A remark on computing the tame kernel of quadratic imaginary fields

GUO XUEJUN   SONG GUANGTIAN

Department of Mathematics
University of Science and Technology of China
Hefei, Anhui 230026, P.R.China

Abstract: In this paper, we discuss a method to compute the tame kernel of number field. Confining to imaginary quadratic field, we prove that $\partial_v : K_2^{S'}F/K_2^S F \longrightarrow k^*$ is bijective when $Nv > 8\delta_D^6$.
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1. Introduction and Notations

Let F be a number field, $\mathcal{O}_F$ be its integers ring and S_∞ denote the set of archimedean places of F. If S is a non empty set of places containing S_∞. we put $\mathcal{O}_S = \{a \in F \mid v(a) \geq 0,\ for\ all\ v \notin S\}$ be the ring of S-integers. Assume that $\mathcal{P}$ is the maximal ideal corresponding to $v \in S$, let $k(v) = \mathcal{O}_S/P$, then the norm $N(v) = |k(v)|$.

We shall put $K_2^S F$ the subgroup of K_2F generated by $\{x,\ y\}$, where $x,\ y \in \mathcal{O}_S^* = U$. We can list the the finite places of F, $v_1,\ v_2,\ ...,\ v_n,\ ...$ so that $N(v_i) \leq N(v_{i+1})$ for all i. Put $S_m = S_\infty \cup \{v_1,\ ...,\ v_m\}$. Let $S = S_m$, $v = v_{m+1} \notin S$, $S' = S_{m+1} = S \cup \{v\}$, $U = \mathcal{O}_S^*$, $k = k(v) = \mathcal{O}_S/P$ and k^* denotes the multiplicative group of k. Let ∂_v be the tame map corresponding to v, in [5] Bass and Tate show that for sufficiently large m, the induced homomorphism

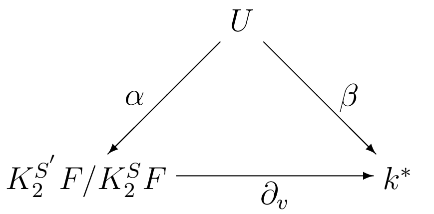
$$\partial_v : \quad K_2^{S'} F/K_2^S F \longrightarrow k^*$$

is an isomorphism. This result implies that

$$K_2\mathcal{O}_F = Ker(\partial_v : \quad K_2^{S_m} F \longrightarrow \coprod_{v \in S_m \setminus S_\infty} k^*(v)).$$

If we can make m relatively small and get sufficiently many relations among $K_2^{S_m} F$, then we can determine the tame kernel of F. Theorem 2.6 of part 2 will give a relatively small m, so the computation of tame kernel will be simplified. Our estimation on the lower bound of m (or equivalently, Nv), is smaller than that given in [1], [2], [3], [4].

let $\mathcal{P}$ be the maximal ideal generated by π, we denote by U_1 the group generated by $(1 + \pi U) \cap U$. Moreover, by [5], there is a commutative diagram



where α maps $u \in U$ to $\{u, \pi\}(mod\ K_2^S F)$ and β is the natural quotient map.

2. Main Results

The following Lemma is based on Lemma 3.2 and Lemma 3.4 of [5].

Lemma 2.1. $\partial_v : K_2^{S'} F/K_2^S F \longrightarrow k^*$ *is bijective if there exist subsets* $W,\ E \subset \mathcal{O}_F \cap U$ *satisfying:*

(a) *the prime ideal of* $\mathcal{O}_S$ *corresponding to* v *is principal,*

(b) W *generates* U *and* $1 \in W$.

(c) *the map* $E \times E \times E \longrightarrow k^* \times k^*$ *sending* $(a,\ b,\ c)$ *to* $(\bar{b}/\bar{a}, \bar{c}/\bar{a})$ *is surjective*

(d) *for any* $e_1,\ e_2,\ e_3,\ e_4 \in E$, $N(e_1e_2 - e_3e_4) < (Nv)^2$

(e) *for any* $w \in W$ *, there exist* $e_1,\ e_2 \in E$ *and* $u_1 \in U_1$ *such that* $e_1w = e_2u_1$.

Proof. Sketch of Proof In the above commutative diagram of part1, to prove ∂_v is bijective, suffices to show that: α and β are surjective and $Ker(\beta) \subset Ker(\alpha)$.

From (a) and (c), the surjectivity of α and β can be proved easily. So we need only to prove $Ker(\beta) \subset Ker(\alpha)$. Since $U_1 \subset Ker(\alpha)$, it suffices to prove $Ker(\beta) \subset U_1$. For any $x \in \ker\beta \subset U$, by (e), we can find w_i, $e_i/e_i^{'}$, u_i, $1 \le i \le t$ such that $\overline{w}_i\overline{e_i^{'}} = \overline{e}_i\overline{u}_i$ and $x = w_1 \cdots w_t$, where $w_i \in W$ and e_i, $e_i^{'} \in E$, $1 \le i \le t$. So

$$x = \frac{e_1 \cdots e_t}{e_1^{'} \cdots e_t^{'}} \cdot \frac{e_1^{'}w_1}{e_1} \cdot \cdots \cdot \frac{e_t^{'}w_t}{e_t}.$$

By condition (e), we see that $e_i^{'}w_i/e_i = u_1 \in U_1$. So it suffices to show that $e_1 \cdots e_t/e_1^{'} \cdots e_t^{'} \in U_1$. Using the condition (c), Tate's proof of Lemma 3.4 of [5] on page 410 can be applied. □

So to make ∂_v bijective, one needs to choose suitable S, W and E such that the conditions in Lemma 2.1 is satisfied. Before doing this, an analogue of the GTT Theorem of [4] is needed(Theorem 2.3).

Let F be a number field with discriminant D and $(F : \mathbb{Q}) = n = s + 2t$. Let σ_1, ..., σ_s, σ_{s+1},$\overline{\sigma}_{s+1}$, ..., σ_{s+t}, $\overline{\sigma}_{s+t}$ be embedding of the field F into $\mathbb{C}$, where σ_1, ..., σ_s are real embedding and σ_{s+1},$\overline{\sigma}_{s+1}$, ..., σ_{s+t}, $\overline{\sigma}_{s+t}$ are complex embedding. For any element $x \in \mathcal{O}_F$, let

$$M(x) := max\{|\sigma_1|,\ ...,\ |\sigma_s|,\ |\sigma_{s+1}|,\ ...,\ |\sigma_{s+t}|\}.$$

Let $V^{s,t} := \mathbb{R}^s \times \mathbb{C}^t$. We have a map

$$\sigma:\ F \longrightarrow V^{s,t}, \sigma(\alpha) = (\sigma_1(\alpha),\ ...,\ \sigma_s(\alpha),\ \sigma_{s+1}(\alpha),\ ...,\ \sigma_{s+t}(\alpha)).$$

For any integral ideal I of $\mathcal{O}_F$, $\sigma(I)$ is a lattice in $V^{s,t}$. For any prime ideal $\mathcal{P}$ of $\mathcal{O}_F$ and $a,\ b \in \mathcal{O}_F$, we can define $f_{a,b}:\ V^{s,t} \times V^{s,t} \times V^{s,t} \longrightarrow V^{s,t} \times V^{s,t}$ by $f_{a,b}(x,\ y,\ z) = (y - x\sigma(a),\ z - x\sigma(b))$. Let $L_{a,b}^{\mathcal{P}} := f_{a,b}^{-1}(\sigma(\mathcal{P}) \times \sigma(\mathcal{P})) \cap (\sigma(\mathcal{O}_F) \times \sigma(\mathcal{O}_F) \times \sigma(\mathcal{O}_F))$.

Lemma 2.2. *vol* $L_{a,b}^{\mathcal{P}} = 8^{-t} \cdot |D|^{3/2} \cdot N(\mathcal{P})^2$.

Proof. Considering the map $f:\ V^{s,t}\times V^{s,t}\times V^{s,t} \longrightarrow V^{s,t}\times V^{s,t}\times V^{s,t}$, $f(x,\ y,\ z) := (x,\ y - x\sigma(a),\ z - x\sigma(b))$, we have $f(L_{a,b}^{\mathcal{P}}) = L_{0,0}^{\mathcal{P}}$. f is a linear transformation and can be seen as a matrix

$$M := \begin{pmatrix} I & * & * \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

where I is an identity matrix. Since $det(M) = 1$, vol $L_{a,b}^{\mathcal{P}}$ =vol $L_{0,0}^{\mathcal{P}}$ = vol $(\sigma(\mathcal{P})\sigma(\mathcal{P})\sigma(\mathcal{O}_F))$ = vol $(\sigma(\mathcal{P}))^2$ ·vol $\sigma(\mathcal{O}_F) = 2^{-3t} \cdot |D|^{3/2} \cdot N(\mathcal{P})^2$. □

Theorem 2.3. *If $\mathcal{P}$ is a prime ideal of the ring $\mathcal{O}_F$ with $N(\mathcal{P}) > (2/\pi)^{3t}|D|^{3/2}$ and $h_1,\ h_2,\ h_3 > 0$ satisfying conditions*

(1) $h_1h_2h_3 = (2/\pi)^{3t/n}|D|^{3/2n}N(\mathcal{P})^{2/n}$,

(2) $h_i^n < N(\mathcal{P})$,

then for any $\overline{a},\ \overline{b} \in (\mathcal{O}_F/\mathcal{P})^*$ there exist $e_1,\ e_2,\ e_3 \in \mathcal{O}_F - \mathcal{P}$ such that

(a) $M(e_i) \leq h_i$, $i = 1,\ 2,\ 3$,

(b) $\overline{a} = \overline{e}_2/\overline{e}_1$, $\overline{b} = \overline{e}_3/\overline{e}_1$, where "bar" means the image in $(\mathcal{O}_F/\mathcal{P})^*$.

Proof. For arbitrary $h = (h_1, h_2, h_3) > 0$, where $h_i > 0,\ i = 1,\ 2,\ 3$. Let $S_h := \{(x, y, z) \in V^{s,t} \times V^{s,t} \times V^{s,t} |\ |x_i| \leq h_1,\ |y_i| \leq h_2,\ |z_i| \leq h_3, i = 1, ..., s+t\}$. We know that

$$vol\ S_h = (2^s\pi^t)^3 h_1^n h_2^n h_3^n = 2^{3(n-t)}|D|^{3/2}N(\mathcal{P})^2 = 2^{3n} \cdot\ vol\ L_{a,b}^{\mathcal{P}}.$$

By Minkowski theorem, there exist a nonzero vector in $L_{a,b}^{\mathcal{P}} \cap S_h$, i.e., for any $a,\ b \in \mathcal{O}_F - \mathcal{P}$, we can find $x_{a,b}^{(i)} \in \mathcal{O}_F$ ($i = 1,\ 2,\ 3$) such that $(\sigma(x_{a,b}^{(1)}), \sigma(x_{a,b}^{(2)}), \sigma(x_{a,b}^{(3)})) \in L_{a,b}^{\mathcal{P}} \cap S_h$. Since $N(x_{a,b}^{(i)}) \leq h_i^n < N(\mathcal{P})$, we know $x_{a,b}^{(i)} \in \mathcal{O}_F - \mathcal{P}$. Let $e_i = x_{a,b}^{(i)}$, where $i = 1,\ 2,\ 3$, we have $e_2 \equiv e_1 a\ (mod\ \mathcal{P})$, $e_3 \equiv e_1 b\ (mod\ \mathcal{P})$. Since e_i is not contained in $\mathcal{P}$, $i = 1,\ 2,\ 3$, we have $\overline{a} = \overline{e}_2/\overline{e}_1$, $\overline{b} = \overline{e}_3/\overline{e}_1$. □

Using Lemma 2.1 and Theorem 2.3, we can do some computation on the tame kernel of an imaginary quadratic field. In the following context, F denotes an imaginary quadratic field with discriminant D. Let $\delta_D = (2/\pi)^{1/2}|D|^{1/4}$ and $E = \{x :\ \ x \in \mathcal{O}_F, 0 < |x| \leq \delta_D N(\mathcal{P})^{1/3}\}$. Confining Theorem 2.3 to an imaginary quadratic field F, we have the following lemma.

Lemma 2.4. *Let F be an imaginary quadratic field with discriminant D. If $\mathcal{P}$ is a prime ideal of the ring $\mathcal{O}_F$ with $N(\mathcal{P}) > \delta_D^6$, then for any $\overline{a}$, $\overline{b} \in (\mathcal{O}_F/\mathcal{P})^*$ there exist $e_1, e_2, e_3 \in \mathcal{O}_F - \mathcal{P}$ such that*

(1) $|e_i| \leq \delta N(\mathcal{P})^{1/3}$, $i = 1,\ 2,\ 3$.

(2) $\overline{a} = \overline{e}_2/\overline{e}_1$, $\overline{b} = \overline{e}_3/\overline{e}_1$, *where "bar" means the image in* $(\mathcal{O}_F/\mathcal{P})^*$.

Remark Using Lemma 2.4, it can be easily seen that the condition (c) of Lemma 2.1 is satisfied when $N(\mathcal{P}) > \delta_D^6$.

Next we need a set $W \subset \mathcal{O}_F \cap U$ such that W generates U and satisfy condition (e) of Lemma 2.1. Let q_F be the least number such that in every class of ideals of $\mathcal{O}_F$ there is an ideal of norm $\leq q_F$. In [1], Browkin proved the following Lemma.

Lemma 2.5. *(Browkin) There is a set $W \subset U$ such that for any $w \in W$, its norm $Nw \leq q_F Nv$, if*
(i) $h_F \leq 2$, *and* $q_F \leq N(\mathcal{P})$, *(ii)* $h_F > 2$ *and* $q_F^2 \leq N(\mathcal{P})$.

One can find the detailed message about W in part 3.2 of [1].

Theorem 2.6. *If $Nv > 8\delta_D^6$, $\partial_v : K_2^{S'} F/K_2^S F \longrightarrow k^*$ is bijective.*

Proof. Since F is an imaginary quadratic field, we have

$$q_F \leq \sqrt{|D|/3} < \frac{2\sqrt{3}}{\pi} q_F < \frac{2}{\pi}\sqrt{|d|} < \delta_D^2.$$

So the condition of Lemma 2.5 is satisfied. We can choose a generating set W of U such that $Nw \leq q_F Nv$. By the remark below Lemma 2.4, the condition (c) of Lemma 2.1 is satisfied. So we need to show that the set W and E defined as above satisfy the conditions (d) and (e) of Lemma 2.1.

For any $e_1,\ e_2,\ e_3,\ e_4 \in E$, $N(e_1e_2 - e_3e_4) = |e_1e_2 - e_3e_4| \leq (|e_1e_2| + |e_3e_4|)^2 \leq 4\delta_D^4 N(\mathcal{P})^{4/3} \leq N(\mathcal{P})^2$. So the condition (d) of Lemma 2.1 is satisfied. We define $E_1 = \{x :\ x \in \mathcal{O}_F, 0 < |x| \leq \delta_D N(\mathcal{P})^{1/4}\} \subset E$. By the virtue of the GTT Theorem of [4], for any $w \in W$, there exist $e_1,\ e_2 \in E_1$ such that $e_1 w - e_2 \in \mathcal{P}$. We need only to show that $\frac{e_1 w}{e_2} \in U_1$. Let $h = N(\mathcal{P})^{\frac{1}{2}}/\sqrt{2}$, $h' = \sqrt{2}\delta_D^2$. By Theorem 2 of [1], there exist $a,\ b \in \mathcal{O}_F$ such that $|a| \leq h,\ |b| \leq h'$, $\overline{a}/\overline{b} = \overline{e}_2/\overline{e}_1$. So $\frac{e_1 w}{e_2} = \frac{e_1 a}{e_2 b} \cdot \frac{wb}{a}$, where $\frac{e_1 a}{e_2 b} \in ker\beta$ and $\frac{wb}{a} \in ker\beta$.

We claim that $\frac{e_1 a}{e_2 b} \in U_1$ and $\frac{wb}{a} \in U_1$.

By Lemma 1 of [6], to prove $\frac{e_1 a}{e_2 b} \in U_1$, it is sufficient to prove that $|e_1 a| + |e_2 b| < N(\mathcal{P})$. Since

$$|e_1 a| + |e_2 b| < \delta_D N(\mathcal{P})^{\frac{1}{4}} (\frac{N(\mathcal{P})^{\frac{1}{2}}}{\sqrt{2}} + \sqrt{2}\delta_D^2) < \frac{N(\mathcal{P})^{\frac{5}{12}}}{\sqrt{2}} \cdot (\frac{N(\mathcal{P})^{\frac{1}{2}}}{\sqrt{2}} + \frac{N(\mathcal{P})^{\frac{1}{3}}}{\sqrt{2}})$$
$$= \frac{1}{2}(N(\mathcal{P})^{\frac{11}{12}} + N(\mathcal{P})^{\frac{3}{4}}) < N(\mathcal{P}),$$

we have $\frac{e_1 a}{e_2 b} \in U_1$.

To prove $\frac{wb}{a} \in U_1$, by Lemma 7 of [1], it suffices to show that $\sqrt{N(\mathcal{P})} > max(2\delta_D^2, \sqrt{2q_F}\delta_D^2 + \frac{1}{\sqrt{2}})$. Since $\delta_D^2 > q_F$, we can replace this condition as $\sqrt{N(\mathcal{P})} > max(2\delta_D^2, \sqrt{2}\delta_D^3 + \frac{1}{\sqrt{2}})$. Obviously, $\sqrt{N(\mathcal{P})} > \sqrt{8\delta_D^6} > 2\delta_D^2$. So we need only to show that $2\sqrt{2}\delta_D^3 > \sqrt{2}\delta_D^3 + 1/\sqrt{2}$, i.e., $\sqrt{2}\delta_D^3 > 1/\sqrt{2}$. This inequation holds in all cases. □

In the table below, we give the estimation of Nv for some cases. The first row is the discriminant of the quadratic imaginary field, the second row is the value of $8\delta_D^6$,

−19	−20	−24	−35	−43	−83	−131	−151
170.95	184.62	242.69	427.40	528.01	1560.80	3094.83	3829.97

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