω if and only if the image by the reciprocity map* $\operatorname{Frob}_{L/K}$ *of the distance ideal $\rho(\mathcal{D}, \mathcal{E})$ is the trivial element of* $\operatorname{Gal}(L/K)$.

In this paper, we extend this result to Eichler orders of level δ (Theorem 2.5). Recall that an Eichler order of level δ is the intersection of two unique maximal orders whose distance ideal is δ (cf [3, Lemma 2.4 of Chapter 2 and Proposition 5.1 of Chapter 3]). As we know that a maximal order is an Eichler order of trivial level, our theorem implies Theorem 1.1 stated above. If the order Ω satisfies (1) of Theorem 1.1, then it is always contained in some maximal order, but is not always contained in some Eichler order of level δ. So it is necessary to impose some local conditions for the existence of any Eichler order of level δ contain a copy of Ω, a feature absent when δ is trivial (Theorem 2.4).

We have the same conditions (1) and (2) as in the above theorem, but we need to modify the condition (3) to:

All prime ideals $\mathcal{P}$ of K which divide the relative discriminant ideal $\operatorname{disc}(\Omega/\mathcal{O}_K)$ and satisfy $\operatorname{ord}_{\mathcal{P}}(\delta) \neq \operatorname{ord}_{\mathcal{P}}(\operatorname{disc}(\Omega/\mathcal{O}_K))$ split in L/K.

Although some different techniques are used, such as the language of ideles, we point out that this paper is based on Chinburg and Friedman's approach, our proof being a simplification of their original argument.

2. Main results

Let $K_{\mathbb{A}}^*$ be the idele group of a number field K and $B_{\mathbb{A}}^*$ the idele group of B. Let $\mathcal{E} \subset B$ be an Eichler order of level δ, $\mathcal{E}_{\mathbb{A}} = \mathcal{E} \otimes_K K_{\mathbb{A}}$ its adele ring and $N(\mathcal{E}_{\mathbb{A}})$ the normalizer of $\mathscr{E}_{\mathbb{A}}$ in $B_{\mathbb{A}}^*$. Recall that each Eichler order $\mathscr{D}$ of level δ is locally isomorphic to $\mathscr{E}$. By the Noether–Skolem Theorem, we know that there is an idele $x \in B_{\mathbb{A}}^*$ such that $D_{\mathbb{A}} = xE_{\mathbb{A}}x^{-1}$. So each Eichler order of level δ can be represented by an idele $x \in B_{\mathbb{A}}^*$. By definition of $N(\mathscr{E}_{\mathbb{A}})$, $\mathscr{D} = \mathscr{E}$ if and only if $x \in N(\mathscr{E}_{\mathbb{A}})$ and $\mathscr{D} \simeq \mathscr{E}$ if and only if $x \in B^* N(\mathscr{E}_{\mathbb{A}})$. If $\mathscr{D} \simeq \mathscr{E}$ (i.e., $\mathscr{D}$ is conjugate to $\mathscr{E}$), we also say that $\mathscr{D}$ and $\mathscr{E}$ are in the same *type*. Recall that there is a natural isomorphism induced by the reduced norm

$$n:\ B^* \backslash B_{\mathbb{A}}^* / N(\mathscr{E}_{\mathbb{A}}) \to K^* \backslash K_{\mathbb{A}}^* / n(N(\mathscr{E}_{\mathbb{A}})).$$

For the definition of "n", we refer to [3, Chapter 1, p. 1] for details. Throughout this paper, a subscript $_{\mathscr{P}}$ on a global object denotes the $\mathscr{P}$-adic completion. Since we assume that B satisfies the Eichler condition, by the proof of Corollary 5.7 of Chapter 3 of [3],

$$n(N(\mathscr{E}_{\mathscr{P}})) = \begin{cases} K_{\mathscr{P}}^* & \text{if } \mathrm{ord}_{\mathscr{P}}\delta \text{ is odd or } \mathscr{P} \text{ divides } \mathrm{disc}(B), \\ \mathscr{O}_{\mathscr{P}}^* K_{\mathscr{P}}^{*2} & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $G = K^* \backslash K_{\mathbb{A}}^* / n(N(\mathscr{E}_{\mathbb{A}}))$. By (2.1), G is a finite group of exponent 2. It is generated by the images $\overline{e_{\mathscr{P}}}$ of local uniformizers $e_{\mathscr{P}} = (1, \ldots, 1, \pi_{\mathscr{P}}, 1, \ldots)$. We assume that $\{\overline{e_{\mathscr{P}_1}},\ \overline{e_{\mathscr{P}_2}}, \ldots, \overline{e_{\mathscr{P}_n}}\}$ is a minimal set of generators of G.

Suppose $\gamma = \prod_{i=1}^n \overline{e_{\mathscr{P}_i}}^{\gamma_i}$ is an element of G, where $\gamma_i \in \{0,\ 1\}$. Every Eichler order $\mathscr{D}$ of level δ is isomorphic to some $\mathscr{E}^{\gamma}$, where $\mathscr{E}^{\gamma}$ is an Eichler order of level δ whose image in G is γ. We define the distance ideal $\rho(\mathscr{D},\ \mathscr{E})$ to be $\prod_{i=1}^n \mathscr{P}_i^{\gamma_i}$. Note that this definition depends on the choice of representatives of the minimal set of generators of G. However, throughout this paper we deal only with the ideal class of $\rho(\mathscr{D},\ \mathscr{E})$ in a quotient group of the fractional ideals group of K. This quotient group is denoted by $T_S(B)$ in [2] and isomorphic to G. The ideal class of $\rho(\mathscr{D},\ \mathscr{E})$ in $T_S(B)$ is independent of the choice of representatives.

Let B be a quaternion algebra over a number field K and $\mathscr{O}_K$ the ring of integers of K. By an integral $\mathscr{O}_K$-order we shall mean one without zero divisors. Let $\Omega \subset B$ be a commutative integral $\mathscr{O}_K$-order and $\Omega \neq \mathscr{O}_K$. Then Ω is not always contained in some Eichler order of level δ. The following lemma reveals that this problem can be reduced to the local case.

Lemma 2.1. *The order Ω can be embedded into some Eichler order of level δ if and only if for any prime ideal $\mathscr{P}$ dividing δ, $\Omega_{\mathscr{P}}$ can be embedded into some local Eichler order $\mathscr{E}_{\mathscr{P}}$ whose level is equal to the $\mathscr{P}$-part of δ.*

Proof. We need only to prove the "if " part.

Since $\Omega \subset B$, we can choose a maximal order $\mathscr{N}$ of B such that $\Omega \subset \mathscr{N}$. For any prime ideal $\mathscr{P}$ dividing δ, fix a local Eichler order $\mathscr{E}_{\mathscr{P}}$ such that the level of $\mathscr{E}_{\mathscr{P}}$ is equal to the $\mathscr{P}$-part of δ and $\Omega_{\mathscr{P}} \subset \mathscr{E}_{\mathscr{P}}$.

By Proposition 5.1 of [3, Chapter 3], we can choose a global Eichler order $\mathscr{F}$ such that

$$\mathscr{F}_{\mathscr{P}} = \begin{cases} \mathscr{E}_{\mathscr{P}} & \text{if } \mathscr{P} | \delta, \\ \mathscr{N}_{\mathscr{P}} & \text{otherwise.} \end{cases}$$

Then the global level of $\mathscr{F}$ is δ and $\Omega \subset \mathscr{F}$. □

In the local case, the optimal embedding problem has been completely solved in Theorem 1.8 of [1]. For the definition of optimal embedding, one can see the first paragraph of [1] for details. If $\Omega_{\mathscr{P}}$ is an integral domain and $L_{\mathscr{P}}$ is its quotient field, then the meaning of optimal embedding $\varphi : \Omega_{\mathscr{P}} \to \mathscr{E}_{\mathscr{P}}$ is that $\varphi(L_{\mathscr{P}}) \cap \mathscr{E}_{\mathscr{P}} = \varphi(\Omega_{\mathscr{P}})$. In this paper, we will identify $\Omega_{\mathscr{P}}$ and $L_{\mathscr{P}}$ with their images under the embedding. The following lemma is a simplified form of Theorem 1.8 of [1].

Lemma 2.2 (Brzezinski [1, Theorem 1.8]). *Let $L_{\mathscr{P}} \supset K_{\mathscr{P}}$ be a quadratic extension of $K_{\mathscr{P}}$ and $B_{\mathscr{P}} \supset L_{\mathscr{P}}$. Let $\Omega_{\mathscr{P}} = \mathcal{O}_{K_{\mathscr{P}}} + \pi^i \mathcal{O}_{L_{\mathscr{P}}}$ be an $\mathcal{O}_{K_{\mathscr{P}}}$-order of $L_{\mathscr{P}}$, where π is a uniformizer of $\mathcal{O}_{L_{\mathscr{P}}}$. Then there exists an optimal embedding $\Omega_{\mathscr{P}} \to \mathscr{E}_{\mathscr{P}}$ for some Eichler order $\mathscr{E}_{\mathscr{P}}$ of level δ only in the following cases*:

(1) *$L_{\mathscr{P}} \supset K_{\mathscr{P}}$ is inert and* $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant 2i$.
(2) *$L_{\mathscr{P}} \supset K_{\mathscr{P}}$ is ramified and* $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant 2i + 1$.
(3) *$L_{\mathscr{P}} \supset K_{\mathscr{P}}$ is split.*

The next lemma asserts that the existence of an optimal embedding is equivalent to the existence of an embedding.

Lemma 2.3. *Let $L_{\mathscr{P}} \supset K_{\mathscr{P}}$ be a quadratic extension of $K_{\mathscr{P}}$ and $B_{\mathscr{P}} \supset L_{\mathscr{P}}$. Let $\Omega_{\mathscr{P}}$ be an $\mathcal{O}_{K_{\mathscr{P}}}$-order of $L_{\mathscr{P}}$. Then there exists an optimal embedding $\Omega_{\mathscr{P}} \to \mathscr{E}_{\mathscr{P}}$ for some Eichler order $\mathscr{E}_{\mathscr{P}}$ if and only if there exists an embedding $\Omega_{\mathscr{P}} \to \mathscr{E}_{\mathscr{P}}$.*

Proof. Obviously we need only to prove the "if" part. By Theorem 1.8(c) of [1], if $L_{\mathscr{P}} \supset K_{\mathscr{P}}$ is split, then there is always an optimal embedding.

If there is an embedding $\Omega_{\mathscr{P}} \to \mathscr{E}_{\mathscr{P}}$, let $\Omega_{\mathscr{P}}' = L_{\mathscr{P}} \cap \mathscr{E}_{\mathscr{P}}$. Then $\Omega_{\mathscr{P}}'$ is an $\mathcal{O}_{K_{\mathscr{P}}}$-order and $\Omega_{\mathscr{P}}' \to \mathscr{E}_{\mathscr{P}}$ is an optimal embedding. Obviously $\Omega_{\mathscr{P}}' \supset \Omega_{\mathscr{P}}$. If $L_{\mathscr{P}} \supset K_{\mathscr{P}}$ is unramified, $\Omega_{\mathscr{P}} = \mathcal{O}_{K_{\mathscr{P}}} + \pi^i \mathcal{O}_{L_{\mathscr{P}}}$ and $\Omega_{\mathscr{P}}' = \mathcal{O}_{K_{\mathscr{P}}} + \pi^{i'} \mathcal{O}_{L_{\mathscr{P}}}$, then $i' \leqslant i$. By Theorem 1.8(a) of [1], $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant 2i'$. Consequently $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant 2i$, so there is an optimal embedding also by Theorem 1.8(a) of [1].

If $L_{\mathscr{P}} \supset K_{\mathscr{P}}$ is ramified, then the proof is similar. □

Theorem 2.4. *Let K be a number field with $\mathcal{O}_K$ the ring of integers of K. Let B be a quaternion algebra over K. And let $\Omega \subset B$ with $\Omega \neq \mathcal{O}_K$ be a commutative $\mathcal{O}_K$-order. Then Ω can be embedded into some Eichler order of level δ if and only if* $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K))$ *for any inert prime ideal $\mathscr{P}$ and* $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K)) - \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\mathcal{O}_{L_{\mathscr{P}}}/\mathcal{O}_{K_{\mathscr{P}}})) + 1$ *for any ramified prime ideal $\mathscr{P}$.*

Proof. As in the proof of Lemma 2.3, we require nothing for split $\mathscr{P}$. The local order $\Omega_{\mathscr{P}} = \mathcal{O}_{K_{\mathscr{P}}} + \pi^i \mathcal{O}_{L_{\mathscr{P}}}$ for some $i \geqslant 0$ and some uniformizer π of $\mathcal{O}_{K_{\mathscr{P}}}$. Since $\mathcal{O}_{L_{\mathscr{P}}} = \mathcal{O}_{K_{\mathscr{P}}}[\omega]$ for some $\omega \in \mathcal{O}_{L_{\mathscr{P}}}$, we have

$$\mathrm{disc}(\Omega_{\mathscr{P}}/\mathcal{O}_{K_{\mathscr{P}}}) = \pi^{2i}\,\mathrm{disc}(\mathcal{O}_{L_{\mathscr{P}}}/\mathcal{O}_{K_{\mathscr{P}}}).$$

If $\mathscr{P}$ is inert, then $\mathrm{disc}(\mathcal{O}_{L_{\mathscr{P}}}/\mathcal{O}_{K_{\mathscr{P}}})$ is trivial. By Lemmas 2.2 and 2.3, there exists an embedding $\Omega_{\mathscr{P}} \to \mathscr{E}_{\mathscr{P}}$ for some Eichler order $\mathscr{E}_{\mathscr{P}}$ if and only if

$$\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant 2i = \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K)).$$

If $\mathscr{P}$ is ramified, the proof is similar. □

Theorem 2.5. *Let K be a number field with $\mathcal{O}_K$ the ring of integers of K. Let B be a quaternion algebra over K. Let $\Omega \subset B$ be a commutative $\mathcal{O}_K$-order and assume B satisfies the Eichler condition (i.e., there is at least one archimedean place which is unramified in B). Assume $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K))$ for any inert prime ideal $\mathscr{P}$ of $\mathcal{O}_K$ and $\mathrm{ord}_{\mathscr{P}}(\delta) \leqslant \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K)) - \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\mathcal{O}_{L_{\mathscr{P}}}/\mathcal{O}_{K_{\mathscr{P}}})) + 1$ for any ramified prime ideal $\mathscr{P}$. Then every Eichler order of B contains a conjugate (by B^*) of Ω except when the following three conditions hold:*

(1) *The order Ω is an integral domain and its quotient field $L \subset B$ is a quadratic extension of K.*
(2) *The extension L/K and the algebra B are unramified at all finite places and ramify at exactly the same (possibly empty) set of real places of K.*
(3) *All prime ideals $\mathscr{P}$ of K which divide the relative discriminant ideal $d_{\Omega/\mathcal{O}_K}$ and satisfy $\mathrm{ord}_{\mathscr{P}}(\delta) \neq \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K))$ split in L/K.*

Suppose now that (1)–(3) hold. Then B has an even number of conjugacy classes of Eichler orders of level δ. The Eichler orders $\mathscr{D}$ containing some conjugate of Ω make up exactly half of these conjugacy classes. If $\mathscr{D}$ and $\mathscr{E}$ are Eichler orders and $\mathscr{E}$ contains a conjugate of Ω, then $\mathscr{D}$ contains a conjugate of Ω if and only if the image by the reciprocity map $\mathrm{Frob}_{L/K}$ of the distance ideal $\rho(\mathscr{D}, \mathscr{E})$ is the trivial element of $\mathrm{Gal}(L/K)$.

Proof. If Ω is contained in K, then (1) fails and every Eichler order contains Ω. The theorem holds obviously. So we will assume that Ω is not contained in K in the following proof.

Let $G = K^* \backslash K_{\mathbb{A}}^* / n(N(\mathscr{E}_{\mathbb{A}}))$. Assume $|G| = 2^n$ and G is generated by the images of local uniformizers $e_{\mathscr{P}_i} = (1, \ldots, 1, \pi_{\mathscr{P}}, 1, \ldots)$, where $i = 1, 2, \ldots, n$. Then there are 2^n types of Eichler orders of level δ. If we can find 2^n types of Eichler orders which contain a conjugate of Ω, then we see that Ω can be embedded into each type of Eichler orders of level δ. In the local case, two Eichler orders of the same level are conjugate by some $x \in B_{\mathscr{P}}^*$.

For each $\mathscr{P}_i$, suppose we can find two local Eichler orders $\mathscr{E}_{\mathscr{P}_i}$ and ${\mathscr{E}_{\mathscr{P}_i}}' = x_i \mathscr{E}_{\mathscr{P}_i} x_i^{-1}$ such that $n(x_i) \notin \mathcal{O}_{\mathscr{P}}^* K_{\mathscr{P}}^{*2}$ and all contain Ω' which is conjugate to Ω. Let $\mathscr{D}$ be a fixed Eichler order of level δ which contains Ω'. For each $\gamma = \prod_{i=1}^{n} \overline{e_{\mathscr{P}_i}}^{\gamma_i} \in G$, let $\mathscr{E}^{\gamma}$ be the global Eichler order of level δ such that its localization at $\mathscr{P}$ is

$$\mathscr{E}^{\gamma}_{\mathscr{P}} = \begin{cases} \mathscr{E}_{\mathscr{P}_i} & \text{if } \mathscr{P} = \mathscr{P}_i \text{ and } \gamma_i = 0, \\ {\mathscr{E}_{\mathscr{P}_i}}' & \text{if } \mathscr{P} = \mathscr{P}_i \text{ and } \gamma_i = 1, \\ \mathscr{D}_{\mathscr{P}} & \text{otherwise.} \end{cases}$$

Then these 2^n global Eichler orders all contain Ω' which is conjugate to Ω. So Ω can be embedded into all types of Eichler orders.

Next in the following four steps, we will choose two appropriate local Eichler orders containing a conjugate of Ω for each $\mathscr{P}_i$.

Step 1: If (1) fails, then $B = M(2, K)$. Since Ω is not an integral domain, Ω must be conjugate to a subring of either

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a,\ b \in \mathcal{O}_K \right\} \tag{2.4}$$

or

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a \in \mathcal{O}_K,\ b \in I \right\}, \tag{2.5}$$

where I is a fractional ideal of K. For each $\mathscr{P}_i$ $(i = 1, 2, \ldots, n)$, denote by π_i a uniformizer of $K_{\mathscr{P}_i}$. For case (2.4) let

$$\mathscr{E}_{\mathscr{P}_i} = \begin{pmatrix} \mathcal{O}_{\mathscr{P}_i} & \mathcal{O}_{\mathscr{P}_i} \\ \pi_i^{\mathrm{ord}_{\mathscr{P}_i}\delta} \mathcal{O}_{\mathscr{P}_i} & \mathcal{O}_{\mathscr{P}_i} \end{pmatrix}, \qquad \widetilde{\mathscr{E}_{\mathscr{P}_i}} = \begin{pmatrix} \mathcal{O}_{\mathscr{P}_i} & \pi_i^{-1} \mathcal{O}_{\mathscr{P}_i} \\ \pi_i^{\mathrm{ord}_{\mathscr{P}_i}\delta + 1} \mathcal{O}_{\mathscr{P}_i} & \mathcal{O}_{\mathscr{P}_i} \end{pmatrix}$$

and for case (2.5) let

$$\mathscr{E}_{\mathscr{P}_i} = \begin{pmatrix} \mathcal{O}_{\mathscr{P}_i} & \pi_i^{x} \mathcal{O}_{\mathscr{P}_i} \\ \pi_i^{\mathrm{ord}_{\mathscr{P}_i}\delta - x} \mathcal{O}_{\mathscr{P}_i} & \mathcal{O}_{\mathscr{P}_i} \end{pmatrix}, \qquad \widetilde{\mathscr{E}_{\mathscr{P}_i}} = \begin{pmatrix} \mathcal{O}_{\mathscr{P}_i} & \pi_i^{x-1} \mathcal{O}_{\mathscr{P}_i} \\ \pi_i^{\mathrm{ord}_{\mathscr{P}_i}\delta - x + 1} \mathcal{O}_{\mathscr{P}_i} & \mathcal{O}_{\mathscr{P}_i} \end{pmatrix},$$

where x is chosen such that $I \subset \pi_i^x \mathcal{O}_{\mathscr{P}_i}$. Since in both cases

$$\widetilde{\mathscr{E}_{\mathscr{P}_i}} = \begin{pmatrix} 1 & 0 \\ 0 & \pi_i \end{pmatrix} \mathscr{E}_{\mathscr{P}_i} \begin{pmatrix} 1 & 0 \\ 0 & \pi_i^{-1} \end{pmatrix},$$

the images of $\mathscr{E}_{\mathscr{P}_i}$ and $\widetilde{\mathscr{E}_{\mathscr{P}_i}}$ are different from each other in G.

Step 2: Assume that (1) holds but (2) fails. Let $K(B)/K$ be the extension corresponding to G. By classfield theory, the quotient field L of Ω is not contained in $K(B)$ (see [3, p. 39] for details). By the Chebotarev density theorem, we can choose primes $\mathscr{P}_i$ which split in L/K and the images of their uniformizers generate G. Let $\mathscr{P}$

be one of the $\mathscr{P}_i$. Since $\mathscr{P}$ splits in L/K, there is a $K_{\mathscr{P}}$-isomorphism $f_{\mathscr{P}}: B_{\mathscr{P}} \to M(2, K_{\mathscr{P}})$ such that

$$f_{\mathscr{P}}(L) \subset \begin{pmatrix} K_{\mathscr{P}} & 0 \\ 0 & K_{\mathscr{P}} \end{pmatrix}.$$

Thus

$$f_{\mathscr{P}}(\Omega) \subset f_{\mathscr{P}}(\mathcal{O}_{\mathscr{L}}) \subset \begin{pmatrix} \mathcal{O}_{K_{\mathscr{P}}} & 0 \\ 0 & \mathcal{O}_{K_{\mathscr{P}}} \end{pmatrix}.$$

Let $\mathscr{E}_{\mathscr{P}_i}$ and $\widetilde{\mathscr{E}_{\mathscr{P}_i}}$ be two Eichler orders such that

$$f_{\mathscr{P}}(\mathscr{E}_{\mathscr{P}_i}) = \begin{pmatrix} \mathcal{O}_{k_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \\ \pi^{\mathrm{ord}_{\mathscr{P}}\delta}\mathcal{O}_{K_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \end{pmatrix}, \qquad f_{\mathscr{P}}(\widetilde{\mathscr{E}_{\mathscr{P}_i}}) = \begin{pmatrix} \mathcal{O}_{K_{\mathscr{P}}} & \pi^{-1}\mathcal{O}_{K_{\mathscr{P}}} \\ \pi^{\mathrm{ord}_{\mathscr{P}}\delta+1}\mathcal{O}_{K_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \end{pmatrix}.$$

Then $\Omega \subset \mathscr{E}_{\mathscr{P}_i} \cap \widetilde{\mathscr{E}_{\mathscr{P}_i}}$.

Step 3: Now assume both (1) and (2) hold, but (3) fails. Let $\mathscr{Q}$ be a prime of K which is inert in L and $\mathrm{ord}_{\mathscr{Q}}(\delta) < \mathrm{ord}_{\mathscr{Q}}(\mathrm{disc}(\Omega/\mathcal{O}_K))$. Since $L \subset K(B)$, we can pick $\mathscr{P}_1, \dots, \mathscr{P}_n$ such that the images of their uniformizers $\overline{\pi_1}, \overline{\pi_2}, \dots, \overline{\pi_n}$ generate G, $\mathscr{P}_i$ splits in L/K for $i > 1$ and $\mathscr{P}_1 = \mathscr{Q}$ is inert in L/K. If $\mathscr{P} = \mathscr{P}_i$ splits in L, we can pick two local Eichler orders containing Ω whose images in G are different from each other as above.

For $\mathscr{P} = \mathscr{Q}$, $L_{\mathscr{P}} \supset K_{\mathscr{P}}$ is unramified. Since $\mathrm{ord}_{\mathscr{P}}(\delta) < \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K))$, we have $\mathrm{ord}_{\mathscr{P}}(\delta) + 1 \leqslant \mathrm{ord}_{\mathscr{P}}(\mathrm{disc}(\Omega/\mathcal{O}_K))$. By Theorem 2.4, there is an embedding

$$\Omega \to \begin{pmatrix} \mathcal{O}_{K_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \\ \pi^{\mathrm{ord}_{\mathscr{P}}\delta+1}\mathcal{O}_{K_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \end{pmatrix}.$$

So Ω can be embedded in

$$\begin{pmatrix} \mathcal{O}_{K_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \\ \pi^{\mathrm{ord}_{\mathscr{P}}\delta}\mathcal{O}_{K_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \end{pmatrix} \bigcap \begin{pmatrix} \mathcal{O}_{K_{\mathscr{P}}} & \pi^{-1}\mathcal{O}_{K_{\mathscr{P}}} \\ \pi^{\mathrm{ord}_{\mathscr{P}}\delta+1}\mathcal{O}_{K_{\mathscr{P}}} & \mathcal{O}_{K_{\mathscr{P}}} \end{pmatrix}.$$

Hence Ω is contained in two Eichler orders whose images in G are different from each other.

Step 4: Now suppose that (1)–(3) hold.

By Theorem 2.4 and the assumptions of Theorem 2.4, Ω is contained in some Eichler order $\mathscr{E}$ of level δ.

Let $\mathscr{D}$ be another Eichler order of level δ. We will prove that there is an Eichler order which is isomorphic to $\mathscr{D}$ containing a conjugate of Ω if and only if $\mathrm{Frob}_{L/K}(\rho(\mathscr{D}, \mathscr{E})) = 1$ in $\mathrm{Gal}(L/K) = \{\pm 1\}$. Choose local uniformizers $\pi_1, \pi_2, \dots, \pi_n$ of F such that their images $\overline{\pi_1}, \overline{\pi_2}, \dots, \overline{\pi_n}$ generate G, $\mathscr{P}_i$ split in L/K for $i > 1$, $\mathscr{P}_1$ is inert in L and $\mathrm{ord}_{\mathscr{P}_1}(\delta) = \mathrm{ord}_{\mathscr{P}_1}(\mathrm{disc}(\Omega/\mathcal{O}_K))$.

Assume that $\rho(\mathscr{D}, \mathscr{E}) = \prod_{i=1}^{n} \mathscr{P}_i^{\gamma_i}$. We know that $\mathrm{Frob}_{L/K}(\rho(\mathscr{D}, \mathscr{E})) = 1$ if and only if $\gamma_1 = 0$. Next, we will prove that Ω can be embedded into an Eichler order which is isomorphic to $\mathscr{D}$ if and only if $\gamma_1 = 0$. If $\gamma_1 = 0$, then $D = \mathscr{E}^{\gamma}$, where $\gamma = \prod_{i=2}^{n} \overline{e_{\mathscr{P}_i}}^{\gamma_i}$. Since $\mathscr{P}_2, \ldots, \mathscr{P}_n$ split in L/K, by arguments as above, we know that Ω can be embedded into an Eichler order which is isomorphic to $\mathscr{D}$. Conversely, if Ω can be embedded into an Eichler order which is isomorphic to $\mathscr{D}$ but $\gamma_1 \neq 0$, then $\Omega_{\mathscr{P}_1} \subset \mathscr{E}_{\mathscr{P}_1} \cap a\mathscr{E}_{\mathscr{P}_1} a^{-1}$, where the norm of a is a uniformizer at $\mathscr{P}_1$. By Lemma 1.3 of [1], there exist suitable elements $l \in L^*_{\mathscr{P}_1}$, b, $b' \in B^*_{\mathscr{P}_1}$ and a maximal order M such that $\mathscr{E}_{\mathscr{P}_1} = M \cap bMb^{-1}$ and $la\mathscr{E}_{\mathscr{P}_1} a^{-1} l^{-1} = M \cap b'Mb'^{-1}$. Since an Eichler order is the intersection of two unique maximal orders, we have $laMa^{-1}l^{-1} = M$ or $laMa^{-1}l^{-1} = b'Mb'^{-1}$. So $la \in N(M)$ or $b'^{-1}la \in N(M)$.

If $la \in N(M)$, then $\mathrm{ord}_{\mathscr{P}_1}(n(la))$ is even by (2.1). Since $L_{\mathscr{P}_1} \supset K_{\mathscr{P}_1}$ is inert, $\mathrm{ord}_{\mathscr{P}_1}(n(l))$ is even. So $\mathrm{ord}_{\mathscr{P}_1}(n(a))$ is even. This is a contradiction since the norm of a is a uniformizer at $\mathscr{P}_1$ by the last paragraph. If $b'^{-1}la \in N(M)$, then by the same arguments we have $\mathrm{ord}_{\mathscr{P}_1}(n(b'))$ is odd which implies the level of $\mathscr{E}_{\mathscr{P}_1}$ is odd. By (2.1), $n(N(\mathscr{E}_{\mathscr{P}_1})) = K^*_{\mathscr{P}_1}$. Hence $\overline{\pi_1}$ is trivial in $G = K^* \backslash K^*_{\mathbb{A}} / n(N(\mathscr{E}_{\mathbb{A}}))$. This is impossible.

So the theorem is proved. □

Acknowledgments

X. Guo is deeply grateful to thank Professor Fei Xu for valuable discussion and the Morningside Center of Mathematics for hospitality.

References

[1] J. Brzezinski, On embedding numbers into quaternion orders, Comment. Math. Helv. 66 (1991) 302–318.

[2] T. Chinburg, E. Friedman, An embedding theorem for quaternion algebras, J. London. Math. Soc. 60 (2) (1999) 33–44.

[3] M.F. Vignéras, Arithmétique des Algèbres de Quaternions, in: Lecture Notes in Mathematics, Vol. 800, Springer, Berlin, 1980.